Reply

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1. Introduction

Hamill et al. (2009, hereinafter H09) presented a critique of our recent work (Ambadan and Tang 2009a, hereinafter AT09). In their comment, two core points are that 1) AT09 incorrectly stated the nature of the measurement function and 2) AT09 should use more-appropriate experimental designs, especially a *state-of-the art* ensemble Kalman filter (EnKF) as a reference benchmark in comparison with the "sigma-point" Kalman filter (SPKF). While we thank Hamill et al. for their constructive criticism we would like to clarify the two issues.

2. Nature of measurement function in EnKF

For the purpose of presentation, we start from the standard EnKF formulation, as follows (Hamill 2006):

$$\mathbf{x}_t^a = \mathbf{x}_t^b + \mathbf{K}[\mathbf{y}_t - h(\mathbf{x}_t^b)], \tag{1}$$

$$\mathbf{K} = \mathbf{P}_t^b\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}_t^b\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1}, \quad \text{and} \tag{2}$$

$$\mathbf{P}_t^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}_t^b, \tag{3}$$

where $\mathbf{P}_t^b$ is the forecast error covariance matrix, $\mathbf{x}_t^b$ represents the system state vector at step t, $\mathbf{R}$ is the observation error covariance matrix, h is a nonlinear measurement function, and $\mathbf{H}$ is the Jacobian matrix of h (i.e., the linearized measurement operator). The forecast error covariance matrix $\mathbf{P}_t^b$ at time step t is approximated using a finite set of model state ensembles (say M) given by

$$\mathbf{P}^b = \frac{1}{M-1}\sum_{i=1}^{M}(\mathbf{x}_i^b - \overline{\mathbf{x}^b})(\mathbf{x}_i^b - \overline{\mathbf{x}^b})^{\mathrm{T}}. \tag{4}$$

Apparently, the $\mathbf{H}$ used in Kalman gain (2) is a linearized operator, thus imposing the assumption of the linearization of nonlinear measurement function in the standard EnKF formulation. The linearization can be done either by analytical analysis like extended Kalman filter or by ensemble members, as proposed by Houtekamer and Mitchell (2001) and Hamill (2006). The latter is often implicit and might not be very straightforward, deserving further analysis.

In Houtekamer and Mitchell (2001) and Hamill (2006), Kalman gain (2) was written by

$$\mathbf{P}^b\mathbf{H}^{\mathrm{T}} = \frac{1}{M}\sum_{i=1}^{M}(\mathbf{x}_i^b - \overline{\mathbf{x}^b})[h(\mathbf{x}_i^b) - \overline{h(\mathbf{x}^b)}]^{\mathrm{T}} \quad \text{and} \tag{5}$$

$$\mathbf{H}\mathbf{P}^b\mathbf{H}^{\mathrm{T}} = \frac{1}{M}\sum_{i=1}^{M}[h(\mathbf{x}_i^b) - \overline{h(\mathbf{x}^b)}][h(\mathbf{x}_i^b) - \overline{h(\mathbf{x}^b)}]^{\mathrm{T}}, \tag{6}$$

where

$$\overline{h(\mathbf{x}^b)} = \frac{1}{M}\sum_{i=1}^{M}h(\mathbf{x}_i^b).$$

Equations (5) and (6) allow direct evaluation of the nonlinear measurement function h in calculating Kalman gain. Mathematically, (5) and (6) approximately hold if and only if

$$\overline{h(\mathbf{x}^b)} = h(\overline{\mathbf{x}^b}) \quad \text{and} \tag{7}$$

$$\mathbf{x}_i^b - \overline{\mathbf{x}^b} = \boldsymbol{\varepsilon}_i, \quad \text{Norm } (\boldsymbol{\varepsilon}_i) \text{ is small for } i = 1, 2, \ldots, M. \tag{8}$$

Under the conditions of (7) and (8), (5) and (6) actually linearize the nonlinear measurement functions *h to* $\mathbf{H}$. Therefore, direct application of the nonlinear measurement function in (5) and (6) in fact imposes an implicit linearization process using ensemble members.

For many realistic atmospheric and oceanic estimation problems, especially for model state estimation,

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TABLE 1. Comparison between left- and right-hand sides (LHS and RHS) of (5) and (6).

$\mathbf{x}_k^a$	Ensemble size	LHS of (5)	RHS of (5)	LHS of (6)	RHS of (6)
10	10 000	3.62	0.48	3.22	0.49

(7) and (8) approximately hold since the nonlinearity of the measurement function h might not be strong and perturbation growth is relatively small. However, in some cases in which either condition is not held, (5) and (6) could cause large errors in estimating the Kalman gain. To demonstrate this, we now consider an example of a one-dimensional nonlinear model with a nonlinear measurement function, as shown below:

state-space model $x_{k+1} = x_k^2 + q_k$, $q_k \sim N(0, Q)$ and
measurement model $y_k = \sin(x_k) + r_k$, $r_k \sim N(0, R)$.

At step k, we have an analysis of the model state, denoted by x_k^a. In the next assimilation cycle, $\mathbf{P}^b\mathbf{H}^{\mathrm{T}}$ and $\mathbf{HP}^b\mathbf{H}^{\mathrm{T}}$ are required to calculate for the Kalman gain. Here we use two approaches: one is to directly calculate them since $\mathbf{H}$ is known and the other is to use (5) and (6). The ensemble is generated by perturbing x_k^a with random numbers drawn from a normal distribution with mean 0 and variance 0.1. Initially x_k^a is arbitrarily set to 10, and the ensemble size is 10 000. A small perturbation and a large ensemble size will help to obtain a relatively stable analysis.

As shown in Table 1, when the measurement function is a nonlinear sine function, (5) and (6) produce large errors. In other words, (5) and (6) hold only if the linearization conditions (7) and (8) are satisfied. The nonlinearity of the measurement functions may exist in some realistic problems, especially in the estimation of model parameters. In the EnKF framework, the parameter estimation is typically processed by defining the parameter as a special or specific system state. This causes the measurement function mapping the observation of real system states to the parameter space to be nonlinear.

Mathematically, a good solution for this issue is to reformulate the Kalman gain. As shown by (9) in AT09, the Kalman gain can be expressed as

$$\mathbf{K} = \mathbf{P}_{\mathbf{x},\tilde{\mathbf{y}}}\mathbf{P}_{\tilde{\mathbf{y}}}^{-1}. \tag{9}$$

Here, $\tilde{\mathbf{y}}$ is defined as the error between the noisy observation and its prediction, given by $\tilde{\mathbf{y}} = \mathbf{y}_t - h(\mathbf{x}_t^b)$. Equation (9) avoids the use of the Jacobian while retaining consistency and accuracy, which allows strong nonlinear measurement functions such as the parameter estimates of strongly nonlinear Lorenz '63 and Lorenz '96 models (Ambadan and Tang 2009b, manuscript submitted to *Nonlinear Processes Geosci.*).

3. Reference benchmark used in compariso

The general focus of AT09 was to introduce the SPKFs to the atmospheric assimilation community. The SPKF concepts were originally derived by Julier et al. 1995 and subsequently developed by many researchers, as mentioned in AT09. The so-called SPKF and its square root variants were originated in the signal-processing community. The SPKF is based on a deterministic sampling approach, whereas EnKF is based on random sampling of ensembles. The essential difference between SPKF and EnKF is the perturbation for generating ensembles and the formulation of Kalman gain. As an early introductory work, AT09 performed two basic SPKF filters: an unscented Kalman filter and a central-difference Kalman filter. For the sake of comparison in the same line, we chose the standard EnKF rather than some recently developed derivatives of EnKF such as the local ensemble transform Kalman filter (Hunt et al. 2007) or ensemble square root filters (Tippett et al. 2003). Note that the Lorenz '63 experimental settings were very similar to those in Evensen (1997).

We agree with H09 that these *state-of-the-art* EnKFs can effectively improve the assimilation analysis. Our motivation in AT09 was to bring SPKF to the attention of the atmospheric assimilation community, leading to further development and application of SPKF in the field of atmospheric assimilation. It might be more appropriate to compare a state-of-the-art EnKF, as mentioned in H09, with a similar SPKF, which is being pursued.

4. Conclusions

In the current EnKF formulation, the measurement function is implicitly assumed to be linear or locally linearized. The direct application of nonlinear measurement operators in the current EnKF formulation, as proposed in Houtekamer and Mitchell (2001) and Hamill (2006), is actually an implicit linearization through ensemble members. In some cases, the implicit linearization of nonlinear operators might lead to large errors of Kalman gain. An alternative treatment of nonlinear measurement function is to reformulate the Kalman gain used in SPKF as presented in AT09.

We agree with H09 that state-of-the-art EnKFs can lead to better assimilation analysis than the standard EnKF used in AT09. However, we think a parallel comparison between EnKF and SPKF in the same line, as performed in AT09, should be allowed. We expect a comparison between a state-of-the-art EnKF and a state-of-the-art SPKF in the near future.

REFERENCES

Ambadan, J. T., and Y. Tang, 2009a: Sigma-point Kalman filter data assimilation methods for strongly nonlinear systems. *J. Atmos. Sci.,* **66,** 261–285.

Evensen, G., 1997: Advanced data assimilation for strongly nonlinear dynamics. *Mon. Wea. Rev.,* **125,** 1342–1354.

Hamill, T. M., 2006: Ensemble-based atmospheric data assimilation. *Predictability of Weather and Climate,* T. N. Palmer and R. Hagedorn, Eds., Cambridge University Press, 124–156.

——, J. S. Whitaker, J. L. Anderson, and C. Snyder, 2009: Comments on ''Sigma-point Kalman filter data assimilation methods for strongly nonlinear systems.'' *J. Atmos. Sci.,* **66,** 3498–3500.

Houtekamer, P. L., and H. L. Mitchell, 2001: A sequential ensemble Kalman filter for atmospheric data assimilation. *Mon. Wea. Rev.,* **129,** 123–137.

Hunt, B. R., E. J. Kostelich, and I. Szunyogh, 2007: Efficient data assimilation for spatiotemporal chaos: A local ensemble transform Kalman filter. *Physica D,* **230,** 112–126.

Julier, S., J. Uhlmann, and H. Durrant-Whyte, 1995: A new approach for filtering nonlinear systems. *Proc. American Control Conf.,* Seattle, WA, IEEE, 1628–1632.

Tippett, M. K., J. L. Anderson, C. H. Bishop, T. M. Hamill, and J. S. Whitaker, 2003: Ensemble square-root filters. *Mon. Wea. Rev.,* **131,** 1485–1490.