A systematic scale analysis is performed for large-scale dynamics over the tropics. It is identified that two regimes are competing: 1) a dynamics characterized by balance between the vertical advection term and diabatic heating in the thermodynamic equation, realized at horizontal scales less than \( L \approx 10^3 \text{ km} \) given a velocity scale \( U \approx 10 \text{ m s}^{-1} \), and 2) a linear equatorial wave dynamics modulated by convective diabatic heating, realized at scales larger than \( L \approx 3 \times 10^3 \text{ km} \) given \( U \approx 3 \text{ m s}^{-1} \). Under the first dynamic regime (balanced), the system may be approximated as nondivergent to leading order in asymptotic expansion, as originally pointed out by Charney.

Inherent subtleties of scale analysis at large scales for the tropical atmosphere are emphasized. The subtleties chiefly arise from a strong sensitivity of the nondimensional parameter to the horizontal scale. This amounts to qualitatively different dynamic regimes for scales differing only by a factor of 3, as summarized above. Because any regime under asymptotic expansion may have a wider applicability than a formal scale analysis would suggest, the question of which one of the two identified regimes dominates can be answered only after extensive modeling and observational studies. Preliminary data analysis suggests that the balanced dynamics, originally proposed by Sobel, Nilsson, and Polvani, is relevant for a wider range than the strict scale analysis suggests.

A rather surprising conclusion from the present analysis is a likely persistence of balanced dynamics toward scales as small as the mesoscale \( L \approx 10^2 \text{ km} \). Leading-order nondivergence also becomes more likely the case for the smaller scales because otherwise the required diabatic heating rate becomes excessive compared to observations by increasing inversely proportionally with decreasing horizontal scales.
associated with deep convection and the vertical advection term to the leading order in the thermodynamic equation [cf. Eq. (7) below]. This furthermore amounts to an elimination of gravity waves to the leading order with an effective neutral stratification. Hence, this leads to a qualitatively different dynamics compared to a popular view of the large-scale tropical atmospheric dynamics dominated by equatorial waves (cf. Gill 1980; Wheeler and Kiladis 1999) because an effective neutral stratification eliminates most of the equatorial waves except for Rossby waves. Consequently, a main question to ask in the present paper is whether large-scale tropical atmospheric dynamics is dominated by balance or waves.

This question is more specifically posed with regard to the so-called convectively coupled equatorial waves (cf. Wheeler and Kiladis 1999).1 For this reason, the scale analysis is mostly limited to the deepest tropospheric mode because of its dominance in these phenomena, although contributions of shallower modes are significant (e.g., Stevens et al. 1977; Haertel et al. 2008). These shallower modes are examined briefly toward the end of the paper.

It transpires that the reason why such a basic issue has never been addressed in published literature is probably because the scale analysis for the large-scale tropical atmosphere is inherently inconclusive. For this reason, subtleties associated with the scale analyses are well emphasized in the present paper. Scale analysis is often better considered as an art rather than as a strict mathematics. Often a few minor factors arbitrarily kept or omitted can easily lead to a difference of orders of magnitude at the end. An obvious example is to simply multiply the factor three by an estimate of an order of magnitude at the end. The scale analysis must always be considered nothing more than a crude estimate for a dominant balance.

At the same time, however, the strength of the scale analysis is emphasized because it can constrain magnitudes of physical variables when it is used in a systematic manner. In the present work, by more or less fixing the magnitude of horizontal winds, systematic estimates for the time scale and the magnitudes of temperature fluctuation and diabatic heating are performed for different characteristic horizontal scales.

To perform the scale analysis as systematically as possible for a wide spectrum, the simplest possible case is considered.2 Neither multiple scales (Bender and Orszag 1978, chapter 11; Pedlosky 1987, section 3.20) nor anisotropies in the horizontal directions (Gill 1980; Stevens et al. 1990) are considered, although they may turn out to be ultimately important. By closely examining the consistency with observations, the present scale analysis also complements a mathematically systematic multiscale asymptotic analysis (e.g., Majda and Klein 2003).

Consequently, the main effort of the paper lies in presenting the ambiguities associated with the scale analysis as clearly as possible and specifying the scales under which either balanced dynamics or equatorial wave dynamics hold. The analysis is performed through sections 2–6. The basic set of equations is presented in the next section with some preliminary discussions, followed by a plan for the analysis. Section 7 is devoted to a preliminary data analysis that supplements the present theoretical analysis. The last section discusses various arising issues in closing the paper.

2. Basic set of equations

For our scale analysis, we take the primitive equation system in pressure coordinates, with \( p \) designating the pressure, as given by

\[
\frac{Dv^*}{Dt^*} = -\nabla_p^* \phi^* - fk \times v^* + F^*,
\]

\[
\frac{\partial \phi^*}{\partial p^*} = -\alpha^*,
\]

\[
\frac{D\theta^*}{Dt^*} + \omega^* \frac{\partial \bar{\theta}}{\partial p^*} = Q^*,
\]

\[
\nabla_p^* \cdot v^* + \frac{\partial \omega^*}{\partial p^*} = 0.
\]

Here, the momentum Eq. (1a) predicts the horizontal wind \( v \) from the given geopotential \( \phi' \) and Coriolis force \( f k \times v \) and the remaining (frictional) force \( F \).3 The Lagrangian time derivative is introduced by \( D/Dt = \delta/dt + v \cdot \nabla_p + \omega \partial/\partial p \) with \( \omega \) the vertical velocity (pressure velocity). The subscript \( p \) in nabla operators indicates that an operation is performed on a constant

---

1 The tropical stratosphere is clearly dominated by equatorial waves. With the absence of moist convection, the possibility of the balanced dynamics proposed by SNP01 is automatically excluded; thus, it is also excluded from consideration in the present paper.

2 It should be emphasized that even under the adopted simplifications, the analysis is hardly exhaustive (as indicated by footnotes throughout the text). Hence, a different scale analysis may still be necessary when the main interests are phenomena other than the so-called convective coupled equatorial waves.

3 Throughout the paper, it is implicitly assumed that by focusing our attention on the free troposphere dynamic, the frictional force \( F \) never becomes a dominant term of the system.
pressure surface. The Coriolis factor is defined by \( f = 2 \Omega \sin \varphi \) with the angular velocity \( \Omega \sim 10^{-4} \text{s}^{-1} \), \( \sim \) indicates an estimate of the order of magnitude for a given term rather than an exact one. Physical constants are also given only by rounded numbers throughout the paper because the orders of magnitudes are the sole concern in the present analysis. Note that as a result all the subsequent estimates implicitly include an uncertainty of a factor of 2–3.

The prime for the geopotential and the other thermodynamic quantities indicates the deviation from a reference climatological state. In the hydrostatic balance (1b), the perturbation specific volume \( \alpha' \) is related to the potential temperature \( \theta' \) by

\[
\alpha' = (R/p')(p'/p_0)^*\theta',
\]

where \( R \sim 300 \text{m}^2 \text{s}^{-2} \text{K}^{-1} \) is the gas constant; \( \kappa = R/C_p \) with \( C_p \) the specific heat with constant pressure; and \( p_0 = 1000 \text{hPa} \) is a reference pressure. In the thermodynamic Eq. (1c), \( Q \) is the total diabatic heating rate and \( \bar{\theta} \) is the reference state for the potential temperature, which is assumed to be a function of pressure only.

An asterisk is added to the variables as well as to the nabla operators in the above equations to indicate that these variables are dimensional, in contrast to the non-dimensional variables introduced later (represented without asterisks). To formally estimate the order of magnitude for each term in each equation, this set of equations is nondimensionalized by introducing a horizontal scale \( L \), a vertical scale (scale height) \( H \), and a velocity scale \( U \).

The vertical scale is fixed to be the scale height \( H = \rho_0/\rho g \sim 10 \text{km} \), with the acceleration of gravity \( g \sim 10 \text{m} \text{s}^{-2} \) and the surface air density \( \rho_0 \sim 1 \text{kg m}^{-3} \). This amounts to considering the deepest tropospheric mode. The case for the shallow modes will be discussed briefly in section 6d. No distinction is made between mean and perturbation winds in defining the velocity scale \( U \). Simply, the absolute magnitude is taken as a representative value for \( U \). The specific values for the remaining two scales \( L \) and \( U \) are left open for later specification.

As a result, the dimensional horizontal derivative, vertical derivative, and horizontal velocity are converted into nondimensional versions by the relations

\[
\nabla^* = \frac{1}{L} \nabla,
\]

\[
\frac{\partial}{\partial p^*} = \frac{\rho_0 g}{H \partial p}, \quad \text{and}
\]

\[
v^* = U v.
\]

Furthermore, the vertical velocity may be nondimensionalized by a scale \( W \) (conveniently given in the unit of m s\(^{-1}\)) as

\[
\omega^* = \frac{W}{\rho_0 g} \omega.
\]

By substituting all these nondimensionalizations, Eq. (1d) reduces to

\[
\frac{U}{L} \nabla_p \cdot \mathbf{v} + \frac{W}{H} \frac{\partial \omega}{\partial p} = 0.
\]

We would expect that these two terms balance; thus, the equation reduces to

\[
\nabla_p \cdot \mathbf{v} + \frac{\partial \omega}{\partial p} = 0.
\]

A required condition for the reduction provides an estimate for the vertical velocity scale \( W \):

\[
W = \frac{H}{L} U. \quad (2)
\]

Note that now and hereafter, to define a scale for a variable [e.g., the vertical velocity scale \( W \) here; see also Eqs. (4) and (13) below], we always assume that a term associated with the variable balances with the most dominant term. Then, later on in section 6, by specifying the orders of magnitudes for all the nondimensional parameters, the consistency of the initial assumption is checked with a given complete set of nondimensional equations. The consistency check sometimes leads to a rescaling of a nondimensional variable (cf. section 4b).\(^4\)

A well-known example in the midlatitude context is that the actual magnitude of the vertical velocity is smaller by a factor of the Rossby number than estimated by Eq. (2). A similar situation is encountered in sections 6a(1) and 6c(1) later. For this reason, the \( W \) scale (2) should be considered an upper limit.

The time derivative is handled slightly differently, and it is tentatively nondimensionalized by the advective time scale \( L/U \):

\[
\frac{\partial}{\partial t^*} \approx \frac{U}{L} \frac{\partial}{\partial t},
\]

and alternative possibilities are considered a posteriori as required. This strategy allows us to consider the possibility of a balanced dynamics (SNP01) apart from wave dynamics. In the latter case, the nondimensional time \( t \) may be multiplied by a nondimensional constant

\(^4\) Note that as a practice the initially introduced variable scale is not redefined.
to allow for a faster time scale [cf. Eqs. (17) and (18) below].

For convenience, the Lagrangian time derivative would also be initially nondimensionally presented as

\[ \frac{D}{Dt^*} = \frac{U}{L} \frac{D}{Dt}. \]

However, this should strictly be considered as a short-handed expression, and it is valid only if the time is scaled by the advective time scale and also only if the vertical velocity is scaled by Eq. (2) in the final scale analysis. Once one of these assumptions breaks down, each of the terms in Lagrangian time derivative must be considered separately.

In the following three sections, preliminary scale analyses are performed on Eqs. (1b), (1c), and (1a). We begin with the hydrostatic balance (1b), the simplest, in the next section. The thermodynamic Eq. (1c) considered in section 4 constitutes a core of our scale analysis because the anticipated balance condition involves the thermodynamics (cf. SNP01). Finally, the momentum Eq. (1a) is analyzed in section 5. This equation involves the highest degree of uncertainties; thus, particular scrutiny is required. A systematic scale analysis is then performed in section 6.

3. Hydrostatic balance

To nondimensionalize the hydrostatic balance (1b), we set the order of magnitude of fluctuations for the geopotential and potential temperature to \( \delta \phi \) and \( \theta \), respectively, leading to the nondimensionalizations

\[ \phi^* = (\delta \phi) \phi', \quad \theta^* = (\delta \theta) \theta'. \]

We also nondimensionalize the pressure \( p \) by the reference pressure \( p_0 \) at the surface; i.e.,

\[ p^* = \frac{p}{p_0}. \]

By substituting all those nondimensionalizations into Eq. (1b), the nondimensional version of the hydrostatic balance is given by

\[ \left( \frac{\delta \phi}{p_0} \right) \frac{\partial \phi'}{\partial \bar{p}} = -\left( \frac{R}{p_0} \delta \theta \right)p^{-1} \theta'. \]

On both sides, the nondimensional variables are expected to scale to order unity. In other words, we would like to reduce Eq. (3) into the form

\[ \frac{\partial \phi'}{\partial \bar{p}} = -p^{-1} \theta'. \]

A required condition is

\[ \frac{\delta \phi}{p_0} = -\frac{R}{p_0} \delta \theta. \]

Then the above condition provides an estimate for the order of magnitude \( \delta \theta \) for the potential temperature in terms of \( \delta \phi \):

\[ \delta \theta = \delta \phi/R. \] (4)

Note that the nondimensionalization of the other equations are performed in a similar manner; details will be omitted hereafter.

4. Thermodynamic equation

In nondimensionalizing the thermodynamic Eq. (1c), we estimate the order of magnitude of the second term by \( W(d\bar{\theta}/dz)_0 \), with \( W \) estimated by Eq. (2) and \( (d\bar{\theta}/dz)_0 = 3 \times 10^{-2} \text{K m}^{-1} \) a typical vertical gradient of the climatological potential temperature. We also designate the magnitude of diabatic heating by \( \delta Q \); thus, \( Q^* = (\delta Q)Q \). As a result, a nondimensional thermodynamic equation is given by

\[ \frac{D\theta'}{Dt} + \gamma \frac{\partial \bar{\theta}}{\bar{p}} = \eta Q, \] (5)

where the two nondimensional parameters have been introduced

\[ \gamma = \frac{\Delta \theta}{\delta \theta}, \quad \eta = \frac{L \delta Q}{U \delta \theta}, \] (6a)

and \( \Delta \theta = H(d\bar{\theta}/dz)_0 \sim 30 \text{ K} \) is a total change of the potential temperature over the troposphere. Note that the stratification parameter \( \gamma \) is related to the Richardson number:

\[ R_i = \left( \frac{H}{U} \right)^2 \frac{g}{\theta_0} \left( \frac{d\theta}{dz} \right)_0 \]

by

\[ \gamma = R_i \left( \frac{U^2}{\delta \phi} \right) \] (6c)

from the definition of the scale height \( gH = R\theta_0 \), the ideal gas law, and the hydrostatic balance (4).

We first note that the vertical advection term is likely to be a dominant term because a typical horizontal fluctuation \( \delta \theta \) (see precise estimates below) is likely to be much smaller than \( \Delta \theta \sim 30 \text{K} \); thus \( \gamma \gg 1 \). The large value of \( \gamma \) (Richardson number) implies that the tropical atmosphere is “strongly stratified” in the sense pointed out by Charney (1963).
The dominant term, the vertical advection, should at least balance with either the local tendency $\partial \theta / \partial t$ or the diabatic heating $Q$. The first possibility is considered only after the momentum equation is nondimensionalized, in section 6. The second possibility is now examined.

### a. Thermodynamic balance

A balance between the vertical advection and the diabatic heating terms, that is,

$$\gamma_0 \frac{\partial \theta}{\partial \rho} \approx \eta Q,$$  \hspace{1cm} (7)

corresponds to the balanced dynamics proposed by SNP01, a major interest of the present paper. This balance (7) is achieved when the condition

$$\gamma = \eta (\gg 1)$$  \hspace{1cm} (8)

is satisfied. The constraint provides an estimate for the advective time scale through

$$L/U = \Delta \theta / \delta Q \sim 3-30 \times 10^5 \text{ s} \sim 3-30 \text{ day},$$  \hspace{1cm} (9)

assuming $\delta Q \sim 1-10 \text{ K day}^{-1} \sim 10^{-4}-10^{-5} \text{ K s}^{-1}$. This further implies that $U \sim 0.3-3 \text{ m s}^{-1}$ when $L \sim 10^3 \text{ km}$, for example. Note that a stronger wind speed is obtained when a stronger diabatic heating is assumed.

### b. Asymptotic nondivergence

The subsequent scale analysis suggests that $\gamma$ could be so large that the leading-order estimate of the vertical advection term does not balance with diabatic heating; that is,

$$\gamma \gg \eta.$$  \hspace{1cm} (10)

This corresponds to a strong stratification (large $\Delta \theta$) and a relatively small diabatic heating rate $\delta Q$. In this case, we have to conclude that the vertical velocity $\omega$ should vanish; that is, $\omega_0 = 0$ to the leading order in an asymptotic expansion of the problem, say

$$\omega = \omega_0 + \varepsilon \omega_1 + \cdots,$$

with $\varepsilon = \eta/\gamma$ an expansion parameter. In other words, the flow is nondiagonal to the leading order. Note that setting $\omega_0 = 0$ in the above expansion amounts to a rescaling of the vertical velocity. Under the rescaling, the balance condition (7) is recovered. Consequently, more generally the balance condition (7) is satisfied whenever

$$\gamma \geq \eta \gg 1.$$  \hspace{1cm} (10)

### 5. Momentum equation

In nondimensionalizing the momentum equation, we first note that the latitude $\varphi$ is rewritten in terms of a nondimensional distance $\tilde{y}$ from the equator as $\varphi = \tilde{y} a / L y$, with the radius of the earth $a \sim 10^4 \text{ km}$. Then an equatorial $\beta$-plane approximation

$$\Omega \sin \frac{L y}{a} \approx \Omega L y$$  \hspace{1cm} (11)

may be introduced by assuming $L y / a \ll 1$. In global modeling contexts, it is common to introduce $\mu = \sin \varphi$, which may be interpreted as a stretched coordinate nondimensionalized by $a$. We may introduce an equivalent stretched coordinate $\tilde{y}$ using

$$\tilde{y} = \frac{L}{L y} \sin \frac{y}{a}$$

under the current nondimensionalization. The equality in Eq. (11) becomes exact when $y$ is replaced by $\tilde{y}$ in the right-hand side.

Nondimensionalization of the momentum equation is performed in the same manner as in the two previous equations. In the due course of this procedure, a non-dimensional $\beta$ parameter is obtained by multiplying the advective time scale $L/U$ by the factor $\Omega (L/a) y$ that appears in $\beta$-plane approximation (11); thus,

$$\beta = \frac{2\Omega L^2}{aU}.$$  \hspace{1cm} (12)

As a result, the nondimensional momentum equation is given by

$$\frac{Dv}{Dt} = -\frac{\delta \phi}{U^2} \nabla \cdot \phi - \beta \tilde{y} \times \mathbf{v} + \frac{L \delta F}{U^2} \mathbf{F}. $$  \hspace{1cm} (13)

Here the magnitudes for the geopotential fluctuation $\delta \phi$ and the forcing $\delta F$ are determined later by balancing the corresponding terms with the most dominant term in Eq. (13) after specifying both the order of magnitude of $\beta$ and a time scale. Recall again that the estimate provides the maximum possible magnitudes for these fluctuations.\(^6\)

---

\(^5\) The nondimensional $\beta$ parameter as introduced in Eq. (12) would also appear in Eqs. (8.5.7a, b) of Pedlosky (1987) if the time were scaled by advection instead of by his choice, $a/2\Omega L$. Under this rescaling, the nondimensional $\beta$ would be obtained by dividing the advective time scale $L/U$ by Pedlosky’s time scale $a/2\Omega L$. This again reduces to Eq. (12).

\(^6\) An alternative possibility is that both the pressure and the frictional forces dominate over all the other terms. Although this situation may realize over a boundary layer, the possibility is excluded in the present study by focusing on the dynamics of the free troposphere.
The actual order of magnitude of friction especially remains a controversial issue (cf. Stevens et al. 1977; Sardeshmukh and Hoskins 1987). A noteworthy example is the relatively large friction assumed by Gill (1980) in his idealized model. Unfortunately, the pure scale analysis here can only provide an estimate for a possible maximum value, not the more likely order of magnitude.

It transpires that the main difficulty of the scale analysis for the large-scale tropical atmosphere arises from the nondimensional parameter $\beta$, defined by Eq. (12). This parameter is quite sensitive to the choice of the horizontal scale $L$ because of a square dependence of the former on the latter; thus, a conclusion from a scale analysis is substantially modified only by a slight change of $L$.

In the next section, we consider the three distinct possibilities:

(i) $\beta \sim 1$: Coriolis force balances with all the other terms;
(ii) $\beta \gg 1$: Coriolis force dominates over the horizontal advection; and
(iii) $\beta \ll 1$: Coriolis force is negligible compared to the other terms.

Note that because both $\Omega \sim 10^{-4}$ $1$ s$^{-1}$ and $a \sim 10^4$ km are already fixed, thus the choice of the value for $\beta$ is constrained by the ratio $L^2/U$ as

$$L^2/U \sim (a/\Omega)\beta \sim 10^{11} \beta \text{ m s}$$

by substituting the former values into Eq. (12).

6. Regime analysis

The basic strategy for the scale analysis here is to identify possible dynamical regimes under the change of the nondimensional parameter $\beta$. The change of $\beta$ mostly amounts to a change of the horizontal scale $L$. By scaling arguments, the degree of fluctuation of the potential temperature $\delta \theta$, the maximum possible diabatic heating rate $\delta Q$, and possible time scales are estimated, and consistency with observations is discussed.

Whenever relevant in the following, two cases are considered in order: first, the case that the time scale is characterized by advection (amounting to a balanced dynamics) and second, the case that the time scale is faster than advection (amounting to a wave dynamics).

a. The case with $\beta \sim 1$

This case may be considered as corresponding to a synoptic scale. By setting $\beta \sim 1$ in Eq. (14), we obtain

$$L^2/U \sim 10^{11} \text{ m s}.$$  

A natural scaling that recovers this estimate would be $L \sim 10^3$ km and $U \sim 10$ m s$^{-1}$. Note that the wind speed is scaled proportionally to the square of the horizontal scale; thus, a modification of the latter by a factor of 3 leads to a wind speed amplified by a factor of 10. Hence, in practice no alternative choice for the scales is evident, although it may be argued that the choice for the wind speed is slightly too large. Note that this is the identical scaling adopted by Charney (1963).

1) Balanced dynamics

When $\beta \sim 1$, the possible maximum values for all the terms in Eq. (13) are of order of unity so that the Coriolis term may balance with all of them. Consequently, the pressure forcing term is scaled as $\delta \phi/U^2 = 1$, leading to the geopotential fluctuation

$$\delta \phi = U^2,$$

and the potential temperature perturbation is estimated from Eq. (4) as

$$\delta \theta = U^2/R \sim 0.3 \text{ K};$$

then we obtain $\gamma \sim 10^2$ from Eq. (6a).

Under this scaling, the advective time scale becomes slightly shorter than the estimate (9): $L/U \sim 10^6$ m/10 m s$^{-1} \sim 10^5$ s $\sim$ 1 day; thus, it also implies a stronger vertical advection. Then the diabatic heating rate must also be increased to maintain the thermodynamic balance condition (8). The required value is obtained by setting $\eta \sim \gamma \sim 10^2$ in Eq. (6b) as $\delta Q \sim 30 \text{ K day}^{-1}$, a rather huge value, although arguably it is still within the ambiguities of the scale analysis.7

Alternatively, if an observationally more typical synoptic value $\delta Q \sim 3 \text{ K day}^{-1}$ is substituted into Eq. (6b), we obtain

$$\eta \sim 10;$$

thus, the inequalities

$$\gamma \gg \eta \gg 1$$

are implied. As a result, the vertical advection term becomes the sole dominant term in Eq. (5) to leading order, implying that the vertical velocity is zero to the leading order, as already discussed in section 4b. It also follows that the wind field must be nondivergent to the leading order, as originally pointed out by Charney (1963). A simple further implication is that the tropical synoptic scale is dominated more by the vorticity than by the divergence (cf. Sardeshmukh and Hoskins 1987).

We propose calling a dynamical regime obtained un-

7 Although this situation may likely happen within mesoscale convective systems, it is much less likely on a synoptic scale.
der scaling (16) asymptotically nondiagonal because it does not mean that divergence is totally negligible. As implied by Charney (1963), local strong divergence associated with convective variability likely brings catalytic effects on the large-scale dynamics at a longer time scale. Note that the scaling (16) also belongs to a general condition (10) for the balance (7), with the diabatic heating still dominating over the temporal tendency.

2) WAVE DYNAMICS

The balanced dynamics holds when $\beta = 1$ as long as the temporal tendency is dictated by the advective time scale. An important alternative possibility is that the local time tendency instead balances with the vertical advection term in the thermodynamics (5).

To realize such a balance, the order of temporal tendency should become the same order of magnitude as the vertical advection term in Eq. (5), so that $\partial/\partial t \sim O(\gamma)$. A required rescaling is accomplished by introducing a fast time scale $\tau$:

$$\frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial \tau}. \quad (17)$$

By assuming that this fast temporal tendency balances with the pressure term in the momentum Eq. (13), the geopotential fluctuation is scaled as $\delta \phi = \gamma U^2$; thus, from Eq. (4) the potential temperature is scaled as $\delta \theta = \gamma U^2 / R$.

Substitution of these results into Eq. (6a) gives an estimate of

$$\gamma^2 \sim \frac{R}{U^2} \Delta \theta \sim 10^2;$$

thus, $\gamma \sim 10$, and $\delta \theta = \gamma U^2 / R \sim 3$ K. By further substituting the estimates into Eq. (6b), $\eta \sim 1$ is obtained with $\delta Q \sim 3$ K day$^{-1}$.

Thus, under this rescaling, the vertical heat advection is balanced by the local temperature tendency and diabatic heating only plays a secondary role. In other words, the gravity wave dynamics dominates over the balanced dynamics. Note that the Coriolis effect does not play a role in the leading order under this rescaling.

The alternative possibility for the gravity wave–dominant dynamics over the balanced dynamics could only be marginally excluded from an observational basis. Synoptic-scale sounding data indicate that temperature perturbation rarely exceeds 1 K below the tropical tropopause (cf. Mapes and Houze 1995; see also Fig. 3a of Mapes et al. 2003). Thus, the estimate $\delta \theta \sim 3$ K with gravity waves appears slightly excessive, although arguably this could still be within the ambiguities of scale analysis.

b. The case with $\beta \gg 1$

This case corresponds to the limit to larger horizontal scales, which may be classified as planetary scale. It turns out that the thermodynamic balance (7) does not realize under this limit; wave dynamics dominate except for the case in which stationary solutions are considered.

To satisfy the condition $\beta \gg 1$, we require

$$L^2 / U \gg 10^{11} \text{ m s}$$

from Eq. (14). This condition is easily satisfied by increasing the horizontal scale and decreasing the wind speed slightly. For example, the choice of $L \sim 3 \times 10^3$ km and $U \sim 3$ m s$^{-1}$ provides $\beta \sim 30$. These are more commonly accepted scaling parameters for reasons that will become immediately clear. Any system characterized by a scale larger than $L \sim 3000$ km also clearly falls into the regime with $\beta \gg 1$.

1) WAVE DYNAMICS

When $\beta \gg 1$, the Coriolis term dominates over horizontal advection in Eq. (13); thus, the momentum equation becomes linear to the leading order. All the remaining terms in Eq. (13) are scaled as $O(\beta)$ under this regime from the following reasoning.

We first note that the pressure forcing would balance with the Coriolis term of $O(\beta)$; thus, the geopotential fluctuation is estimated as

$$\delta \phi = \beta U^2 = \frac{2\Omega L^2 U}{a},$$

which also leads to an estimate of the potential temperature fluctuation as

$$\delta \theta = \frac{2\Omega L^2 U}{aR} \sim 1 \text{ K}$$

from Eq. (4). Although these are the maximum possible estimates for both the geopotential and the potential temperature fluctuations, they would likely represent the actual case because a pressure perturbation would naturally develop in response to a given Coriolis force (via a mass redistribution); that is, both terms are likely to compete in an equilibrium.$^8$ We also assume the fric-

---

$^8$ The word equilibrium is used in contrast to an initial transient phase that could be controlled by choosing an initial condition accordingly. For example, a geopotential perturbation may be taken to be small by an initial condition. Although this may remain true during an initial transient phase, the geopotential perturbation would eventually grow to a level comparable with the Coriolis term.
tional term $F$ is of $O(\beta)$, but only as a possible maximum value. The actual magnitude of $F$ could be smaller depending on a frictional process actually occurring in a given system.

Likewise, we would expect that the temporal tendency in the momentum equation also competes with the Coriolis term; thus, $\partial/\partial t \sim O(\beta)$. It forces us to introduce a fast time scale $\tau$ by setting

$$\frac{\partial}{\partial t} = \frac{\beta}{\partial \tau},$$

(18)

which is equivalent to nondimensionalizing the time by $a/2\Omega L$ [cf. Eq. (8.5.6) in Pedlosky 1987].

The resulting nondimensional momentum equation is

$$\left( \frac{\partial}{\partial \tau} + \frac{1}{\beta} \nabla \cdot \nabla_p + \omega \frac{\partial}{\partial \rho} \right) \mathbf{v} = -\nabla_p \phi' - \mathbf{j} \mathbf{k} \times \mathbf{v} + \mathbf{F},$$

(19)

which is an approximation typically adopted in studies of the large-scale tropical dynamics (e.g., Matsuno 1966; Gill 1980; Lindzen and Nigam 1987).

Now, in the thermodynamic Eq. (5), the two nondimensional parameters are estimated as

$$\gamma \sim \eta \sim 30$$

by substituting the given scales for $L$, $U$, and $\delta\theta$ along with $\delta Q \sim 3$ K day$^{-1}$ into Eqs. (6). Furthermore, by comparing with the estimated value for $\beta$, we establish the scaling $\gamma \sim \eta \sim \beta$, implying that all the three terms in the thermodynamic Eq. (5) are of the same order. As a result, the standard picture of tropical atmospheric dynamics consisting of linear equatorial waves forced by convective heating (cf. Webster 1982; Holton 1992, chapter 11; Wheeler and Kiladis 1999) is recovered. Importantly, the thermodynamic balance (7) is not realized under this planetary-scale regime with $\beta \gg 1$.

2) Balanced dynamics?

It is important to emphasize that although the time scale defined by Eq. (18) is nothing more than the fastest possible scale, nonetheless that would probably be the time scale that most naturally arises in the given system. Note that Eq. (19) reduces to a steady Gill problem (Gill 1980) by setting $\partial/\partial \tau = 0$. It is tempting to consider a system evolving more slowly with time, say by an advective time scale $L/U \sim 10^6$ s $\sim$ 10 days, comparable to an intraseasonal scale, with $\partial/\partial t = O(1)$. In that case, the thermodynamic balance (7) is also recovered.

Can such a balanced dynamics actually be realized at the scale with $\beta \gg 1$? We would argue that it is less likely the case from the following argument based on the scale analysis.

Imagine that we initialize a system under the two regimes $\beta \sim 1$ and $\beta \gg 1$, respectively, with a strict thermodynamic balance indicated by Eq. (7). A main conclusion from section 6a is that regardless of the details of the associated initial wind field, the system would evolve consistently under a close thermodynamic balance (7) throughout the time when $\beta \sim 1$. The fastest possible time scale in the momentum equation is the advective time scale under this regime, as actually chosen in the scale analysis. That tendency is much smaller than the two dominant terms in the thermodynamic equation; thus, the thermodynamic balance is well maintained consistently with time.

The situation is different when $\beta \gg 1$. A spontaneous evolution of the system (again defined as the fastest possible time scale in the momentum equation) is by factor of $\beta$ faster than the advective time scale. That tendency turns out to be comparable with the two other terms in the thermodynamic equation. Thus, the thermodynamic balance is not spontaneously maintained even it is well posed initially.

In other words, to maintain a balanced condition under $\beta \gg 1$, a much stronger constraint must be imposed in the initial condition. Not only must the system be under the thermodynamic balance, it must also be dynamically well balanced so that fast waves would never be generated in subsequent evolution. It is hard to imagine that such a situation arises in a natural atmosphere, although the possibility is not completely excluded. A certain filtering mechanism must exist that efficiently eliminates the spontaneous generation of fast waves for such a regime to be realized.

c. The case with $\beta \gg 1$

This case corresponds to the limit of small horizontal scale. This is best classified as a mesoscale because the scale analysis here would not be applicable to the convective scale, whose system would no longer be hydrostatic.

1) Balanced dynamics

In this case, we require

$$L^2/U \ll 10^{11} \text{m s}.$$

A possible choice that satisfies this constraint is $L \sim 10^2$ km, $U \sim 10$ m s$^{-1}$, which leads to $\beta = 10^{-2}$. The esti-
mate for the magnitudes of the other terms in the momentum equation remains the same as the case with 
\( \beta \sim 1 \) [section 6a(1)] and especially \( \delta \phi = U^2 \); thus, again \( \delta \phi = U^2/R \sim 0.3 \) K. It further recovers the estimate \( \gamma \sim 10^3 \).

This is rather an interesting result, implying that even in the mesoscale regime, the vertical advection term dominates over the horizontal advection in the thermodynamics over the tropics. This further indicates a relative robustness of the tropical thermodynamic balance (7).

A similar problem as the case with \( \beta \sim 1 \) [section 6a(1)] is encountered by estimating the magnitude of the diabatic heating rate that is required to balance vertical advection estimated by \( \gamma \):

\[
\delta Q = \frac{U \delta \theta}{L} \approx \frac{U \delta \theta}{L} \gamma \sim \frac{10 \text{ m s}^{-1} \times 0.3 \text{ K}}{10^3 \text{ m}} \times 10^2 \sim 3 \times 10^{-3} \text{ K s}^{-1} \sim 300 \text{ K day}^{-1},
\]
a huge value for the assumed \( L \), although it could still be an acceptable value within the ambiguities of choice of characteristic scale. To keep the scale analysis consistent with the observed less intensive diabatic heating rate, on the other hand, an asymptotically nondiagonal condition must be assumed even at the mesoscale, just as for the synoptic scale in section 6a(1). Note that the effect of the vertical advection becomes increasingly larger with decreasing horizontal scales; thus, it becomes harder to balance by the diabatic heating. To this extent, asymptotic nondivergence is an increasingly important constraint with decreasing horizontal scales.

**2) Wave Dynamics**

An alternative possibility that the gravity wave dynamics takes over the balanced dynamics may also be considered as in the case with \( \beta \sim 1 \) [cf. section 6a(2)]. Note that in the limit of \( \beta \ll 1 \), the Coriolis term does not play a role to leading order. The rescaling is performed by introducing a fast time scale (17), leading to \( \delta \phi = \gamma U^2 \), \( \delta \theta = \gamma U^2/R \), \( \gamma \sim 10 \), and \( \delta \theta \sim 3 \) K as in section 6a(2). Airborne observations (e.g., Wei et al. 1998 and references therein) show that the temperature anomaly rarely exceeds 3 K even within convective cores. A very high-frequency (VHF) radar observation (Vincent et al. 2004) shows that convectively generated gravity waves may have a vertical velocity amplitude as high as 1 m s\(^{-1}\). However, even assuming a frequency near the Väisälä frequency, this corresponds to a temperature amplitude of only 0.3 K. Compared with these observations, the estimate \( \delta \theta \sim 3 \) K still appears to be excessive even for the mesoscale. Hence, the gravity waves would be better dealt with as higher-order effects in an asymptotic expansion of mesoscale convective dynamics, although it is likely that higher-order gravity waves play a catalytic role in the leading-order balanced dynamics (cf. Mapes 1993).

**d. The case for shallow modes**

Finally, modifications of the results are briefly discussed when modes shallower than \( H \sim 10 \) km are considered. The only nondimensional parameter affected by this modification is \( \gamma \) as defined by Eq. (6a).\(^9\) The stratification parameter \( \gamma \) becomes proportionally smaller as a shallower system is considered. As a result, the balance condition (7) is less satisfied, as indicated for 2-day waves by Haertel et al. (2008). It is also expected to be less nondiagonal for the regime with \( \beta \sim 1 \).

**7. Data analysis**

The orders of magnitude of variables obtained by the scale analysis can also be verified by data. The present section reports such a preliminary attempt. Simply, two analyses are presented both dependent on the time \( \Delta t \) and horizontal \( \Delta x \) scales. The first is an estimate for the characteristic wind speed \( U \) at 850 hPa. The second is a measure of a degree that the thermodynamic balance (7) is satisfied.

In performing this type of analysis, we should keep in mind that a standard global analysis dataset is likely to be more contaminated by model physics over the tropics than in middle latitudes, especially by a cumulus parameterization. For this reason, a gridded dataset over the Large-Scale Array (LSA) domain during the Tropical Ocean and Global Atmosphere (TOGA) Coupled Ocean–Atmosphere Response Experiment (COARE) Intensive Observing Period (IOP, 1 November 1992–28 February 1993) is adopted instead for the present analysis. The dataset was processed at the Colorado State University and is available online (http://tornado.atmos.colostate.edu/togadata/gridded.html), where the reader is referred for details.

To quantify the scale dependence of a variable, say, \( \chi(x, y, t) \), the variable is first moving-averaged over a time span \( \Delta t \), longitude \( \Delta x \), and latitude \( \Delta y (= \Delta x) \) to obtain a space-time averaged value:

\[^9\] A direct use of an alternative definition (6c) is problematic because \( H \) is implicitly assumed as a scale height in the derivation.
Then an order of magnitude of the given variable \( \bar{x}(x, y, t, \Delta t, \Delta x) \) for a given scale \( \Delta t, \Delta x \) is estimated by its root-mean-square; i.e.,

\[
\langle \chi^2 \rangle_{\Delta t, \Delta x}^{1/2} = \left[ \frac{1}{L_x L_y T} \int_0^T \int_0^{L_y} \int_0^{L_x} \chi^2(x, y, t, \Delta t, \Delta x) \, dx \, dy \, dt \right]^{1/2}.
\]

Data collected for the LSA TOGA COARE IOP covered a domain spanning 40° and 20° in longitude and latitude, respectively, for a 4-month period with resolution of 1° and 6 h in space and time, respectively. Hence, the possible range for averaging is 1°-20° in space and 6 h to 4 months in time. The former approximately corresponds to \( \Delta x = 100-2000 \) km by measuring the latitude in the unit of distance (km).\(^{10}\) The results for the 850-hPa level are presented here.

The estimate of the characteristic wind speed \( U \) for a given scale \( \Delta t \) and \( \Delta x \) are given by setting \( \chi = v_H \) in the above, where \( v_H \) is the wind vector. The obtained result \( \langle \chi^2 \rangle_{\Delta t, \Delta x}^{1/2} \) is plotted in Fig. 1 as a function of the time scale \( \Delta t \) and the horizontal scale \( \Delta x \). The wind scale \( U \) is quite homogeneous when either a time or a space averaging is performed separately, with a value of \( U = 3-5 \) m s\(^{-1}\). The choice of \( U = 10 \) m s\(^{-1}\) thus may lead to an error of a factor of 2–4 throughout the analysis. On the other hand, when both time and space averaging are performed simultaneously, a decrease of the wind scale in long-time, large-scale limits is noticeable.

As an example demonstrating the possibilities for estimating the relative contributions of two terms in an equation, we plot in Fig. 2 a measure \( R \) of the thermodynamic balance (7) for a given scale as

\[
R = \left\{ \frac{\left\langle \left( \frac{\partial \theta}{\partial t} \right)^2 \right\rangle}{\left\langle \left( \omega \frac{\partial \theta}{\partial \phi} \right)^2 \right\rangle} \right\}^{1/2} (\Delta t, \Delta x).
\]

---

\(^{10}\) The space averaging is further extended to 40° in longitude only, but the conclusions below are not modified by this extension.
The numerator and the denominator in the definition (20) are calculated at 850 hPa. The analysis is also repeated for 500 and 250 hPa, and similar results are obtained, but with a decreasing tendency of $R$ with height. Note that they are both likely to decrease with increasing scales both in space and time, but at a different rate. Thus, Eq. (20) measures a change of relative magnitude of these two terms in changing scales. The measure $R$ is small when the thermodynamic balance is satisfied.

For all the horizontal scales, the measure $R$ is high when computed from instantaneous data, but it rapidly decreases when data are averaged over a few days to order $10^{-1}$. It remains the same order or less for longer time scales, implying that the thermodynamic balance (7) is well satisfied over time scales longer than a few days. We interpret this to mean that high-frequency variability associated with moist convection is rapidly damped by averaging over a few days, although a relatively large value for $R$ in short time scales is likely results at least partially from sampling errors of data (cf. Haertel 2002; Mapes et al. 2003).

On the other hand, the scale analysis of the last section predicts that the regime transits from the balanced dynamics to the wave dynamics over the horizontal scale $L \sim 1000$–3000 km, and thus the balance measure $R$ should increase with increasing horizontal scales $L \sim \Delta x$. The most surprising result from the present analysis is that we see no such tendency by increasing the horizontal scale up to $\Delta x = 2000$ km. For time scales less than 40 days, the measure $R$ is almost independent of the horizontal scale. More surprisingly, for time scales greater than 70 days, the measure $R$ actually decreases with increasing horizontal scales.

The result suggests a wider relevance of the thermodynamic balance condition (7), especially toward larger horizontal scales, than suggested by the strict scale analysis in the previous section. Nevertheless, it is still to be seen whether $R$ eventually begins to increase above the scale $L \sim 3000$ km. The limited domain covered by the LSA TOGA COARE IOP dataset unfortunately does not permit us to address this question here.

Qualitatively, the relative invariance of $R$ with increasing horizontal scale may be explained in terms of a pulselike nature of tropical convective variability. Implications of the pulselike behavior in temporal direction associated with the $1/f$ noise are extensively discussed by Yano et al. (2004). A major consequence is the persistence of variability over all the time scales. The same conclusion also applies to spatially pulselike

---

11 However, recall the discussions in section 6b(2).
features, which could explain a constant $R$ on scales less than 70 days.

8. Discussion

A systematic scale analysis is performed for the large-scale dynamics over the tropics. A nondimensional $\beta$ parameter is identified as a key parameter for performing such a systematic analysis. The magnitude of $\beta$ is determined mostly by the horizontal scale $L$, although the wind scale $U$ may be modified by a factor. Once the value of $\beta$ is specified, the scale analysis provides estimates for the degree of fluctuation of the potential temperature $\delta \theta$, the maximum possible diabatic heating rate $\delta Q$, and possible time scales. Consistency with observations excludes certain identified regimes. Conversely, observations provide additional constraints on the scaling, notably the asymptotically nondivergent tendency over scales less than $L \sim 10^3$ km.

Guided by such a systematic analysis, two possibilities are distinguished: first, a balanced dynamics characterized by the balance between the vertical advection and diabatic heating terms in thermodynamic equation and second, a system consisting of linear equatorial waves modulated by convective diabatic heating. It is important to emphasize that these two major regimes are objectively identified purely guided by the scale analysis using the nondimensional $\beta$ as the sole external parameter. The obtained results are summarized in Table 1.

The case of balanced dynamics in the tropics constitutes an interesting possibility, originally proposed by SNP01. This regime is found to be possible under a formal scale analysis for scales less than $L \sim 10^3$ km, with $U \sim 10$ m s$^{-1}$. The proposed balance condition (7) is also observationally well known for the tropical synoptic scales (e.g., Frank and McBride 1989; Mapes and Houze 1995; Yano 2001, especially Fig. 1). The balance system may furthermore be considered as nondivergent to leading order as originally pointed out by Charney (1963). This particular possibility will be considered further both observationally and theoretically.

However, the possibility that a dominance of gravity waves breaks the balance (7) is not easily excluded. The two regimes are simply distinguished by a relatively small difference in the potential temperature fluctuations: $\delta \theta \sim 0.3$ K for the balanced dynamics and $\delta \theta \sim 3$ K for the gravity wave dynamics. The latter estimate is high compared to typically observed temperature perturbations (cf. Mapes and Houze 1995), although the discrepancy is likely to be within a range of ambiguities in scale analyses.

A more subtle aspect in the scale analysis is that the two major regimes (balance and waves) are separated only by a factor of 3 in the horizontal scale $L$. This results from a strong sensitivity of the nondimensional $\beta$ parameter (12) to $L$, leading to a tenfold increase when $L$ is increased only by a factor of 3. As a result, when the horizontal scale is slightly increased, then the $\beta$ effect suddenly dominates over the whole dynamics and the balance dynamics is replaced by linear equatorial wave dynamics.

If the result is taken literally, the tropical atmosphere rather abruptly transits from a balanced regime below $\sim 1000$–3000 km to a wave regime above. However, an asymptotic expansion often presents a much wider

---

12 Of course, only in an asymptotic sense. Those waves are certainly nonlinearly modulated at longer time scales.
applicability than a simple scale analysis would suggest (cf. Bender and Orszag 1978). Consequently, a possibility that a wider applicability of the balanced dynamics above the horizontal scale of 3000 km, for example, is not excluded.

For this reason, in practice, it would be hard to distinguish between the two regimes purely from a horizontal scale, especially over the transition scales at 1000–3000 km. The present scale analysis, nevertheless, strongly suggests that both regimes are equally feasible. This point is to be emphasized considering the current dominant view of a tropical large-scale dynamics based on equatorial waves.

Our preliminary data analysis indicates that the balance condition (7) is in fact satisfied well above the scale $L \sim 10^3$ km. A recent composite analysis (Haertel et al. 2008) suggests that the Madden–Julian oscillation (MJO), which possesses a characteristic scale well above 3000 km, is also well under the thermodynamic balance (7). It may turn out that those so-called convectively coupled equatorial waves (cf. Wheeler and Kiladis 1999) may better be interpreted in terms of the balanced dynamics.

Probably a more surprising result is a tendency for asymptotic nondivergence [cf. Eq. (16)] that is likely to persist toward the smaller scales, including the mesoscale $L \sim 10^2$ km. A required diabatic heating rate that balances with the leading-order nonvanishing vertical advection is already excessive over the synoptic scale compared to the observations, and it further increases inversely proportionally with decreasing horizontal scales. Hence, it becomes more and more excessive unless an asymptotic nondivergence is assumed.

Acknowledgments. The present work may be considered for the first author as a realization of a long-lasting homework requested by Noriyuki Nishi more than a decade ago. Careful readings by Peter Bechtold and Riwal Plougouven and comments by Brian Mapes, Chidong Zhang, and Joan Alexander are much appreciated. Editorial review by Duane Stevens and careful reading by Steve Sherwood were particularly helpful in preparing a revised text. Comments by Pat Haertel on the LSA TOGA COARE IOP data are also much appreciated.

REFERENCES


