Reply

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(Manuscript received and in final form 24 May 2007)

We would like to thank Yao and Hughes (2008, hereafter YH08 for their comments) on our manuscript Teixeira et al. (2007, hereafter TRJ07). YH08 agree with TRJ07 about the need to study in detail the role and behavior of truncation errors in numerical integrations of nonlinear systems and recognize the relevance of our efforts.

YH08 criticize the assumption of TRJ07 that the trajectory computed numerically using the smallest time step is the closest to the correct trajectory of the ordinary differential equations. This appears to be a reference to statements made in the main body of the text and in particular in the captions of Figs. 2 and 4 in TRJ07. These figures show the separation (L2 norm error) of numerically computed trajectories for different time steps from a reference trajectory. Ideally this reference trajectory should be the correct (analytic) solution of the differential equations. Since the analytic solution of the Lorenz (1963) system of differential equations (or of the quasigeostrophic model from Fig. 4) is not known, the trajectory computed with the smallest time step was used as a proxy. This is appropriate in the sense that the trajectory computed using the smallest time step is typically closer to the correct (analytic) solution than trajectories computed using larger time steps. In practice, in terms of the analysis presented in TRJ07 it is sufficient that the solution with the shortest time step remains the closest to the analytic solution in the time interval that is being analyzed (while the solutions obtained with larger time steps decouple from the one with the shortest time step). This reasoning is implied in TRJ07 and the decoupling time formula discussed there provides an estimate of the typical separation time. The smallest time step integration typically remains the best proxy for the longest time, and larger time step solutions typically separate before smaller time step solutions. In particular, in TRJ07 it is discussed how, for one of the case studies, the errors calculated using $\Delta t = 10^{-6}$ Lorenz time units (LTU) as the reference value are in general similar to those obtained using 10^{-7} LTU as the reference value (except for $\Delta t = 10^{-6}$ LTU, which has zero error for this case). This confirms that the characteristics of these particular results shown in TRJ07 are not sensitive to this reference value in this time period.

However, YH08 are correct in cautioning that the smallest time step solution is not guaranteed to be the closest to the correct solution, inasmuch as the error from a numerical integration that uses a smaller time step can be at times larger than the error from a numerical integration that uses a larger time step.

The essential mathematical problem is that although the evolution operator of a numerical integration is homotopic to the evolution operator of the ordinary differential equations in finite time, these mappings are not guaranteed to be conjugate as dynamical systems. Consequently, the dynamics (i.e., trajectories) of the

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DOI: 10.1175/2007JAS2523.1

numerical integration is not the same as that of the ordinary differential equations. YH08 mention a mechanism for this failure and hence provide a concrete example of their warning.

Note that any conclusions from this type of study depend on the manner in which the error is defined and on the nature of the models that are being analyzed. More complex models, such as the quasigeostrophic model and the Navy Operational Global Atmospheric Prediction System (NOGAPS) that are analyzed in detail in TRJ07, show a different truncation error growth behavior, as is illustrated in TRJ07.

YH08 also describe what they call "explosive" growth of errors. It is our opinion that care should be taken in using the word explosive in this context. In YH08 the term explosive should not be understood in its usual mathematical sense of finite blow-up (escape) time. Error growth is indeed not uniform over the Lorenz attractor (e.g., Smith et al. 1999) and this can be seen for example in the spikes, jumps, and separation of trajectories that occur in the figures shown in TRJ07. In particular, Fig. 3b shows how for time steps 0.01 and 0.0001 LTU the solution has a completely different behavior (for that time window and those initial conditions) than the one obtained with a time step of 0.001 LTU.

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