A New Algorithm for Low-Frequency Climate Response

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ABSTRACT

The low-frequency response to changes in external forcing for the climate system is a fundamental issue. In two recent papers the authors developed a new blended response algorithm for predicting the response of a nonlinear chaotic forced-dissipative system to small changes in external forcing. This new algorithm is based on the fluctuation–dissipation theorem and combines the geometrically exact general response formula using integration of a linear tangent model at short response times and the classical quasi-Gaussian response algorithm at longer response times. This algorithm overcomes the inherent numerical instability in the geometric formula arising because of positive Lyapunov exponents at longer times while removing potentially large systematic errors from the classical quasi-Gaussian approximation at moderate times.

Here the new blended method is tested on the low-frequency response for a T21 barotropic truncation on the sphere with realistic topography in two dynamical regimes corresponding to the mean climate behavior at 300- and 500-hPa geopotential height. The 300-hPa regime has robust strongly mixing behavior with a nearly Gaussian equilibrium state distribution, whereas the 500-hPa regime is weakly chaotic with slowly decaying time autocorrelation functions and strongly non-Gaussian climatology. It is found that the blended response algorithm has significant skill beyond the classical quasi-Gaussian algorithm for the response of the climate mean state and its variance. Additionally, the predicted response of the T21 barotropic model in the low-frequency regime for these functionals does not seem to be affected by the model’s structural instability. Thus, the results here suggest the use of the fluctuation–dissipation theorem–based blended response algorithm for more complex nonlinear geophysical models.

1. Introduction

The low-frequency response to changes in external forcing for various components of the climate system is a central problem of contemporary climate change science. In particular, in the atmosphere the response of the low-frequency teleconnection patterns such as the Arctic Oscillation (AO), the North Atlantic Oscillation (NAO), and the Pacific–North America pattern (PNA), which steer the storm tracks, is a fundamental issue. Leith (1975) suggested that if the climate system satisfied a suitable fluctuation–dissipation theorem (FDT), then climate response to small external forcing could be calculated by estimating suitable statistics in the present climate. For the general FDT, see Deker and Haake (1975), Risken (1989), and Majda et al. (2005, hereafter MAG05). The topic of this paper is the use of a new algorithm (Abramov and Majda 2007, 2008, hereafter AM07, AM08) for calculating the FDT response to address the above issues in a prototype setting for the atmosphere.

In practice, Leith (1975) proposed that the response of the global climate due to changes in external parameters could be computed through an approximation, the quasi-Gaussian fluctuation–dissipation theorem (qG-FDT) for appropriate variables in the climate system. To predict global climate response in this setting, one observes the climatology of a model for a sufficiently long time under the tacit assumption that the dynamics is close to its statistical equilibrium and then, under the assumption of a Gaussian statistical
equilibrium state, applies the quasi-Gaussian fluctuation–dissipation formula to predict mean climate response to small changes of the physical parameters of the dynamics; these estimates are modeled without actually simulating an appropriate scenario of climate development for those changes of parameters, which usually poses a computational problem of substantial complexity. Leith’s suggestion has inspired others such as Bell (1980), Carnevale et al. (1991), North et al. (1993), Gritsun and Dymnikov (1999), Gritsun (2001), Gritsun et al. (2002), Gritsun and Branstator (2007), and Gritsun et al. (2008) to apply the qG-FDT for idealized climate models with various approximations and numerical procedures. Bell (1980) considered a special truncation of the barotropic vorticity equation and studied the response of this truncation to small kick of trajectory at initial time, reporting that the predictions of classical FDT for the response of the system hold extremely well. Carnevale et al. (1991) generalize the quasi-Gaussian FDT formula to a non-Gaussian but smooth equilibrium state, also documenting reasonably high precision of the classical Gaussian FDT formula for a model suggested by Orszag and McLaughlin (1980) but its failure for the classical three-dimensional “Lorenz butterfly” model (Lorenz 1963). North et al. (1993) studied the fluctuation–dissipation of a general circulation model in response to a step-change in solar forcing. Gritsun and Dymnikov (1999), Gritsun (2001), and Gritsun et al. (2002) developed a systematic procedure to compute the complete response operator in matrix form for different response times; they also presented a robust algorithm of verification for the FDT response prediction through a series of direct numerical simulations with the actual perturbed system. Gritsun and Branstator (2007) and Gritsun et al. (2008) have applied qG-FDT to a comprehensive general circulation model for computing the response of the mean state and various quadratic functionals, such as variance and meridional eddy flux, with interesting results. In particular, the response to tropical heating is considered in these papers; qG-FDT typically has high correlation skill, above 0.8 in the upper troposphere for linear and 0.7 for quadratic functionals, with amplitude errors in the 25%–50% range. In fact, this skill is sufficiently high for several successful applications of linear response modeling based on qG-FDT. On the other hand, the skill of qG-FDT in this setting can often deteriorate significantly in the lower-middle troposphere (Gritsun and Branstator 2007), and the reasons for these discrepancies are unclear at the present time.

The response of the forced-dissipative 40-mode Lorenz 96 (L96) model (Lorenz 1995; Lorenz and Emanuel 1998; Abramov and Majda 2004) has also been studied by MAG05 within the framework of the quasi-Gaussian FDT. It has been found in MAG05 that although the quasi-Gaussian FDT works reasonably well for the forced-dissipative L96 model in strongly chaotic dynamical regimes, for weakly chaotic regimes the predictions of the classical FDT become much worse. Although it is suitable for dynamical systems with a nearly canonical Gaussian equilibrium state, the fluctuation–dissipation theorem in its classical formulation is only partially successful for nonlinear dynamical systems with forcing and dissipation. The major difficulty in this situation is that the statistical equilibrium in the limit as time approaches infinity in this case is typically described by a Sinai–Ruelle–Bowen (SRB) probability measure, which is usually not smooth (Eckmann and Ruelle 1985; Ruelle 1997, 1998; Young 2002). In this context, AM07 and AM08 developed and tested novel computational algorithms for predicting the mean response of functionals of a chaotic dynamical system to small changes in external forcing via the fluctuation–dissipation theorem; the new algorithms include the short-time FDT (ST-FDT) and the blended short-time/quasi-Gaussian (ST/qG-FDT) algorithm. Unlike the earlier work in developing FDT-type computational strategies for chaotic nonlinear systems with forcing and dissipation, the new FDT methods developed in AM07 and AM08 are based on the theory of Sinai–Ruelle–Bowen probability measures, which commonly describe the equilibrium state of such dynamical systems. The ST-FDT algorithm takes into account the fact that the dynamics of chaotic nonlinear forced-dissipative systems often reside on chaotic fractal attractors, where the classical quasi-Gaussian formula of the fluctuation–dissipation theorem often fails to produce satisfactory response prediction, especially in dynamical regimes with weak and moderate chaos and slower mixing. It has been discovered in AM07 and AM08 that the ST-FDT algorithm is an extremely precise response approximation for short response times but is numerically unstable for longer response times. On the other hand, the classical qG-FDT method is numerically stable for all times; however, for a non-Gaussian climate it can produce significant errors at intermediate response times despite its guaranteed accuracy at extremely short times (MAG05). The blended ST/qG-FDT method employs the ST-FDT method at short response times and the qG-FDT method at longer response times, thus combining the advantages of both methods. The new response algorithms were tested in AM07 for the 40-mode L96 model in a wide range of dynamical regimes varying from weakly to strongly chaotic and to fully turbulent for both linear and quadratic functionals. The results in AM07 demonstrated a remarkably high
level of precision for the blended response algorithms in all regimes for the L96 test bed for both linear and quadratic functionals. Furthermore, the ST/qG-FDT algorithm utilizes a linear tangent model at short times to calculate the response. All of these facts from AM07 suggest that the ST/qG-FDT algorithm is potentially useful in operational long-term climate change predictions for both linear and nonlinear response functions.

Here we test the new blended ST/qG-FDT response algorithm on the T21 barotropic model on the sphere with realistic topography and two different constant forcing fields, which are chosen to mimic the dynamical properties of the actual airflow at 500- and 300-hPa geopotential height. The same model and regimes were used by Abramov et al. (2005) to study the predictability for climate low-frequency variability. Here we address the response of the low-frequency teleconnection patterns in these models to external forcing. The 300-hPa constant forcing is developed by Franzke et al. (2005) and has a dynamically robust regime with a nearly Gaussian equilibrium state and rapidly mixing time autocorrelation functions. On the other hand, the 500-hPa constant forcing, developed originally by Selten (1995), has a strongly non-Gaussian equilibrium state and slowly decaying time autocorrelation functions. The dynamical system behavior in these two regimes of the model here roughly corresponds to the two extremes of behavior at low frequencies in more complex contemporary climate models. We test both the new blended ST/qG-FDT and classical qG-FDT climate response algorithms in these two different dynamical regimes and verify their predictions against the directly measured ideal response (Gritsun and Dymnikov 1999; AM05; AM07; AM08), demonstrating the high skill and general superiority of the blended ST/qG-FDT algorithm in predicting the climate response in both situations.

Among other observations, a remarkable result here is the apparent insensitivity of the directly measured ideal low-frequency response to the structural changes of the equilibrium statistical state under changed external forcing. The T21 barotropic model, being a complex chaotic nonlinear dynamical system, is obviously structurally unstable in a geometric sense, which means that the modifications of external forcing parameters, no matter how small, cause structural changes in the topology of the statistical equilibrium state. It is impossible to discover these structural changes “remotely” through the tangent approximation, offered by the ST/qG-FDT because such bifurcations can only be captured by “running into them” (i.e., through the directly perturbed ideal response method). Yet, as presented below, the ideal response generally quite closely follows the ST/qG-FDT prediction. This behavior suggests that the climate response of simple linear and nonlinear response functions is not sensitive to the structural changes in a complex nonlinear climate state and can be accurately described by an FDT-type response approximation. The results presented here and in AM07 with the new blended response algorithm very strongly contrast with the pessimistic viewpoint advocated recently by McWilliams (2007) regarding predictive skill due to structural instability of climate models.

2. Climate response algorithms

Here we proceed under the general assumption that the climate model is numerically represented as a chaotic nonlinear dynamical system of ordinary differential equations for a state vector $x \in \mathbb{R}^N$, given by

$$\frac{dx}{dt} = f(x),$$

where $x$ in our case is a vector of principal components (PCs) corresponding to the kinetic energy EOFs. Let $A(x)$ be a linear or nonlinear response function that is a part of the model climatology (in the climate response tests below, $A(x)$ is set to represent the response of the climate mean state or its variance). The average value $\langle A \rangle$ of $A(x)$ for the dynamics in (1) is computed as a time average along a single long-term trajectory of (1):

$$\langle A \rangle = \lim_{r \to \infty} \frac{1}{r} \int_0^r A[x(t)] \, dt. \quad (2)$$

We assume that the dynamical system in (1) is perturbed by small external forcing as

$$\frac{dy}{dt} = f(y) + \delta f(t). \quad (3)$$

In our framework, the external forcing represents the collection of altered geophysical parameters (such as solar radiative forcing) that cause climate change. Apparently, the solution $y(t)$ will cause $\langle A \rangle$ to deviate from its original value, which represents the climate response to change in the external parameters:

$$\delta \langle A \rangle = \lim_{r \to \infty} \frac{1}{r} \int_0^r \{A[y(t)] - A[x(t)]\} \, dt. \quad (4)$$

The formula in (4) provides a straightforward estimate for the climate response, based on the distance between two different numerically simulated long-term trajectories of the perturbed and unperturbed models, respectively. However, it requires separate long-term numerical simulation with the climate model for each
perturbation, which can be numerically expensive in the case of multiple climate change scenarios. Also, it computes the climate response at infinite time, and provides no information about transient time-dependent behavior of climate response when the changes in external forcing are applied to the dynamics.

Provide that the change in external forcing is small, the fluctuation–dissipation theorem offers a more versatile way to construct an approximation for $\delta(A)$:

$$\delta(A)(t) = \int_0^t R(t')\delta\mathbf{f}(t-t')dt', \quad (5)$$

where $t = 0$ is the initial time when the forcing change is introduced into the dynamical system. Here, $R(t)$ is the response operator, which directly relates the changes in external parameters $\delta\mathbf{f}$ to the climate response $\delta(A)$. Note that $R(t)$ does not depend on the magnitude and direction of $\delta\mathbf{f}$ and, once computed, provides the climate response for a wide range of magnitudes and directions $\delta\mathbf{f}$. Also, one can study the singular directions of $R$ to determine which changes in $\delta\mathbf{f}$ will result in a most catastrophic (or otherwise, most insignificant) response of $\delta(A)$ (Gritsun and Dymnikov 1999; MAG05). Additionally, the formula in (5) provides the time-dependent climate response for finite values of $t$, which allows the study of transient climate change effects. In the situation when the forcing $\delta\mathbf{f}$ is a constant (which is “turned on” at time $t = 0$) and does not depend on space and time, a simplified version of (5) is used:

$$\delta(A)(t) = R(t)\delta\mathbf{f}, \quad R(t) = \int_0^t R(t')dt'. \quad (6)$$

In the numerical experiments throughout the paper, a constant forcing $\delta\mathbf{f}$ is used.

Evaluating the response operator $R(t)$ or its constant forcing counterpart $R(t)$ is a nontrivial problem. Below we outline several mathematical and numerical methods to compute suitable approximations for $R$ in a variety of dynamical regimes.

a. Quasi-Gaussian FDT method

Under the assumption of nearly Gaussian probability distribution with mean state $\mathbf{x}$ and covariance matrix $\sigma^2$ for the long-term climatological equilibrium, the response operator $R(t)$ is evaluated as a time autocorrelation function (Deker and Haake 1975; Risken 1989; MAG05; AM07; AM08; Gritsun et al. 2008):

$$R_{qG}(t) = \lim_{r \to \infty} \frac{1}{r} \int_0^r A[\mathbf{x}(t+t')]\sigma^{-2}[\mathbf{x}(t') - \mathbf{x}] dt'. \quad (7)$$

The formula in (7) is the quasi-Gaussian FDT response operator. When $\mathbf{x}$ is a vector of PCs, its climatology has zero mean state and identity covariance matrix, whereas the response of the mean state or its variance. However, it is efficient only for dynamical regimes with a nearly Gaussian equilibrium state and robust mixing dynamics (MAG05; AM07; AM08). For a strongly non-Gaussian climatology with slow decay of correlation functions, the qG-FDT response algorithm rapidly loses its precision.

b. Short-time FDT method

A wide variety of practical geophysical models are complex nonlinear chaotic forced-dissipative dynamical systems with statistical attractors that are not smooth. The equilibrium states on such attractors are typically Sinai–Ruelle–Bowen probability measures (for details, see Eckmann and Ruelle 1985; Ruelle 1997, 1998; Young 2002). AM07 and AM08 reworked the FDT formula in such a way that an approximation to the equilibrium state is not required to compute the response, thus adapting the response formula to an arbitrary equilibrium state. This method is exact for a general dynamical system and is called short-time FDT because of its inherent numerical instability at longer times for chaotic dynamical systems. With this method, the time autocorrelation function is given by

$$R_{ST}(t) = \lim_{r \to \infty} \frac{1}{r} \int_0^r \nabla A[\mathbf{x}(t+t')]T'_{\mathbf{x}(t')} dt', \quad (9)$$

where $T'_{\mathbf{x}}$ is a tangent map at $\mathbf{x}$ to $t$, which is computed by solving the equation

$$\frac{dT}{dt} = \nabla \mathbf{f} T. \quad (10)$$

The ST-FDT response formula in (9) provides the precise response approximation for an arbitrary climatological equilibrium state, regardless of the dynamical robustness and mixing. The main drawback of the ST-FDT method is that the unstable Lyapunov directions of the tangent map in (10) grow exponentially in time and
create numerical instability in (9) for long response
times. Also, the ST-FDT method is computationally
more expensive than the qG-FDT method. The detailed
description of the ST-FDT method and its numerical
implementation for computing (9) and (10) are given in
AM07 and AM08 with illustrative detailed applications.

c. Blended ST/qG-FDT method

In the previous two sections we observed that the ST-
FDT response operator and the qG-FDT approxima-
tion have somewhat opposite advantages and draw-
backs. Whereas for shorter response times the ST-FDT
approximation is superior to the qG-FDT response
operator because of its precise geometric interpretation
of the response formula in (9), it eventually blows up at
longer times because of exponential growth in positive
Lyapunov directions. On the other hand, the qG-FDT
response operator is free of numerical instabilities, but
it produces larger errors for weakly mixing non-Gaussian
regimes. However, there is a simple way to avoid the
blowup and increase the overall accuracy by blending
the ST-FDT response with the qG-FDT response at
later times before the numerical blowup occurs, thus
combining the best properties of both responses
through the following formula:

$$ R_{ST/qG} = (1 - \xi) R_{ST} + \xi R_{qG}, \quad (11) $$

where $\xi = \xi(t)$ is a blending function, chosen so that it is
zero for small values of $t$, and thus the response at early
times is computed through the highly accurate $R_{ST}$
operator; however, $\xi(t)$ assumes the value of 1 shortly
before the numerical instability manifests itself in $R_{ST}$.
The simplest and most straightforward choice of the
blending function is the Heaviside step function

$$ \xi(t) = H(t - T_{cutoff}), \quad (12) $$

where $T_{cutoff}$ is the cutoff time chosen just before the
numerical instability occurs in $R_{ST}$. The blended re-
response operator $R_{ST/qG}$ combines the advantages of the
ST-FDT for shorter times and qG-FDT for longer
times. The Heaviside step-function (12) is used as a
blending function for (11) throughout the paper and is
developed and tested extensively in AM07. In particu-
lar, the universal empirical strategy $T_{cutoff} = 3 T_{lyap}$,
where $T_{lyap}$ is the largest Lyapunov exponent, is utilized
in AM07 for the L96 model with widely varying external
forcing parameters for linear and quadratic functionals
and yields excellent universal response performance.

Note, however, that the choice of the blending function
$\xi(t)$ is not limited to the Heaviside function, and in fact
can be chosen systematically to minimize errors be-
tween the ideal and predicted response. Detailed
strategies for (12) are discussed in the application below
for the T21 model.

3. Climatology of the T21 barotropic model

The model equations are

$$ \frac{\partial \xi}{\partial t} + \nabla^T \psi \cdot \nabla q = \left( -\frac{1}{\tau} + K \Delta^3 \right) \xi + \xi^*, $$

$$ \xi = \Delta \psi, \quad q = \xi + 2 \Omega \sin \theta (1 + h/H). \quad (13) $$

The physical units for the model length and time are
meters and days, respectively. The model variables are as
follows: $\phi, \theta$, and $t$ are the longitudinal, latitudinal,
and time coordinates, respectively; $\psi = \psi(\phi, \theta, t)$ is the
streamfunction; $\xi = \xi(\phi, \theta, t)$ is the relative vorticity;
$q = q(\phi, \theta, t)$ is the potential vorticity; and $h = h(\phi, \theta)$ is
the realistic earth topography (shown in Fig. 1) scaled
by the height of the troposphere ($H = 10 000$ m). The
physical parameters are the following: $\tau = 15$ days is
the Ekman friction time scale, $\Omega = 2 \pi$ radian day$^{-1}$
is the angular velocity of the earth’s rotation, and the
day selective damping coefficient $K = 3.38 \times 10^{-9} \text{ m}^6$
$\text{day}^{-1}$ is chosen so that the damping time scale for
wavenumber 21 is 3 h.

The model equations in (13) are projected onto the
triangular-shaped array of spherical harmonics with
maximum wavenumber 21 (standard T21 barotropic
truncation). The T21 truncation is further restricted to
the Northern Hemisphere with no flow across the
equator; therefore, the truncated flow is completely
described by 231 spectral coefficients. The standard
fourth-order Runge–Kutta method is used to integrate
the model in time.

The function $\xi^*$ is $\xi^*(\phi, \theta)$ is the time-independent
forcing, which is chosen to roughly match the properties
of model climatology with observations. In this work,
two different types of steady forcing are used in the
barotropic model to mimic a realistic climatology at
500- and 300-hPa heights in the troposphere.

For the quadratically nonlinear T21 barotropic
model, the tangent map equation needed in (10) for
the blended response algorithm is the linearized model
truncation

$$ \frac{d\Delta T}{dt} = \nabla^T \cdot \nabla T + \nabla^T \Delta T \cdot \nabla \psi $$

$$ + \left( -\frac{1}{\tau} + K \Delta^3 \right) T, \quad T(0) = I, \quad (14) $$

for which the numerical implementation is readily
available through a straightforward modification of
In (14), the operators $\nabla T$, $\Delta T$, $\nabla^2 \Delta T$, and $\Delta^3 T$ are to be understood in the sense of their finite spectral representations applied to the columns of the tangent map $T$. In our setting, the tangent map Eq. (14) is solved using the fourth-order Runge–Kutta method with the same time step as for the T21 barotropic model itself.

**a. Barotropic model climate for 500-hPa height**

This type of climate is proposed by Selten (1995). Here, $\psi_{CL}$ denotes the climatological mean field computed from 10 winters from European Centre for Medium-Range Weather Forecasts (ECMWF) daily analysis of 300-hPa vorticity fields rescaled to 500 hPa via multiplication by a factor of 0.6. The first guess for the forcing is determined through the formula proposed by Roads (1987):

$$\xi_{500hPa}^{\psi_{CL}} = \nabla^\perp \psi_{CL} \cdot \nabla \xi_{CL} + \left( \frac{1}{\tau} - K \Delta^3 \right) \xi_{CL} + \langle \nabla^\perp \psi'_{CL} \cdot \nabla \xi'_{CL} \rangle. \quad (15)$$

**Fig. 1.** (top) Realistic earth orography (m day$^{-1}$). (bottom) Steady vorticity forcing (m day$^{-1}$) for the (left) 500- and (right) 300-hPa geopotential height.
where $\psi'_{\text{CL}}$ and $\zeta'_{\text{CL}}$ are the deviations of the 10-day running mean from the climatological mean state and the angle brackets denote the time average. Then, model runs for 2000 days are performed, and each subsequent iteration of forcing is obtained by replacing the previous transient eddy forcing with the one from the current run:

$$
\xi^{(n)}_{\text{500hPa}} = \xi^{(n-1)}_{\text{500hPa}} + \langle \nabla \psi'_{\text{CL}} \cdot \nabla \xi^{(n-1)} - \nabla \psi'_{\text{CL}} \cdot \nabla \zeta'_{\text{CL}} \rangle.
$$

The third iteration of (16) is implemented in the model and shown in Fig. 1.

b. Barotropic model climate for 300-hPa height

This type of climate is proposed by Franzke et al. (2005). Here the same observational data is used as in 500-hPa climate, but no additional rescaling is performed. The first guess for the forcing is determined through the same formula as in (15). Model runs for 3600 days are performed so that each subsequent iteration of forcing is obtained by replacing the model climatological mean state and transient eddy forcing from the previous run with the observed climatological mean state and transient eddy forcing:

$$
\xi^{(n)}_{\text{300hPa}} = \xi^{(n-1)}_{\text{300hPa}} + \xi_{\text{CL}} - \langle \zeta^{(n-1)} \rangle + \langle \nabla \psi'_{\text{CL}} \cdot \nabla \zeta'_{\text{CL}} \rangle
$$

The forcing is implemented after 30 iterations and shown in Fig. 1.

c. Climatological and dynamical properties

Here we demonstrate the climatological data for a single 10 000-yr-long run of the T21 model with the 500- and 300-hPa forcing. The zero velocity field is taken as the initial condition for the forward time integration, and an initial time interval of 100 yr is skipped before beginning the time averaging. Hereafter, the statistical steady states resulting from the steady forcing in sections 3a and 3b are called the 500- and 300-hPa climates, respectively, in a slight abuse of terminology. The reader is urged to remember, however, that properties of statistical steady states in the barotropic model are governed by the artificially created constant forcing rather than by the actual atmospheric height.

The climatological mean state and standard deviation in the North Polar stereographic projection are shown in Fig. 2. Indeed, the 500-hPa mean state looks like a version of the 300-hPa mean state scaled by 0.6. The biggest difference between the two mean states is the sub-equatorial band of high pressure in the 500-hPa climate over Africa, the Indian Ocean, and the western part of the Pacific Ocean, which is missing in the 300-hPa climate. The variances of both types of climate also resemble each other, although to a smaller extent than the mean states. The average variability of the 500-hPa climate is nearly 3 times as small as that of the 300-hPa climate, as might be expected from the change in height. However, the relative variability over the North Pole for the 500-hPa climate is higher than that for the 300-hPa climate (the contour lines are denser near the North Pole). In addition, the variability for the 300-hPa climate increases rather uniformly from the equator to the North Pole, whereas the 500-hPa climate has a pronounced local maximum over the Europe.

The Lyapunov characteristic exponents for both the 500- and 300-hPa regimes are shown in Fig. 3. Here we observe that the two dynamical regimes, which correspond to different geopotential height, also have different dynamical properties: whereas the more dynamically robust 300-hPa climate has 14 positive Lyapunov exponents (with the largest one at 0.1373, which corresponds to the Lyapunov characteristic time of roughly 1 week), the less chaotic 500-hPa climate has 7 positive Lyapunov exponents (with the largest one at 5.086 × 10⁻², which corresponds to the Lyapunov characteristic time of roughly 3 weeks). Also, the Kolmogorov–Sinai entropy (the sum of positive Lyapunov exponents), which is a bulk measure of information loss on the attractor (Eckmann and Ruelle 1985; Young 2002) differs significantly between the two regimes, being 0.7878 for the 300-hPa climate and 0.1655 for the 500-hPa climate.

The empirical orthogonal functions are computed in the energy metric (for details, see Abramov et al. 2005). The projections on the EOFs are denoted as principal components, with such normalization that each PC has unitary climatological variance. Variances of the EOFs in the energy metric for both types of climate are shown on Fig. 4. In this study of low-frequency variability, further attention is restricted to the first four energy metric EOFs. For the 500-hPa climate, the first four EOFs contain about 45% of total variance, of which roughly one half is concentrated in the first EOF. For the 300-hPa climate there is no large gap in variance between the first and second EOFs, but nevertheless the first four EOFs have about 40% of the total variance. In physical space, EOFs 1–4 for 500- and 300-hPa types of climate are shown in Figs. 5 and 6, respectively. The EOFs for the 500-hPa climate are characterized by smaller-scale structures than those for the 300-hPa climate. This is especially noticeable for the second and fourth EOFs, in which the wave train over Southeast Asia and the Pacific Ocean in the 500-hPa climate is replaced in the 300-hPa climate by one huge cyclone for the second EOF and a cyclone–anticyclone pair for the fourth EOF. Because of
these differences, the EOFs offer different zonal-blocking mechanisms depending on the type of barotropic model climate: the first EOFs for both types of climate strongly project on the Arctic Oscillation, but whereas the second EOF for the 300-hPa climate projects on the Pacific–North American (PNA) pattern, the third EOF for the 500-hPa climate resembles the North Atlantic Oscillation (NAO; Crommelin 2003). In addition, it is worth noting that for the 500-hPa climate EOF 2 has significant projection on the NAO whereas EOF 4 has a strong projection on a Rossby wave train resembling the ducted mode (Branstator 2002). For the 300-hPa climate, EOF 4 has features of a superposition of two low-wavenumber Rossby wave trains. Of course, the realistic fidelity of the T21 barotropic model is far from that of high-resolution multilayer models because of the absence of baroclinic instability; nevertheless, these realistic features of the two barotropic model climates provide an interesting setting for testing the new climate response algorithms (AM07; AM08).

Fig. 2. (top) The climatological mean state and (bottom) climatological std dev (variability) for the (left) 500- and (right) 300-hPa geopotential height.
The time autocorrelations of the first four principal components for the 500- and 300-hPa types of climate are shown in Fig. 7. The rescaling of forcing from 300 to 500 hPa slows down mixing and is manifested by the significantly larger autocorrelation times for the PCs for the 500-hPa model climate. This is expected because the model damping coefficients stay the same and the forcing is scaled down. However, this situation is natural; the higher-velocity fields in the upper troposphere homogenize spontaneous structures much faster than in the lower troposphere, which corresponds to shorter mixing times. Note that the oscillatory correlation functions in Fig. 7 are a reflection of the Rossby wave train evident in the corresponding EOF.

Probability density functions (PDFs) of the first four principal components for 500- and 300-hPa types of climate are shown in Fig. 7. The 300-hPa climate has smooth and nearly Gaussian PDFs for the first four PCs, whereas the PDFs for the 500-hPa climate are skewed, and the PDF for PC1 exhibits signs of bimodality. Similar trends are found in the joint two-dimensional probability density functions of the first four principal components for the 500- and 300-hPa types of climate (for details, see Abramov et al. 2005; Crommelin 2003; Franzke et al. 2005).

Given the differences between the two climates, we expect the classical qG-FDT climate response algorithm to produce relatively good climate response prediction in the dynamically robust and mixing 300-hPa regime and have much less skill in the weakly chaotic 500 hPa
climate. On the other hand, the blended ST/qG-FDT response algorithm, being a better approximation in a geometric sense, is expected to perform comparably for both types of climate.

4. Low-frequency climate response for the mean state

Here we present the response of the climate mean state to changes in external forcing, which corresponds to the response function $A(x) = x$, where $x$ is the vector of the principal components, corresponding to the kinetic energy EOFs. Both the qG-FDT and blended ST/qG-FDT response operators are computed along a single 1000-yr-long solution of the T21 barotropic model (13). The time discretization step used in the Runge–Kutta routine is 4 h for the 500-hPa regime and 3 h for the 300-hPa regime. The qG-FDT and ST/qG-FDT climate response operators are verified against the ideal response operator, which is computed by performing a series of runs with a 10 000-member ensemble.
solution that is directly perturbed by small changes in forcing. Forcing magnitudes with a strength that is 2% of the climate forcing are utilized to calculated this ideal response by direct numerical experiment with the barotropic model. These experiments are supplemented with others involving weaker and stronger forcings of strengths of 1% and 5% as checks for statistical accuracy. A detailed description of the computational algorithm for the ideal response operator and its performance is given in Gritsun and Dymnikov (1999), MAG05, AM07, and AM08. Both the forcing and response are considered in the kinetic energy EOF space; that is, the forcing is applied to a certain PC, and then the mean response is measured in terms of average PC deviations. Although both the qG-FDT and ST/qG-FDT response operators are automatically computed for the complete EOF space, here we demonstrate only parts of the FDT response operators that correspond to the low-frequency subspace of EOFs 1–4, which are the main focus here in a climate change scenario. The relative errors in figures and tables are computed as the ratio of the $L_2$-error norm between the ideal and the
corresponding FDT operator to the norm of the ideal response operator itself. There is the usual caveat in implementing errors in the table that when the ideal response in a particular EOF is itself small, then such relative errors are less meaningful and high skill of the FDT response only requires a similar small magnitude and trend.

a. Mean state response for 300-hPa climate

In this section we present the results for the blended ST/qG-FDT response operator in the 300-hPa regime for the climate mean state response function \(A(x) = x\), with the Heaviside step-function playing the role of the blending function. Here we observe that the empirical rule \(T_{\text{cutoff}} = 3T_{\text{lyap}}\), used by the authors for L96 in AM07, is valid for the T21 model with 300-hPa climate. Accordingly, in the results below, we use this rule for the ST/qG-FDT response, with \(T_{\text{cutoff}} = 22\) days.

In Figs. 8–11 and Table 1 we show the blended ST/qG-FDT response (\(T_{\text{cutoff}} = 22\) days) and the classical qG-FDT response and compare them with the ideal response at PCs 1–4. In general, the blended ST/qG-FDT yields an average error reduction of a factor of 2–3 over the classical qG-FDT operator for these PCs, and in some cases (PC2–PC3, PC4–PC1) the error is reduced by a factor of 7. There are, however, a few cases in which the classical qG-FDT operator yields slightly better results than the blended ST/qG-FDT, in particular when both the forcing and the response occur at PC1. However, at long times this improvement is very likely due to nonlinearities in the PC1 ideal response because of its relatively large magnitude. There are also a few response entries in which the blended ST/qG-FDT and classical qG-FDT response yield roughly same precision, which is due to an extremely good performance of the classical qG-FDT operator in this dynamically robust and mixing regime.

The forcing at the AO mode produces a large and significant response up to 40 days for all four low-frequency EOFs, as shown in Fig. 8, but the weakest response amplitude occurs for the PNA in the model (EOF 2); both response algorithms have high skill here. On the other hand, forcing the PNA produces the weakest overall response in all four low-frequency patterns (see Fig. 9). Forcing EOF 4 with the strong northern quadrupole yields a response that projects strongly on the AO mode, EOF 1, as shown in Fig. 11. Figures 9–11 show oscillatory behavior in the response operator to constant forcing, as the duration time for this forcing varies; the blended response algorithm tracks these changes exceptionally well quantitatively, whereas generally qG-FDT qualitatively tracks these trends with skill.

b. Mean state response for 500-hPa climate

Here we present the results for the blended ST/qG-FDT response operator in the 500-hPa regime for the climate mean state response function \(A(x) = x\), with
the Heaviside step-function playing the role of the blending function. During the course of numerical simulations we observed that, unlike the 300-hPa regime, the ST-FDT blowup times for the 500-hPa regime can be very different depending on the forcing and response PCs. Taking into account these observations, here the blending ST/qG-FDT method is used by choosing the cutoff time individually for each displayed response, based on the a posteriori observed errors produced by both the ST-FDT and qG-FDT response operators. In Figs. 12–15 and Table 2 we show the ideal, qG-FDT, and blended ST/qG-FDT response operators for PCs 1–4. Note that for this case the response operators are calculated for forcing durations up to 100 days as compared with 40 days for the 300-hPa climate. Observe that the blended ST/qG-FDT operator typically delivers much better precision than the standalone qG-FDT operator, with errors differing by factors ranging from roughly 1 (forcing at PC1, response at PC3) to 16 (forcing and response at PC4). In general, the blended ST/qG-FDT algorithm is clearly superior to the classical qG-FDT algorithm for the 500-hPa regime because of the strong non-Gaussianity of the equilibrium state and weak mixing.

From Fig. 12, forcing at the AO mode (EOF 1) yields a very large-amplitude response at both the AO and NAO mode (EOF 3). Also note that the blended ST/qG-FDT algorithm captures the response at the ducted mode (EOF 4), but the qG-FDT gives completely erroneous results with the opposite sign. Forcing the NAO mode (EOF 3) gives a strong response at EOF 2 and the ducted mode but a weaker response at AO; here the blended algorithm tracks the response with very high skill but qG-FDT has very poor skill as depicted in Fig. 14. Similar remarks apply for the skill in forcing the ducted mode, EOF 4 (see Fig. 15).

5. Low-frequency climate response for the variance

Here we present the response of the climate variance to changes in external forcing, which corresponds to the response function $A_i(x) = x_i^2$, where $x$ is the vector of the PCs corresponding to the kinetic energy EOFs. Both the qG-FDT and blended ST/qG-FDT response operators as well as the ideal response operator are computed in exactly the same fashion as described at the beginning of section 4. Similar to section 4 for the mean state response, here the results are shown for PCs 1–4. The relative errors in figures and tables are computed as the ratio of the $L_2$ error norm between the ideal and corresponding FDT operator to the norm of the ideal response operator itself. Here, however, the actual response of the variances in the barotropic model is much smaller than the mean state response in both climates, probably because of the lack of baroclinic instability in the basic model; thus, a large relative error does not necessarily indicate a lack of skill in the variance response.
In this section we present the results for the blended ST/qG-FDT response operator in the 300-hPa regime for the climate variance response function $A_i(x) = x_i^2$, with the Heaviside step-function playing the role of the blending function. Here we use the same empirical blending rule $T_{\text{cutoff}} = 3T_{\text{lyap}}$, found by the authors (AM07) for the L96 model, and with high skill for the mean state response already reported in section 4 for this climate model.

In Figs. 16–19 and Table 3 we show the blended ST/qG-FDT ($T_{\text{cutoff}} = 22$ days) and classical qG-FDT responses and compare them with the ideal response for

$\text{FIG. 9. As in Fig. 8, but for PC 2.}$

$\text{FIG. 10. As in Fig. 8, but for PC 3.}$
PCs 1–4. In general, the blended ST/qG-FDT yields average error reduction by a factor of 2–3 over the classical qG-FDT operator for these PCs, and in some cases the error is reduced by a factor of 12–13. There are, however, a few cases in which the classical qG-FDT operator yields somewhat better results than the blended ST/qG-FDT. However, this improvement is very likely due to the loss of precision in the ideal response because the magnitude of the response itself is much smaller than for the mean state (typically the magnitude of the variance response does not exceed the value of 1). Also, because of the low magnitude of the ideal response, the relative errors in figures and Table 3 sometimes reach utterly disproportionate values exceeding 100%. This happens when the norm of the difference between the ideal and the corresponding FDT response operators is greater than the norm of the ideal response operator itself. In these cases, quite often simply predicting an appropriate small response indicated significant skill.

The largest response in the variances occurs when the AO (EOF 1) is forced (see Fig. 16) and the blended algorithm has significant skill here. The variance response to forcing the PNA (EOF 2) and EOF 3 is surprisingly weak in all components, and the superiority with high skill of the blended response algorithm compared with qG-FDT in computing these weak responses is evident (Figs. 17 and 18). The variance response from forcing the Rossby wave train mode in EOF 4 is largest on the PNA pattern and is well captured by the blended response algorithm (see Fig. 19).

**b. Variance response for 500-hPa climate**

Here we present the results for the blended ST/qG-FDT response operator in the 500-hPa regime for the

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**TABLE 1. Climate mean response relative errors of the ST/qG-FDT and qG-FDT algorithms, 300-hPa climate.**

<table>
<thead>
<tr>
<th>Forcing/response</th>
<th>ST/qG-FDT response</th>
<th>qG-FDT response</th>
<th>Error ratio of qG-FDT to ST/qG-FDT response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PC 1</td>
<td>PC 2</td>
<td>PC 3</td>
</tr>
<tr>
<td>PC 1</td>
<td>11%</td>
<td>13%</td>
<td>6%</td>
</tr>
<tr>
<td>PC 2</td>
<td>12%</td>
<td>6%</td>
<td>4%</td>
</tr>
<tr>
<td>PC 3</td>
<td>40%</td>
<td>10%</td>
<td>17%</td>
</tr>
<tr>
<td>PC 4</td>
<td>2%</td>
<td>12%</td>
<td>54%</td>
</tr>
</tbody>
</table>
climate variance response function \( A_i(x) = x_i^2 \), with the Heaviside step-function playing the role of the blending function. During the course of numerical simulations we observed that, unlike the 300-hPa regime, the ST-FDT blowup times for the 500-hPa regime can be very different depending on the forcing and response EOFs. Taking into account these observations, here (as in section 4) the blending ST/qG-FDT method is used by

**Fig. 12.** Climate response of the mean state for PCs 1–4 when forced at PC 1; ideal, qG-FDT, and ST/qG-FDT response, 500-hPa geopotential height. ST/qG-FDT cutoff time \( T_{\text{cutoff}} = 65, 40, 65, 65 \) days for the response at PC 1, 2, 3, and 4, respectively.

**Fig. 13.** As in Fig. 12, but for forcing at PC 2; \( T_{\text{cutoff}} = 40, 85, 65, 75 \) days for the response at PC 1, 2, 3, and 4, respectively.
choosing the cutoff time individually for each displayed response, based on the a posteriori observed errors produced by both the ST-FDT and qG-FDT response operators. In Figs. 20–23 and Table 4 we show the ideal, qG-FDT, and blended ST/qG-FDT response operators for PCs 1–4. Observe that the blended ST/qG-FDT operator typically delivers better precision than the standalone qG-FDT operator, with errors differing on average by a factor of 2–3 and sometimes reaching as much as 15. In general, the blended ST/qG-FDT

![Fig. 14](image1)

![Fig. 15](image2)

FIG. 14. As in Fig. 12, but for forcing at PC 3; $T_{\text{cutoff}} = 50, 80, 80, 65$ days for the response at PC 1, 2, 3, and 4, respectively.

FIG. 15. As in Fig. 12, but for forcing at PC 4; $T_{\text{cutoff}} = 70, 45, 65, 65$ days for the response at PC 1, 2, 3, 4, respectively.
algorithm is clearly superior to the classical qG-FDT algorithm for the 500-hPa regime because of the strong non-Gaussianity of the equilibrium state and weak mixing. As with the climate variance response of the 300-hPa climate, the relative errors can reach huge values of more than 100% because of the low magnitude of the ideal response.

The variance response onto itself from Fig. 20 from forcing the AO mode is large and well captured by the blended algorithm as well as by the much weaker response on the other EOFs. The variance response from forcing the NAO mode, EOF 3, is surprisingly small for all the EOF modes except the ducted mode, EOF 4 (see Fig. 22); here the qG-FDT algorithm gives an extremely large false variance response on EOFs 1–2 with much higher skill for the blended response algorithm. Similar behavior occurs for these algorithms with the variance response to forcing at EOFs 2 and 4 (depicted in Figs. 21 and 23).

6. Conclusions

In two recent papers (AM07; AM08) the authors developed a new blended ST/qG-FDT response algorithm for predicting the climate response of a nonlinear chaotic forced-dissipative system to small external forcing. This new algorithm combines the geometrically exact general ST-FDT response formula at short response times using integration of a linear tangent model and the classical qG-FDT response algorithm at longer response times. This blended algorithm overcomes the inherent numerical instability in the ST-FDT formula arising because of positive Lyapunov exponents at

<table>
<thead>
<tr>
<th>Forcing/response</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC 1</td>
<td>26%</td>
<td>11%</td>
<td>15%</td>
<td>10%</td>
</tr>
<tr>
<td>PC 2</td>
<td>4%</td>
<td>6%</td>
<td>56%</td>
<td>25%</td>
</tr>
<tr>
<td>PC 3</td>
<td>37%</td>
<td>24%</td>
<td>6%</td>
<td>6%</td>
</tr>
<tr>
<td>PC 4</td>
<td>32%</td>
<td>15%</td>
<td>50%</td>
<td>7%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Forcing/response</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC 1</td>
<td>73%</td>
<td>22%</td>
<td>44%</td>
<td>119%</td>
</tr>
<tr>
<td>PC 2</td>
<td>14%</td>
<td>36%</td>
<td>487%</td>
<td>69%</td>
</tr>
<tr>
<td>PC 3</td>
<td>43%</td>
<td>70%</td>
<td>82%</td>
<td>77%</td>
</tr>
<tr>
<td>PC 4</td>
<td>89%</td>
<td>128%</td>
<td>126%</td>
<td>109%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Forcing/response</th>
<th>PC 1</th>
<th>PC 2</th>
<th>PC 3</th>
<th>PC 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC 1</td>
<td>2.8</td>
<td>2.9</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>PC 2</td>
<td>3.5</td>
<td>6</td>
<td>8.7</td>
<td>2.8</td>
</tr>
<tr>
<td>PC 3</td>
<td>1.2</td>
<td>2.9</td>
<td>14</td>
<td>13</td>
</tr>
<tr>
<td>PC 4</td>
<td>2.8</td>
<td>8.5</td>
<td>2.5</td>
<td>16</td>
</tr>
</tbody>
</table>

FIG. 16. Climate response of the variance for PCs 1–4 when forced at PC 1; ideal, qG-FDT, and ST/qG-FDT response, 300-hPa geopotential height. ST/qG-FDT cutoff time $T_{cutoff} = 22$ days.
longer times while delivering consistently better response prediction in the observed response time range. Section 2 describes the ST/qG-FDT and qG-FDT climate response algorithms.

The goal here is to develop a skillful algorithm to study the low-frequency climate response to external forcing through FDT algorithms in a prototype idealized atmospheric model. In the work here we tested the new blended algorithm on the T21 barotropic truncation on the sphere with realistic topography in two dynamical regimes corresponding to the mean climate behavior at 300- and 500-hPa geopotential height. It is

FIG. 17. As in Fig. 16, but for forcing at PC 2.

FIG. 18. As in Fig. 16, but for forcing at PC 3.
shown in section 3 that these two regimes have somewhat opposite statistical dynamical properties: whereas the 300-hPa regime has robust strongly mixing behavior with nearly Gaussian equilibrium state distribution, the 500-hPa regime is weakly chaotic with slowly decaying time autocorrelation functions and strongly non-Gaussian climatology. The choice of these two different dynamical regimes (which both mimic realistic climate dynamics at different tropospheric heights) allows for an extensive test of both ST/qG-FDT and qG-FDT methods. In sections 4 and 5 the ST/qG-FDT and qG-FDT response algorithms are tested for the T21 barotropic model with realistic topography in the 300- and 500-hPa climate regimes. It is found that the blended ST/qG-FDT algorithm is highly skillful and generally superior to the classical qG-FDT algorithm for the observed time range of the response. Unlike the qG-FDT, the accuracy of the ST/qG-FDT blended algorithm does not deteriorate for the response of linear and quadratic functions for both dynamical regimes of weakly and strongly chaotic behavior because the ST/qG-FDT is represented by a much more precise geometric response formula for early response times, when significant response errors can occur. Furthermore, the response of the T21 barotropic model for the studied response functions does not seem to be affected by the model’s structural instability in a precise geometric sense; that is, the directly perturbed ideal response generally follows the tangent ST-FDT prediction at early response times for the climate mean state and variance response. At later times, the errors between the ideal and FDT response cannot be clearly attributed to the structural instability because of the presence of other error-producing factors, such as the developing nonlinearity of the ideal response and errors in finite time sampling of numerical integration of the FDT response operators. The main additional

### Table 3. Climate variance response relative errors of the ST/qG-FDT and qG-FDT algorithms, 300-hPa climate.

<table>
<thead>
<tr>
<th>Forcing/response</th>
<th>Relative errors of the ST/qG-FDT response</th>
<th>Relative errors of the qG-FDT response</th>
<th>Error ratio of qG-FDT to ST/qG-FDT response</th>
</tr>
</thead>
<tbody>
<tr>
<td>PC 1</td>
<td>7%  5%  12%  7%</td>
<td>56%  14%  8%  27%</td>
<td>8  2.8  0.67  3.9</td>
</tr>
<tr>
<td>PC 2</td>
<td>16%  15%  23%  36%</td>
<td>68%  25%  23%  38%</td>
<td>4.3  1.7  1  1.1</td>
</tr>
<tr>
<td>PC 3</td>
<td>51%  20%  83%  23%</td>
<td>343% 237% 529% 178%</td>
<td>6.7  12  6.4  7.7</td>
</tr>
<tr>
<td>PC 4</td>
<td>41%  5%  17%  16%</td>
<td>41%  65%  56%  50%</td>
<td>4  13  3.3  3.1</td>
</tr>
</tbody>
</table>

**FIG. 19.** As in Fig. 16, but for forcing at PC 4.
overhead of the ST/qG-FDT computation is the tangent map, which puts the blended ST/qG-FDT formula between the classical qG-FDT response and the full ideal response in terms of computational expense. The results here demonstrate significant skill for the blended algorithm on two different types of large-dimensional dynamical systems that sit at the extremes of statistical behavior in low-frequency regimes for contemporary climate models. Thus, these results suggest the use of the ST/qG-FDT response algorithm.
for predicting the low-frequency climate response of complex nonlinear geophysical models and for assessing their structural stability at low frequencies. Furthermore, the predicted response of the mean and covariance can be used to recalibrate the EOFs for nonequilibrium perturbed dynamics. Future research in this direction will be centered on developing systematic adaptive algorithms for choosing optimal blending functions for the climate response in a variety of practical settings, as well as testing the blended response algorithms for more sophisticated climate models with more realistic large-scale features.
of the real atmosphere, ocean, and coupled atmosphere–ocean. This is possible because the new blended algorithms utilize integration of a linear tangent model for the short time response in a precise fashion (AM07; AM08) and such numerical algorithms are already available in many climate models. It remains to be seen whether less precise algorithms for the short time response introduce significant numerical errors that will swamp the skill demonstrated here on the barotropic model. To apply the ST/qG-FDT formula for the present climate models, an important extension of the FDT needs to be made to incorporate nonautonomous seasonal variations; such an extension has been developed recently by Majda and Wang (2008, manuscript submitted to Comm. Math. Sci.) In addition, climate models have nondifferentiable features due to moist convection processes, which usually occur on much shorter time scales. Thus, there is a possibility of applying blended response algorithms to reduced stochastic models for low-frequency components of the system.

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