Interaction of Upper-Tropospheric Turbulence and Gravity Waves as Obtained from Spectral and Structure Function Analyses

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(Manuscript received 29 October 2007, in final form 28 December 2007)

ABSTRACT
Spectral and structure function analyses of horizontal velocity fields observed in the upper troposphere and lower stratosphere during the Severe Clear Air Turbulence Collides with Air Traffic (SCATCAT) field program, conducted over the Pacific, were carried out in an effort to identify the scale interactions of turbulence and small-scale gravity waves. Because of the intermittent nature of turbulence, these analyses were conducted by clearly separating out the cases when turbulence did or did not occur in the data. In the presence of turbulence, transitional power spectra from $k^{-2}$ to $k^{-5/3}$ were found to be associated with gravity waves and turbulence, respectively. The second-order structure function analysis was able to translate these spectral slopes into $r$ and $r^{2/3}$ scaling, consistent with the Monin and Yaglom conversion law, in physical space, which presented clearer pictures of scale interactions between turbulence and gravity waves. The third-order structure function analysis indicated the existence of a narrow region of inverse energy cascade from the scales of turbulence up to the gravity waves scales. This inverse energy cascade region was linked to the occurrence of Kelvin–Helmholtz instability and other wave-amplifying mechanisms, which were conjectured to lead to the breaking of small-scale gravity waves and the ensuing generation of turbulence. The multifractal analyses revealed further scale breaks between gravity waves and turbulence. The roughness and intermittent properties were also calculated for turbulence and gravity waves, respectively. Based on these properties, turbulence and gravity waves in a bifractal parameter space were mapped. In this way, their physical and statistical attributes were clearly manifested and understood.

1. Introduction

The Severe Clear Air Turbulence Colliding with Aircraft Traffic (SCATCAT) experiment was conducted as part of the Winter Storms Reconnaissance Program during February 2001 by scientists from the National Oceanic and Atmospheric Administration (NOAA) and National Center for Atmospheric Research (NCAR). The experiment was intended to improve understanding of clear-air turbulence (CAT) generation mechanisms in the upper troposphere and lower stratosphere, where commercial aircraft are routinely routed. On 17–18 February, the NOAA G-IV aircraft executed a stack of four constant-altitude flights, at 10.1-, 10.7-, 11.4-, and 12.5-km levels, over the north-central Pacific Ocean. Koch et al. (2005) have described this mission in detail. Figure 1, reproduced from their paper, schematically shows these flight tracks (black arrowed lines, with yellow segments indicating the locations where turbulence was observed) overlain on isentropic surfaces. The isentropic surfaces and wind isotachs constructed from dropsonde measurements conducted during the flight indicate the presence of an upper-level front associated with a strong jet streak. The SCATCAT flights were conducted right above this jet core in the left exit region of the jet. Gravity waves are apparent along the flight tracks just above the altitude of the jet core. The diagnostic turbulence kinetic energy function (DTF) is calculated and plotted as contours, shaded with colors of orange, yellow, and green. The DTF is a model turbulence diagnostic variable, which is defined as

$$\text{DTF} = (0.75 - 0.52 \text{Re}_f) \frac{0.75 S^2 K_m}{N},$$

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DOI: 10.1175/2007JAS2660.1

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where $R_i = R_i K_h/K_m$ is the flux form of Richardson number $R_i$, $K_h$ and $K_m$ are the eddy diffusivity coefficients for heat and momentum, respectively, $S = [(\partial u/\partial z)^2 + (\partial v/\partial z)^2]^{1/2}$ is the vertical wind shear, and $N$ is the Brunt–Väisälä frequency (for details, see Marroquin and Stankov 1991). Because $R_i \leq 1$, the DTF will typically present larger, positive values for smaller values of static stability and $R_i$ (particularly for the occurrence of turbulence when $R_i \leq 1/4$) and larger values of vertical wind shear. From Fig. 1, one can see that the DTF diagnostic displays large values aligned with the frontal zone and within regions of substantial vertical wind shears, but misses some of the regions of turbulence associated with gravity wave activity experienced along the flight tracks (e.g., the yellow segments at 10.7- and 11.4-km altitudes).

In addition to the dropsonde measurements, in-flight observations of horizontal and vertical velocity, temperature, ozone concentration, and other variables were made. These in-flight observations were sampled at a frequency of 25 Hz (indicating a horizontal spatial scale of 9.2 m given a constant aircraft speed of 230 m s$^{-1}$). As a result, the fluctuations on the small scales of gravity waves and turbulence could be adequately resolved. Lu et al. (2005a,b) studied the polarization of the small-scale gravity waves and their interaction with turbulence in terms of Stokes polarization parameters using this dataset. In particular, Lu et al. (2005b) found that the intermittent generation of turbulence was related to high temporal variability of polarization energy associated with small-scale gravity waves, and that the turbulence intensity surges tended

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**Fig. 1.** The four stacked legs of the G-IV tracks (at 10.1-, 10.7-, 11.4-, and 12.5-km altitudes; black lines with arrows depicting the aircraft path), and those segments of the legs (yellow highlighting) for which moderate or greater turbulence was diagnosed in the flight-level data. Also shown is the vertical cross section of wind magnitude (5 m s$^{-1}$ isotachs; blue lines), potential temperature (2-K isentropes; black lines), and DTF turbulence diagnostic computed from dropsondes (note release times at bottom of display) from 2326:06 UTC 17 Feb through 0024:02 UTC 18 Feb 2001. The jet core is highlighted by winds in excess of 80 m s$^{-1}$ (maximum of 100 m s$^{-1}$), and DTF values are contoured at 0.6 and 1.0 m$^2$ s$^{-3}$ (yellow and red areas, respectively). Note the distance scale and the aircraft-flight altitudes (km) at the top and the side of the display. [After Koch et al. (2005).]
to simultaneously bring down the level of the polarized energy of gravity waves. Koch et al. (2005) and Lane et al. (2004) showed that the gravity waves experienced an energy cascade from large to small scales, and continuously down to the scales of turbulence. They pointed out that such an energy cascade could be realized via nonlinear wave steepening (Weinstock 1986, 1987; Dewan 1979, 1997) and the consequential reduction of static stability and the increase of wind shear (i.e., the reduction of the local Richardson number to a subcritical value). However, many aspects of the interaction of turbulence and gravity waves have not yet been fully explored from these studies. These include the spectral and structure function characteristics associated with these small-scale gravity waves and turbulence, the transition and interaction between these two flow regimes identified from the structure function analysis, and their multifractal properties.

Spectral and structure function analysis methods have been widely used to analyze the atmospheric and oceanic energy spectra. For the analysis of turbulence, the Kolmogorov theory (Kolmogorov 1941) has been the benchmark for identifying the existence of turbulence. For scales that are larger than the usual characteristics of 3D turbulence, many climatological and case studies have been carried out to characterize their power spectra and compare them to those of turbulence (Vinnichenko et al. 1980; Nastrom et al. 1984; Nastrom and Gage 1985; Gage and Nastrom 1986; Bacmeister et al. 1996). Because turbulence is highly intermittent both in space and time, it is important when studying these spectra to clearly separate out cases both with and without the presence of turbulence. Also, because of the complexity of turbulence, the energy cascade is different between 2D and 3D turbulence (Kraichnan 1967, 1971; Charney 1971; Lilly 1983, 1989; Tung and Orlando 2003). In particular, Tung and Orlando (2003) discussed the controversies in several classic theories about the directions of energy fluxes in relation to the transition of flow regimes from planetary, to synoptic, subsynoptic, meso-, and dissipation scales. It is necessary to understand the direction of energy flux in order to clearly identify various physical processes responsible for turbulence. With the recent theoretical development of the third-order structure function analysis (Lindborg 1999; Lindborg and Cho 2001) and the studies of scaling of mesoscale gravity waves and turbulence (Cho and Lindborg 2001; Cho et al. 2001) in mind, we perform spectral and structure function analyses of the horizontal velocity field observed during SCATCAT. In particular, we are interested in the interaction between gravity waves and turbulence, and the energy cascade between the two flow regimes.

Davis et al. (1994) and Marshak et al. (1997) demonstrated a systematic way to analyze multifractal processes using the multiscaler structure function and singular measure methods. These analyses will render two important parameters: the Hurst parameter and intermittency, which are physically meaningful for waves and turbulence in the upper troposphere. In this study, we will also carry out the multiscaler structure function and singular-measure analyses for upper-tropospheric velocity observations following their method.

2. The analysis method and its theoretical background

Using dimensional analysis, the Kolmogorov theory (Kolmogorov 1941) predicts that for 3D isotropic turbulence the power spectral density of kinetic energy should follow the $-5/3$-slope law, that is,

$$ E(k) = C_k k^{2/3} k^{-5/3}, $$

where $E$ denotes the power spectral density (PSD), $k$ is the wavenumber, $C_k$ is a universal constant (the Kolmogorov constant), and $\varepsilon$ is the energy dissipation rate. The spectral region where PSD preserves this slope is considered to be the turbulent inertial range within which the energy input from larger scales passes through to dissipation scales without any storage. This power law has been observed in many atmospheric, oceanic, and laboratory turbulence studies.

Some larger-scale features also approximate this slope. For example, Dewan (1979, 1997) argued that atmospheric gravity waves possess a similar inertial range. Nastrom et al. (1984) and Nastrom and Gage (1985) analyzed commercial airplane data to show that the energy spectrum transition from the large synoptic scale to mesoscale and turbulence follows a $k^{-3}$ to $k^{-5/3}$ behavior change. However, intermediate slopes between these two values are quite possible. For example, Garrett and Munk (1972, 1975) discussed the $k^{-2}$ spectral slope for internal gravity waves in the ocean. The Bolgiano scaling for a buoyancy subrange (Bolgiano 1959) actually predicts a $k^{-11/5}$ power law. Skamarock (2004) found that mesoscale numerical weather prediction models reproduce the observed $k^{-3}$ wavenumber dependence characteristic of the mesoscale and its transition to a less steep $k^{-5/3}$ slope observed at larger scales. A general consensus is that the spectral power law for scales larger than the turbulence inertial subrange should lie between $k^{-1}$ and $k^{-3}$ (Cho et al. 2001). Therefore, for any nonstationary, self-similar (self-affine) process, a general power law can be written as

$$ E(k) \propto k^{-\beta} \quad \text{where} \quad 1 \leq \beta \leq 3. $$
An estimate of this generalized power law will be given in section 6, following the results of Davis et al. (1994) and Marshak et al. (1997).

The Kolmogorov theory can also be derived in physical space, in which case the power law is expressed in terms of a structure function constructed from velocity or passive scalar variables. For example, the second-order structure function of the velocity field is defined as (e.g., Monin and Yaglom 1975; Lesieur 1993)

$$S_2(r) = \langle (u(x+r) - u(x))^2 \rangle = \langle (\delta u(r))^2 \rangle,$$

where $r$ is a separation distance along the $x$ direction and the angle brackets denote an ensemble average. It is not difficult to show (e.g., Orszag 1977) that for the inertial range the power law in physical space should be

$$S_2(r) = 4.82C_k \varepsilon^{2/3}r^{-5/3},$$

which is to say that the computed second-order structure function scales as the 2/3 power of the spatial distance. In fact, Monin and Yaglom (1975) show that given a power law for the structure function $r^n$, the spectral representation of this power law should be converted as $k^{-(1+\alpha)}$. According to this conversion law, it is a simple matter to compare and check the data for spectral and structure function analyses.

It is further argued in Lindborg (1999), Cho and Lindborg (2001), and Lindborg and Cho (2001) that the third-order structure function provides a more useful analysis of atmospheric turbulence than that from the second-order structure function. Not only does the third-order structure function remove the arbitrariness of the universal constant appearing in (4), but it also provides the direction of energy flux as either a downscale or upscale cascade. The third-order structure function (for the diagonal part) can be scaled by

$$S_3(r) = \langle (\delta u_L(r))^3 \rangle + 2(\delta u_L(r)\delta u_T(r))^2) = -\frac{4}{3} \varepsilon r,$$

where $u_L$ and $u_T$ denote the two components of the horizontal velocity field: the longitudinal (along the path) and transverse (normal to the path) velocity, respectively. Although the third-order structure function defined as such scales as $r$, it can take on either positive or negative values. The sign for the computed third-order structure function bears the following meaning: the energy cascade is downward (from large to small scales) if negative; and upward (from small to large scales) if positive. It is this property of the structure function that provides identifications in the present study for various important physical processes involved in the interaction of gravity waves and turbulence.

3. The data and the spectral analysis

The observed longitudinal (along the flight track) and transverse (perpendicular to the flight track) horizontal velocity fields were carefully analyzed for the four flight tracks carried out during the SCATCAT mission at altitudes of 10.1, 10.7, 11.4, and 12.5 km. Because analysis showed that the 10.7-km track had significant deviations in direction, this leg was removed from further analysis. The data for the other three flight tracks were unidirectional and at a constant altitude. To avoid contamination during aircraft ascent/descent and turns during the transition between two flight tracks, data at the beginning and ends of these legs were truncated. Gravity wave activity and turbulence associated with these observations were spectrally analyzed and numerically modeled in Koch et al. (2005), Lane et al. (2004), and Lu et al. (2005a,b). In particular, gravity wave signals were identified at all levels, whereas turbulence was found only at the 10.1- and 11.4-km levels. The lack of turbulence activity at 12.5 km may be due to the fact that gravity waves were weakening at this level, and hence the wave-induced shear instability was not significant enough to lower the Richardson number below its critical value for turbulence to occur (Fua et al. 1982). The higher Richardson number in this region was the result of increased static stability at this higher altitude. Differences among the three flight levels (two have gravity waves with observed turbulence and one has gravity waves but without observed turbulence) pose an interesting question: can spectral and structure function analyses of these data illuminate the different physics of turbulence and gravity waves and their interaction?

Spectral analyses of data at the three levels were performed and the power spectral densities are displayed in Figs. 2a–c, where the PSD of longitudinal velocity is shown as a function of a 2π-normalized horizontal wavenumber ($k/2\pi$) in log–log coordinates. Here we have used Taylor’s hypothesis (Monin and Yaglom 1975) to transform the time series data into spatial distributions based on the fact that the aircraft traveled at a constant speed of 230 m s$^{-1}$. For reference, two spectral slopes are plotted in these figures: the blue line indicates the $k^{-5/3}$ slope and the red line indicates the $k^{-2}$ slope. The smoothed spectra are shown in a light-green color, for comparison with the two reference slopes. From these figures it can be seen that the resolvable features range from hundreds of kilometers ($k \sim 10^{-5} m^{-1}$) to tens of meters in wavelength ($k \sim 10^{-1} m^{-1}$). Small-scale gravity waves dominate the spectral domain from wavelengths ($\lambda$) of a few kilometers to tens of kilometers (Lu et al. 2005a), while tur-
bulence can be identified in the spectrum from tens of meters to a few hundred meters (Koch et al. 2005; Lu et al. 2005b). It is quite evident from the figures that when turbulence occurs at 10.1- and 11.4-km levels, there are significant fluctuations of spectral power starting from $10^{-3} \text{ m}^{-1}$ ($\lambda \approx 1$-km wavelength) to a larger wavenumber, whereas such energetic fluctuations are missing in the 12.5-km level where Koch et al. (2005) suggests an absence of turbulence activity. Such identified turbulence spectra in Fig. 2a (10.1 km) and Fig. 2b (11.4 km) are very well depicted by the Kolmogorov scaling ($k^{-5/3}$ slope), known as the turbulent inertial range, whereas at the gravity wave spectra, the scaling is closer to the $k^{-2}$ slope. At 12.5-km level (Fig. 2c), where turbulence is missing, the spectrum follows the $k^{-2}$ slope entirely, and the $k^{-5/3}$ slope does not provide a good fit for any part of the spectrum until at the very end (at high wavenumber) of the spectrum.

4. Second-order structure function analysis

It is clear from the analysis in the previous section that the PSDs exhibit different distributions when turbulence is either present or absent. The Monin and Yaglom (1975) conversion law allows us to compare and check if these results are consistent with structure function analysis.

Following Cho et al. (2001), we compute the structure function in the following steps: 1) Using Taylor’s hypothesis with a constant aircraft speed, we first convert the sampled time series data into a spatial distribution. 2) We then determine the longitudinal and transverse velocity differences at spatial intervals of $r = (230 \text{ m s}^{-1}/25 \text{ Hz})^{2n}$, for integers $n = 0, 1, \ldots, N$, where $N$ is determined by the maximum data length (in the current case $N = 14$). 3) We compute the structure function by taking an ensemble average of the difference of the horizontal velocity field for a given spatial interval.

In Fig. 3 we plot the second-order structure functions at three different altitudes, as functions of spatial scale $r$. The second-order structure function we present here is the average of $(\delta u_x)^2$ and $(\delta u_y)^2$, an approximate measure of the isotropic second-order structure function. Each curve represents the ensemble average de-
fined in (3). The error bars show the deviation from these ensemble means. The error bounds were calculated based on the standard deviation (Cho et al. 2001). They typically displayed errors that were < ±10% of the averaged values, which indicate that the calculated structure functions are relatively robust. More importantly, the change in structure function slope with spatial scale \( r \) is unaffected by the magnitude of the uncertainties in the estimates.

One can see from the figure that at the 12.5-km level (where turbulence is not observed), the second-order structure function (black curve) follows the \( r \) scaling for most of the spatial scales. The inertial range occurs at the very end of the scale range. The \( r \) scaling for the most part is consistent with spectral analysis of the \( k^{-2} \) slope, which is identified as the gravity wave subrange. The very narrow range of \( r^{2/3} \) at the tail of this curve (about a few tens of meters) is consistent with the turbulence inertial range scaling with the \( k^{-5/3} \) slope, but was nearly missing (or at least not clearly identifiable) in the spectral analysis. This attribute is one advantage of a structure function analysis over a spectral analysis, because it provides a clearer picture of the scale interactions and physical processes.

At 10.1- (red curve) and 11.4- (blue curve) km levels, the presence of turbulence seems to have an effect to stretch the inertial range upscale, almost to the kilometer scale. The stronger the turbulence intensities [stronger at the 10.1-km level than at the 11.4-km level, as inferred from pilot reports and the wavelet time–frequency analysis in Koch et al. (2005)], the greater the extent of upscale inertial range shift that the structure function exhibits. At scales greater than 1 km, data at all levels display gravity wave scaling of \( r \), which agrees closely with the spectral scaling of the \( k^{-2} \) slope.

5. Third-order structure function analysis

Thus far, we have discussed scale interaction between gravity waves and turbulence using spectral analysis and the analysis of the second-order structure functions. To further our understanding of the possible physical processes associated with the two-flow regimes, we also computed and analyzed the third-order structure functions for the aircraft observational data collected at the three different altitudes. In Koch et al. (2005), we presented a third-order structure function for the 10.1-km level only. In this section, we complete the third-order structure function analyses by clearly separating processes with and without turbulence activity. In this way, the interaction of upper-level turbulence and gravity waves can be understood better.
In Fig. 4 we plot the third-order structure functions for the observed horizontal velocity fields at 12.5 (black curve), 11.4 (blue curve), and 10.1 (red curve) km. Here we computed the diagonal part of the third-order structure function (Lindborg and Cho 2001; Cho and Lindborg 2001). The solid and dashed segments in the colored curves for the structure functions at 10.1- and 11.4-km levels indicate the directions of the energy flux, as discussed at the end of section 2. The solid segments are values possessing negative signs (but plotted as their absolute values) in the calculated $S_3(r)$, representing a downscale energy cascade, and the dashed segments are values with positive signs in the $S_3(r)$, representing an upscale (inverse) energy cascade. Several immediately identifiable features are shown in this figure. First, an extended inertial range with the approximate slope $r$ (verified with a linear regression of these points for this segments, shown as the two dotted lines), consistent with the scaling result in Eq. (5), exists in the data at 10.1 and 11.4 km where turbulence was observed. The extended inertial range occupies spatial scales from a few tens of meters to a few hundreds of meters (close to the order of a kilometer). Second, for data with observed turbulence, the gravity wave subrange obeys an $r^2$ slope. When there is no observed turbulence (at 12.5-km level, e.g.), the gravity waves obey an $r^3$ slope, which is consistent with the second term on the right-side of Eq. (4) in Cho and Lindborg (2001) for large-scale processes (not discussed in this paper). Third, the extended gravity wave scaling at 12.5 km has squashed the inertial range into the tail of the curve with relatively narrow spatial scales, extending only to about 30–40 m with relatively large variances. Finally, the absence of turbulence generation at 12.5 km and relatively weak turbulence activity at 11.4 km may be related to a reduced energy input at a very large scale (relatively flat curves at the largest spatial scale, ~100 km).

In addition to the above observations, the most striking result of Fig. 4 is that at 12.5 km, where no turbulence is generated, there seems to be an “open” path for energy to cascade all the way from large scales down to the dissipation scale (the solid curve), whereas when turbulence does occur (at the 10.1- and 11.4-km levels), there is a very narrow scale (the dashed segments) relative to which energy converges from both the large and small scales. We believe that a plausible physical explanation for the presence of such an inverse energy cas-
cascade is that a Kelvin–Helmholtz instability is present, which leads to gravity wave breaking and hence an energy cascade that continues down through the turbulent inertial range and dissipates at the tail. A set of physical reasons supporting this postulation can be put forward. First, an inverse energy cascade is usually associated with relatively large (compared to 3D isotropic turbulence) quasi-two-dimensional structures. The Kelvin–Helmholtz wave is a flow regime that belongs to this 2D structure (Lesieur 1993). Second, the Kelvin–Helmholtz waves usually occur at scales from a few hundred meters to a few kilometers. The dashed regions in the curves for the 10.1- and 11.4-km levels are certainly consistent with these scales. Third, the Kelvin–Helmholtz instability is known to be conducive to the generation of turbulence. That the turbulent inertial range occurred at scales just below the inverse energy cascade, in contrast to the situation at 12.5 km where there is no inverse cascade (and hence no turbulence generation) until the dissipation range, further supports this conclusion.

The generation of turbulence resulting from wave breaking may, in turn, possibly force gravity waves whose vertical wavelength matches the depth of the turbulent layer (Scinocca and Ford 2000; Bühler et al. 1999). Such a process would lead to an energy cascade upscale. Because the requisite vertical wavelength would be ~0.5–1.0 km, as given by the depth of the DTF turbulence diagnostic, as well as the layer of Richardson number Ri < 1.0 computed from dropsondes and as predicted by the cloud-revolving model shown in Koch et al. (2005), the forced gravity waves would necessarily be high-frequency ones. As discussed by Achatz (2007a,b), high-frequency gravity waves, even those that are statically and dynamically stable, are characterized by such strong horizontal buoyancy and velocity gradients that they can be destabilized by the growth of modal and nonmodal gravity waves well below the traditional Miles–Howard threshold of Ri = 0.25. The energetics of three-dimensional instabilities driven by “density gradient dynamics” and linear/nonlinear growth of modal and nonmodal small-scale gravity waves have been discussed by Lombard and Riley (1996), Fritts et al. (2006), and Achatz (2007a,b). Such mechanisms might also be responsible for wave breaking and energy cascades both downscale and upscale.

Errors were also calculated for the third-order structure function (drawn as error bars in Fig. 4). The errors showed large values in the transitions between the scales. For example, the third-order structure functions for 10.1- and 11.4-km levels were shown to have large errors between slope transitions from $r^2$ to $r$, at the same locations where the energy flux displayed convergence from both small and large scales (the dashed segments). Large errors also occurred in the tail range of the third-order structure function for the 12.5-km level. These errors indicate a great deal of uncertainty either associated with scale transitions in these physical processes or in the dissipative processes of gravity waves. Away from these scale-break regions, errors were typically small, which again indicated that the slopes of these structure functions are statistically robust.

6. Multiorder structure function analysis and singular measures

In the previous sections, we examined the spectral and the second- and third-order structure function properties of SCATCAT data for three different altitudes. Scale breaks were identified from these analyses. In particular, the spectral analysis revealed a turbulence spectral slope of $-5/3$ for high wavenumbers and a spectral slope of $-2$ for low wavenumbers. The second-order structure functions presented a consistent scaling in the physical space for these two distinctive scales of motions. A further analysis of the third-order structure function indicated an inverse energy flux transport process, separating these two scales. These results suggested that the signals measured at 10.1- and 11.4-km levels during the SCATCAT field campaign display some multifractal characterizations (in the sense of nonmonoscaling).

Presumably, these two scales are related to different physical flow regimes. What various physical properties can be revealed from a multifractal analysis for waves and turbulence at the upper troposphere? How do we characterize these two physical processes? To answer these questions, we conduct further analyses using the multiorder structure functions and singular measures.

Davis et al. (1994) and Marshak et al. (1997) demonstrated a systematic way to analyze self-similar (or self-affine) processes using the multiorder structure function and singular measure methods. These analyses will render two important parameters: the Hurst parameter and intermittency, which are physically meaningful for waves and turbulence in the upper troposphere. Following their method, we carry out multiorder structure function and singular measure analyses for upper-tropospheric velocity observations.

a. Smoothness/roughness property associated with CAT and gravity waves

Similar to (3), the $q$th-order structure function can be defined as
where \( q \) can be any real number (limit for positive only for this study). The \( q \)-th order structure function is simply a \( q \)-th moment of a generalized correlation function. Notice that the definition (6) is slightly different than (3) and (5) in that all moments are calculated with an ensemble average of their absolute values, so that there is no sign change for \( S_q(r) \) when \( q \) is an odd number. Assuming that the process \( u(x) \) is scale invariant and self-similar (self-affine), then the \( q \)-th order structure function can be scaled as

\[
S_q(r) = C_q r^\zeta(q) \quad \text{for} \quad q \geq 0, \tag{7}
\]

where \( C_q \) is a proportional constant, and \( \zeta(q) \) is the scaling power function. According to Frisch (1991) and Marshak et al. (1997), a monotonic, nonincreasing function can be defined using this power function:

\[
H(q) = \frac{\zeta(q)}{q}, \tag{8}
\]

where \( H(q) \) provides a set of scaling parameters with respect to a given set of \( q \). The most frequently cited two scaling parameters are \( H(2) = H_2 = \zeta(2)/2 \) and \( H(1) = H_1 = \zeta(1) \) (Marshak et al. 1997). The parameter \( H_2 \) gives an estimate for the general spectral power law [Eq. (2)] by using the Monin and Yaglom (1975) conversion law, which results in

\[
1 \leq \beta = 2H_2 + 1 \leq 3. \tag{9}
\]

The upper bound can be obtained when the analyzed process is close to being an everywhere differentiable signal. The parameter \( H_1 \), on the other hand, gives

\[
0 < H_1 = \zeta(1) < 1, \tag{10}
\]

which is the well-known self-similar (self-affine) exponent or Hurst parameter. The Hurst parameter measures the roughness (nonstationarity) of the signal in data, with 0 representing the roughest functional series, such as white noise, and 1 representing an infinite smoothness function (Marshak et al. 1997).

Following Kerr et al. (2001), we use the longitudinal velocity (the velocity component along the flight direction) data to calculate the \( q \)-th order structure function, with \( q = 0, 0.25, 0.5, \ldots, 5 \). For this multifractal analysis, only the time series taken at the 10.1- and 11.4-km levels (where both turbulence and gravity waves exist) are analyzed below. The procedure is essentially the same as the one for calculating the second- or third-order structure functions, except that the ensemble average of the \( q \)-th order structure function is replaced by an ensemble average of the absolute of the \( q \)-th order structure function, as noted above in (6). Other details have already been discussed in section 4.

Figures 5a,b show the first- to fifth-order structure functions, plotted on log–log coordinates, for the longitudinal velocity data at the 10.1- (Fig. 5a) and 11.4- (Fig. 5b) km levels. From these figures, particularly for the low moments of \( q = 1 \) or 2, one can see that two scalings (with different slopes) are present for these structure functions: above and below about 1 km. These two scalings represent a transition of turbulence scale to mesoscale (gravity waves), which is consistent with the analyses in the previous sections (viz., Fig. 3). Another apparent feature present in the third- (and higher) order structure function is the breaks between the two scalings. Note that a narrow zone of the breaks that occurred in the third-order structure function is
consistent with the analysis in section 5, even though these breaks involved no sign change (in Fig. 5), as contrasted to those of Fig. 4 for different samples but the same set of data. The scale breaks are relatively sharp for data at the 11.4-km level, and they occur at a horizontal scale between 1 and 2 km, whereas the relative broad breaks for data at the 10.1-km level extend from 300 m to 2 km in the horizontal scale. This difference indicates the severity of CAT activity, as discussed in the previous sections.

The Hurst parameters for these two scalings can be calculated by performing a least squares line fit to the first-order structure function. The two datasets (those for 10.1- and 11.4-km levels, respectively) result in the Hurst parameter for the turbulence scale \(H_1 = 0.30 \pm 0.03\) and for the gravity wave scale \(H_2 = 0.49 \pm 0.2\) (the error range comes from the estimate of the two cases). These results seem to be physically reasonable because the gravity wave packet has a smoother motion than turbulence. With an approximation of \(\langle \delta u(r)^q \rangle \approx \langle \delta u(r)^2 \rangle^{q/2}\) and (7)–(8), one obtains (Marshak et al. 1997) \(H(q) = H(2)\). Substituting this relation into (9) with values of \(H_1 = 1/3\) and \(H_2 = 1/2\) and using the Monin–Yaglom conversion law, one can straightforwardly verify that these Hurst parameters correspond to \(\beta = 2\) for the gravity wave and \(\beta = 5/3\) for the turbulence power laws [see Eq. (2)].

b. Intermittency/singularity property associated with CAT and gravity waves

The other multifractal statistic describes intermittency in nonstationary data. This statistical analysis is called singularity measures (Davis et al. 1994; Marshak et al. 1997). To demonstrate how this analysis is conducted, let us first write down the second-order structure function as

\[
|\Delta u(\eta, x)|^2 = |u(x + \eta) - u(x)|^2, \quad 0 \leq x \leq L - \eta,
\]

where \(L\) is the maximum data length and \(\eta\) is typically four times the Nyquist wavelength, that is, in the present case, \(\eta = 4\ell = 36.8\) m. A normalized, nonnegative measure can then be defined for this second-order structure function as

\[
\varepsilon(\eta, x) = \frac{1}{N} |\Delta u(\eta, x)|^2,
\]

where \(N = \langle |\Delta u(\eta; x)|^2 \rangle\) is an ensemble average of (11). The next step is to calculate the measure over a scale \(r\) \((\eta < r \leq L)\), where the following spatial average is performed:

\[
e(r, x) = \frac{1}{r} \int_{x}^{x+r} \varepsilon(\eta; x') \, dx', \quad \eta \leq x \leq L - r.
\]

The \(q\)th order of this measure can be obtained by taking an ensemble average, that is, \(\langle \varepsilon(r, x)^q \rangle = \langle \varepsilon(r)^q \rangle\). The self-similarity (or self-affinity) of the fluctuation will render this \(q\)th-order measure as

\[
\langle \varepsilon(r)^q \rangle \propto r^{-K(q)}, \quad \text{for} \quad q \geq 0.
\]

Knowing the \(K(q)\) exponent, a generalized dimension can be obtained as

\[
D(q) = 1 - \frac{K(q)}{q - 1}.
\]

The variation of \(D\) with \(q\) gives an indication of the degree of multifractality of a fluctuation. For \(D(q) = 1\), the fluctuation demonstrates a monoscaling. Otherwise, \(D(q) < 1\) for \(q > 0\); we have a highly singular system (Marshak et al. 1997). When \(q = 0\), \(D(0)\) is called the fractal dimension; when \(q = 1\), \(D(1)\) is called the information dimension; and when \(q = 2\), \(D(2)\) is called the correlation dimension, etc. (Carreras et al. 2000). Because we try to use as few multifractal parameters as possible to capture some physical properties of a fluctuation, only \(D(1)\) is considered, similar to the use of \(H = \xi(1)\) for a smoothness measure. It can be understood that the more information dimensions there are, the less singular the fluctuations, thus causing the fluctuation to be more like white noise. The intermittency parameter is defined to measure the singularity of a fluctuation. The more singular a fluctuation, the larger the intermittency and the fewer the information dimensions, resulting in the fluctuation being more like the Dirac delta function. With this interpretation, the intermittency can be understood as an information codimension of the fractality of a process, that is, \(1 - D(1)\). Let us denote the intermittency parameter as \(C_1\). Then, the intermittency of a fluctuation can be estimated as

\[
C_1 = C(1) = 1 - D(1) = 1 - \lim_{q \to 1} D(q) = \lim_{q \to 1} \frac{K(q)}{q - 1} = K'(1),
\]

where we have used the fact that \(K(1) = 0\). In fact, it is straightforward to verify that both \(K(0) = K(1) = 0\), using relations (12) and (14). The value of the intermittency parameter is typically in the range of \(0 \leq C_1 \leq K'(1) \leq 1\).

Using (11), (12), and (13), we calculated the \(q\)th-order singular measure as a function of the horizontal spatial scale \(\langle \varepsilon(r)^q \rangle\). We then plot the singular measure.
in log–log coordinates, which is shown in Figs. 6a,b, respectively, for data at 10.1- and 11.4-km levels. The solid, dashed, and dotted lines are the least squares line fit to the calculated data points, plotted in log–log coordinates.

higher moments, the break scales stretch slightly upscale, which separate mesoscales (solid line for \( r > 2 \) km) from the upper-tropospheric turbulence (CAT) scale (dashed line for \( 100 \text{ m} < r < 2 \text{ km} \)). Although there is an additional scale break present at about 100 m, this may not be physical. It may be a numerical artifact, not only because the singular measure calculations at that scale are close to the Nyquist wavelength, but also because of an electronic noise signal in the aircraft data processor prohibiting information to be retrieved at frequencies > 6 Hz (scales smaller than 36 m, which is quite close to the 100-m cutoff scale). In addition, such breaks did not show their presence in the structure function analyses. The physical interpretation of this additional break, if it is real, can be attributed to the scale transition from CAT (100 m < \( r < 2 \text{ km} \)) to the dissipation range (dotted line for \( r = 100 \text{ m} \)). Nevertheless, owing to uncertainty about this additional break, we only concentrate on the two scales for CAT and the gravity wave packet. The emphasis on the gravity wave packet here, rather than on a monochromatic gravity wave, is because a perfect wave form (monochromatic), which is infinitely smooth, nonsingular, and stationary, is hardly observed in nature. Wave signals presented in the observational data represent a combination of various transient waves and waves with different wavelengths, which form a wave packet. Wave packets can have some degree of multifractality because of their transient nature.

The line fitting of a singular measure renders slopes of self-similar (or self-affine) turbulence and gravity wave processes. One can plot these slopes as a function of \( q \), that is, \( K(q) \) according to Eq. (14). Figures 7a,b show the slopes \( K(q) \) as functions of \( q \) for data at the 10.1- and 11.4-km levels, respectively. To accurately capture the derivatives, we conducted this calculation with more data points (i.e., \( q = 0, 0.25, 0.5, \ldots, 5 \)) than shown in Figs. 6a,b. The diamonds indicate \( K(q) \) for the turbulence scale and the asterisks indicate \( K(q) \) for the wave scale. The tangent lines represent the derivatives for \( K(q) \) at the point \( q = 1 \), that is, \( K'(1) \). The solid slope is for waves and the dashed slope is for turbulence. The values of intermittency parameters for the 10.1-km level data are \( CT = 0.20 \) and \( CW = 0.12 \), respectively, for turbulence and wave packets. The corresponding values for the 11.4-km level data are \( CT = 0.36 \) and \( CW = 0.24 \), which are larger than those for the 10.1-km level. These results are consistent with the analyses in Fig. 16a in Koch et al. (2005), where both turbulence and wave packet signals showed relatively stronger singularity at the 11.4-km level than at the 10.1-km level. Furthermore, in relation to the notion of the level of “intermittency” being situation dependent,
the simultaneous enhancement of intermittency at the 11.4-km level (compared to the 10.1-km level) for both turbulence and wave packets indicates that the two are connected dynamically. This result is consistent with that in Lu et al. (2005b), where they found that a surge of turbulence activity is accompanied by an increase in wave energy, using the Stokes parameter analysis. Finally, it is noted from these results that turbulence is always more "intermittent" than the associated wave packets. This fits into a customary understanding of wave and turbulence, though wave packets can also be intermittent. Such an analysis or measure is capable of quantifying this physical characteristic of waves and turbulence.

c. Mapping of CAT and gravity waves in the bifractal parameter space

With the two selected multifractal parameters, a parameter space \( (H_1, C_1) \) can be constructed so that one is able to map various physical processes into this parameter space, to compare their physical and/or statistical attributes.

Because the natural range for both \( H_1 \) and \( C_1 \) is \((0, 1)\), the parameter space constitutes a bifractal plane \( (H_1, C_1) \in R^2 = (0, 1) \times (0, 1) \). The abscissa, representing the Hurst parameter, marks processes with increasing smoothness (or long-term correlation) from 0 to 1. The ordinate, representing the intermittency parameter, points to the direction of processes with increasing singularity/intermittency from 0 to 1 (see Fig. 8). The four corners of the parameter space, that is, \((0, 0)\), \((0, 1)\), \((1, 1)\), and \((1, 0)\), are occupied by well-known mathematical models or functions. These are, respectively, white noise, the Dirac delta function, the Heaviside step function, and the infinite differentiable function (Davis et al. 1994; Marshak et al. 1997). Along the Hurst parameter axis, \( H_1 = 0.5 \) typically marks a standard Browning motion and \( H_1 = 0.33 \) represents the fractal Browning motion (fBm).

Considering analyses of in-flight data for both layers, the singularity and intermittency associated with CAT and the gravity wave packet are plotted in the bifractal parameter space (Fig. 8). For a comparison, we also plot values of the Hurst parameter and intermittency for marine stratocumulus (Sc) clouds and plasma turbulence. The localization of marine Sc (shown with three small clouds in Fig. 8) comes directly from studies by Davis et al. (1994) and Marshak et al. (1997). The results from these studies were derived from analyses of cloud liquid water content (LWC) for data collected during the First International Commission on Cloud Physics Regional Experiment (FIRE) in 1987 and Atlantic Stratocumulus Transition Experiment (ASTEX) in 1992. The localization of plasma turbulence is derived from two separate plasma physics studies. Yu et al. (2003) investigated the values for the Hurst parameter in plasma turbulence, resulting in \( H_1 \) in a range from 0.55 to 0.78 (shown with crosses). Another plasma physics study by Carreras et al. (2000) came up with values for intermittency in plasma turbulence ranging from 0.15 to 0.30 (shown with a vertical line of square dots).

It is interesting to note that marine Sc is much less intermittent than CAT, but possesses about the same roughness level (or correlation range) as CAT. Gravity waves are more intermittent than marine Sc, but smoother than Sc. Plasma turbulence has a comparable...
intermittency level as its atmospheric counterpart, but is much smoother (or has longer correlation) even than small-scale atmospheric waves (i.e., wave packets).

Finally, we would like to emphasize the fact that the bifractal parameter space provides an important and useful diagnostic tool. Not only can we conduct a data analysis (such as this study), we can also use this method to diagnose models and parameterization schemes. For example, spectral and structure function properties from various numerical models can be analyzed and plotted in the bifractal parameter space. Various parameterization schemes, such as gravity wave drag, turbulence closure, and cloud-resolving models, etc., can all be compared inside this parameter space, along with observed signals for similar processes. Furthermore, scale breaks and energy spectrum transitions within atmospheric models (such as those studied by Tung and Orlando 2003; Skamarock 2004; Takahashi et al. 2006) can clearly be identified from multifractal analyses and compared in the bifractal parameter space.

7. Concluding remarks

In this study, we have applied both spectral and structure function analysis methods to a set of aircraft observational data collected at three altitudes (10.1, 11.4, and 12.5 km) in an effort to investigate the interaction between turbulence and gravity waves in a strong upper-level jet/front system. These two flow regimes, with adjacent time and spatial scales, were previously suggested to have causality relations, on the bases of wavelet, cross-spectral, and Stokes parameter analyses (Koch et al. 2005; Lu et al. 2005b).

The spectral analysis of the observed horizontal velocity field showed a transitional spectral slope, from \( k^{-2} \) at scales corresponding to gravity waves to the classic inertial range of \( k^{-5/3} \) for turbulence, when both physical processes were present. When turbulence was not observed, the spectrum presented a pure \( k^{-2} \) slope. It is reasonable to believe that such a \( k^{-2} \) spectrum represents a transitional subrange for the classic \( k^{-5/3} \) to \( k^{-3} \) paradigm (Nastrom and Gage 1985; Gage and Nastrom 1986). It should be pointed out, however, that the gravity waves that we analyzed here are small-scale internal gravity waves with spatial scales ranging from a few kilometers to a few tens of kilometers (Lu et al. 2005a,b; Koch et al. 2005; Lane et al. 2004), not the inertia–gravity waves of several hundred kilometers in scale.

The second-order structure function analysis of the same velocity field confirmed the results of spectral analysis, but presented clearer evidence of scale interactions between turbulence and gravity waves. In particular, the converted scaling into physical space corresponds to spectral power laws for turbulence and gravity waves (Monin and Yaglom 1975). While the structure function scaling tended to shift the inertial range into a very narrow range at small scales when turbulence was absent (viz., at the 12.5-km level), stretching of the inertial range upscale can be clearly seen in the structure function curves at the other two levels, where turbulence did exist.

The most intriguing results come from the third-order structure function analysis. At larger scales (from about 700 m to 1 km) of the stretched inertial ranges at the 10.1- and 11.4-km levels, a narrow region of an inverse energy cascade appears. This inverse energy cascade region was not found at the 12.5-km level. In this study, we attribute this region of an inverse energy cascade as evidence for the occurrence of Kelvin–Helmholtz instabilities, which lead to the breaking of gravity waves, the generation of turbulence, and the ensuing emission of envelope-scale gravity waves (Bühler et al. 1999; Scinocca and Ford 2000). The breaking of small-scale gravity waves may not be limited to the Kelvin–Helmholtz instability. Other wave instabilities may also be possible, for example, either the linear or nonlinear growth of modal and nonmodal
internal gravity waves studied in Lombard and Riley (1996), Fritts et al. (2006), and Achatz (2007a,b).

We also conducted multifractal analyses to better understand the physical and statistical attributes of upper-tropospheric turbulence and gravity waves. The Hurst parameter and intermittency parameter were calculated for the 10.1- and 11.4-km levels where both turbulence and gravity waves were detected. Scale breaks between the two were again present, which further supports the findings in the analyses of the spectra and second- and third-order structure functions. This scale separation resulted in different localizations in a bifractal parameter space for turbulence and gravity waves. This may help in better parameterizing subgrid-scale effects caused by the interaction of gravity waves and turbulence in a numerical model.

Finally, we stress that although we analyzed a high-resolution (25 Hz) dataset (with each flight level, there were about 30,000 observations), and many interesting physics were revealed from this study, only one experiment case was investigated here. To validate and generalize these results, large climatological datasets need to be analyzed.

Acknowledgments. The vertical cross-sectional analysis and quality control aircraft data were provided by Brian Jamison and Edward Tollerud. Discussions about various aspects of this work with them helped to clarify some key concepts. We thank Mariusz Pagowski, who conducted the NOAA internal review, and Ann Reiser, who conducted a technical edit of the manuscript. We also thank Professor K. K. Tung for his excellent editorialship, and two anonymous reviewers for their constructive comments and suggestions. Some of the motivation for this work also came from our discussions with Jerry Weinstock and Shaun Lovejoy.

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