NOTES AND CORRESPONDENCE

Equatorially Bounded Zonally Propagating Linear Waves on a Generalized β Plane

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ABSTRACT

Meridionally confined zonally propagating wave solutions to the linear hydrostatic Boussinesq equations on a generalized equatorial β plane that includes the ‘‘nontraditional’’ Coriolis force terms associated with the poleward component of planetary rotation are calculated. Kelvin, Rossby, inertia–gravity, and mixed Rossby–gravity modes generalize from the traditional model with the dispersion relation unchanged. The effects of the nontraditional terms on all waves are the curving upward with latitude of the surfaces of constant phase and the equatorial trapping width of the solutions (the equatorial radius of deformation) increasing by order $(\Omega/N)^2$ compared to the traditional case, where Ω is the planetary rotation rate and N the buoyancy frequency. In addition, for the Rossby, inertia–gravity, and mixed Rossby–gravity modes, there is a phase shift of $O(\Omega/N)$ in the zonal and vertical velocity components relative to the meridional component, and their spatial structures are further modified by differences of $O(\Omega/N)^2$. For the Rossby and inertia–gravity waves, the modifications depend also on the phase speed of the wave. In the limit $N \gg \Omega$, the traditional approximation is justified and lines of constant phase in the y–z plane become horizontal, whereas for $N \ll \Omega$ phase lines become everywhere almost parallel to the planetary rotation vector. In both limits, the phase lines are perpendicular to the dominant restoring force—respectively, gravity and the centrifugal force associated with the solid-body rotation of the atmosphere at rest in the rotating frame.

1. Introduction

Most studies of large-scale geophysical problems do not consider the effect of the Coriolis force due to the poleward component of planetary rotation. Its neglect is an element of a suite of approximations known as the ‘‘traditional approximation’’ (Eckart 1960) linked to the shallowness of the atmosphere and ocean compared to their horizontal scale. The particular exclusion of the nontraditional Coriolis terms is motivated by the desire to retain conservation of energy and absolute zonal angular momentum that are otherwise no longer satisfied when making the shallow atmosphere approximation (Phillips 1966; Lorenz 1967), but this obstacle does not arise on the β plane, where energy and angular momentum conservation still hold without the traditional approximation (Grimshaw 1975).

The neglected terms are proportional to the cosine of latitude and are therefore most significant near the equator, where the rotation vector is entirely in the poleward direction. Since they couple the zonal and vertical velocities, they are least easily dismissed for motions with the vertical scale significant compared to the horizontal scale. White and Bromley (1995) showed that they are small like $2\Omega H \cos\phi/U$, compared to the next smallest terms, where Ω is the planetary rotation rate, U and H characteristic horizontal velocity and height scales, and ϕ latitude. In terms of wave modes in a stratified atmosphere or ocean, it has been argued that the neglected terms are small to the extent that $(\Omega/N)^2$ is small, where N is the buoyancy frequency (Phillips 1968).

The effect of the nontraditional Coriolis force terms on gravity waves on an f plane has been considered by several authors (Thuburn et al. 2002; Kasahara 2003; Gerkema and Shrira 2005a), who have shown that the nontraditional terms affect the dispersion relation only for waves

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that propagate in latitude. Gerkema and Shrira (2005b) studied gravity waves on a nontraditional midlatitude β plane, considering primarily meridionally propagating modes. Bretherton (1964), Stern (1963), and Harlander and Maas (2007) calculated low-frequency, axially symmetric solutions on a nontraditional equatorial β plane that are trapped near the equator in the sense that energy is not radiated out of the equatorial region. A summary of the literature on the importance of the nontraditional Coriolis terms can be found in Gerkema et al. (2008).

The present note addresses the opposite extreme case: zonally and vertically propagating, equatorially confined waves. This problem was treated in the traditional system by Matsuno (1966) in the context of zonally propagating waves in a single shallow layer and by Lindzen (1967) for vertically propagating waves. These waves fall into two main categories: the fast eastward and westward propagating inertia–gravity waves and the slow, westward propagating equatorial Rossby waves. In addition, there are two waves particular to the equatorial region: the nondispersive, eastward-propagating equatorial Kelvin wave and the so-called mixed Rossby–gravity wave, which is fast and resembles an inertia–gravity wave in its eastward propagating branch and is slow and resembles a Rossby wave in the short wavelength limit of its westward propagating branch.

The family of equatorially confined waves is of central importance to the understanding and interpretation of equatorial phenomena. The variability in convective activity and temperature in the equatorial region was investigated by Wheeler and Kiladis (1999) by studying outgoing longwave radiation (OLR) data from satellites. They found a strong correspondence between the variability of the OLR data viewed in the wavenumber–frequency domain and the low frequency part of the dispersion diagram of the confined equatorial waves. The quasi-biennial oscillation in the equatorial stratosphere is generally understood to be forced by vertically propagating Kelvin, mixed Rossby–gravity, and inertia–gravity waves, as first proposed by Holton and Lindzen (1972), and equatorially confined waves are essential to understanding both the Madden–Julian oscillation (Zhang 2005) and the El Ninõ–Southern Oscillation (Suarez and Schopf 1988).

While it has been argued that the nontraditional Coriolis terms are most significant near the equator and their effect on midlatitude gravity waves and equatorial inertial stability has been studied in some detail, to this point their effect on the important class of equatorially confined horizontally and vertically propagating waves has not. That is the purpose of the present note.

In the next section, solutions to the linear hydrostatic Boussinesq equations on the nontraditional equatorial β plane are presented. The equations are first solved for the case of no meridional velocity, yielding Kelvin wave modes whose spatial structure is different than in the traditional case. The Rossby, inertia–gravity, and mixed Rossby–gravity modes are then found using a change of variables suggested by the spatial structure of the Kelvin wave. The structure of the mixed Rossby–gravity wave is examined in detail. Remarkably, the dispersion relations of all of the waves in the nontraditional system are unchanged from in the traditional system.

2. Zonally propagating waves

Consider the inviscid Boussinesq equations on a generalized equatorial β plane, including all Coriolis force terms to order y and with constant background stratification, linearized about a state of rest:

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \beta y v - \gamma w, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} - \beta y u, \tag{2.1b}$$

$$0 = -\frac{\partial p}{\partial z} - \frac{\rho'}{\rho_m} g + \gamma u, \tag{2.1c}$$

$$\frac{\partial \rho'}{\partial t} = \frac{\rho_m}{g} N^2 w, \tag{2.1d}$$

and

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \tag{2.1e}$$

where $[0, \gamma, \beta y] \equiv 2\mathbf{\Omega}_0$ is twice the planetary rotation vector (the terms with $\gamma = 2\Omega$ represent the nontraditional Coriolis force, and $\beta = 2\Omega/a$, where a is the planetary radius, is the meridional gradient of the traditional Coriolis parameter) and the total density is $\rho = \rho_m + \rho_0(z) + \rho'(x, y, z, t)$, with both ρ' and ρ_0 much smaller than ρ_m and $N^2 \equiv (-g/\rho_m)\, d\rho_0/dz = \text{const}$. Our objective is to find equatorially confined zonally and vertically propagating wave solutions to (2.1). The boundary conditions to be applied are periodicity in the x direction, the vanishing of all fields as $y \to \pm\infty$, and no normal flow at the flat bottom surface. No explicit upper boundary condition will be enforced for the monochromatic solutions discussed here but, for a vertically propagating wave packet, Lindzen (1967) would be followed and a radiation condition—that no energy propagate toward the surface from infinity—would be applied. Harlander and Maas (2007) used the same set of equations with no-normal-flow conditions on rigid upper and lower boundaries and concentrated on the limiting case $N = 0$, that of an homogeneous fluid.

Equation (2.1c) neglects the linearized vertical acceleration term $\partial w/\partial t$. We are thus considering relatively slow motions and/or motions with the horizontal scale large compared to the vertical scale. For typical synoptic horizontal and vertical scales, $L = 10^6$ m and $H = 10^4$ m in the atmosphere or $L = 10^5$ m and $H = 10^3$ m in the ocean, the vertical acceleration term is much smaller than the nontraditional term γu in (2.1c) if the time scale is much greater than several minutes. That the solutions are consistent with the neglect of the vertical acceleration is confirmed a posteriori.

a. Kelvin wave

First, consider the case of $v \equiv 0$. Because of (2.1e), we can introduce a streamfunction $\psi(x, y, z, t)$ for flow in the (x, z) plane such that $u = -\partial\psi/\partial z$ and $w = \partial\psi/\partial x$. Setting $v \equiv 0$ in (2.1), eliminating ρ' and p, and rearranging gives

$$-\beta y\left(\frac{\partial^3\psi}{\partial z^2\partial t} + \gamma\frac{\partial^2\psi}{\partial x\partial z}\right) - (N^2+\gamma^2)\frac{\partial^2\psi}{\partial x\partial y} = 0, \qquad (2.2a)$$

and

$$\beta y\left(N^2\frac{\partial^2\psi}{\partial x\partial z} - \gamma\frac{\partial^3\psi}{\partial z^2\partial t}\right) - (N^2+\gamma^2)\frac{\partial^3\psi}{\partial y\partial z\partial t} = 0. \qquad (2.2b)$$

Assume a traveling wave solution to (2.2) of the form

$$\psi = \mathrm{Re}\{\hat\psi(y,z)\exp[i(kx-\omega t)]\}, \qquad (2.3)$$

where $\hat\psi$ may be complex. Substituting (2.3) into (2.2a) and assuming a separable form

$$\hat\psi = \hat\psi_0 Y(y)Z(z)$$

yields

$$\frac{1}{Z}\left(\frac{\omega}{k}\frac{d^2Z}{dz^2} - \gamma\frac{dZ}{dz}\right) = \left(\frac{N^2+\gamma^2}{\beta y Y}\right)\frac{dY}{dy} = -\mu, \qquad (2.4)$$

where μ is the complex separation constant. Solutions to (2.4) are

$$Y = \exp\left[\left(\frac{-\mu\beta}{N^2+\gamma^2}\right)\frac{y^2}{2}\right], \quad Z = \exp(i\nu z), \qquad (2.5)$$

where the vertical wavenumber ν satisfies

$$-\frac{\omega}{k}\nu^2 - i\gamma\nu + \mu = 0. \qquad (2.6)$$

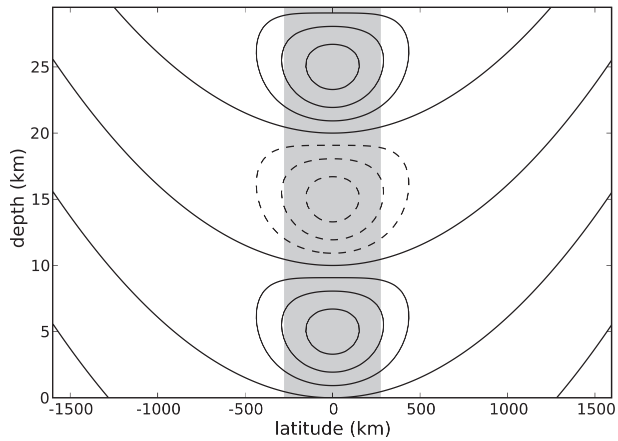

FIG. 1. Contours of zonal velocity amplitude for the standing Kelvin wave with vertical wavelength 20 km. The buoyancy frequency is $N = 5 \times 10^{-4}$ s^{-1}. Dashed contours are out of phase by π with solid contours; the shaded region is within one deformation radius of the equator.

Hence,

$$\hat\psi = \hat\psi_0\exp\left\{-\left[\frac{\beta\nu^2(\frac{\omega}{k})}{N^2+\gamma^2}\right]\frac{y^2}{2}\right\}\exp\left\{i\nu\left[z - \left(\frac{\gamma\beta}{N^2+\gamma^2}\right)\frac{y^2}{2}\right]\right\}. \qquad (2.7)$$

Substituting (2.7) into (2.3) and (2.2b) yields the dispersion relation

$$\frac{\omega^2}{k^2} = \frac{N^2}{\nu^2}. \qquad (2.8)$$

For $\hat\psi$ to be bounded as y tends to $\pm\infty$, $\omega/k \equiv c_K > 0$. Therefore, hydrostatic Kelvin waves are nondispersive and propagate eastward, as in the traditional system. The bounded solutions are thus

$$\psi = \hat\psi_0\exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\exp\left\{i\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\}, \qquad (2.9)$$

where $\alpha \equiv N^2/(N^2+\gamma^2)$ and $\lambda^{-1} \equiv \gamma\beta/(N^2+\gamma^2)$ measure, respectively, the widening of the equatorial trapping and the upward curvature with latitude of the lines of constant phase. The equatorial trapping width of the corresponding equatorial wave is usually interpreted as the equatorial radius of deformation for a given vertical mode. An appropriate definition of the equatorial radius of deformation for vertical mode ν is evidently $\sqrt{N/(\alpha\beta\nu)}$ in the nontraditional system. The deformation radius is therefore *larger* in the nontraditional system.

Figure 1 shows contours of zonal velocity amplitude for a standing Kelvin wave (i.e., the superposition of a wave with $\nu = \nu_0$ and a wave with $\nu = -\nu_0$) with vertical wavelength 20 km. A very low (but not unrealistic for the deep ocean) background stratification of $N = 5 \times 10^{-4}\ \mathrm{s}^{-1}$ has been chosen to highlight the differences due to the nontraditional terms. Dashed contours are out of phase by π with solid contours.

For typical earth scales, the effect of the nontraditional terms is quite small. An average value of N for the atmosphere is $2 \times 10^{-2}\ \mathrm{s}^{-1}$ and for the ocean is $2 \times 10^{-3}\ \mathrm{s}^{-1}$, which are both relatively large compared to $\gamma \approx 1.5 \times 10^{-4}\ \mathrm{s}^{-1}$. In the deep ocean, N is typically considerably lower, and on Jupiter, where $\gamma \approx 3.5 \times 10^{-4}\ \mathrm{s}^{-1}$, the ratio γ/N might be as large as 20% assuming $N \approx 2 \times 10^{-3}\ \mathrm{s}^{-1}$ (Vasavada and Showman 2005).

b. *Rossby, inertia–gravity, and mixed Rossby–gravity waves*

We now return to (2.1) and look for solutions with v not identically zero. Eliminating ρ' and w by differentiating (2.1c) with respect to t and z and substituting from (2.1d) and (2.1e) gives

$$\frac{\partial^3 p}{\partial z^2 \partial t} - N^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \gamma \frac{\partial^2 u}{\partial z \partial t} = 0, \qquad (2.10)$$

and eliminating w between (2.1a) and (2.1e) gives

$$\frac{\partial^2 u}{\partial z \partial t} + \frac{\partial^2 p}{\partial x \partial z} - \beta y \frac{\partial v}{\partial z} - \gamma\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \qquad (2.11)$$

We solve (2.1b), (2.10), and (2.11) by assuming a solution of the form

$$[u, v, p] = \mathrm{Re}\{(\hat{u}(y), \hat{v}(y), \hat{p}(y)) \exp[i(kx + \nu z - \omega t)]\}, \qquad (2.12)$$

where $\hat{u}$, $\hat{v}$, and $\hat{p}$ are complex (henceforth, the Re notation will be suppressed). This yields a system of coupled ODEs:

$$i\omega\hat{v} - \frac{d\hat{p}}{dy} - \beta y \hat{u} = 0, \qquad (2.13a)$$

$$-i\omega\nu^2\hat{p} + N^2\left(ik\hat{u} + \frac{d\hat{v}}{dy}\right) + \omega\nu\gamma\hat{u} = 0, \qquad (2.13b)$$

and

$$-\omega\nu\hat{u} + k\nu\hat{p} + i\nu\beta y\hat{v} + \gamma\left(ik\hat{u} + \frac{d\hat{v}}{dy}\right) = 0, \qquad (2.13c)$$

which may be combined to give the single equation for $\hat{v}$,

$$\frac{d^2\hat{v}}{dy^2} + 2i\nu\lambda^{-1} y \frac{d\hat{v}}{dy} + \left(\frac{1}{N^2 + \gamma^2}\right)\left[\nu^2\omega^2 - (k^2N^2 - i\gamma\beta\nu) - \frac{\beta k N^2}{\omega} - \nu^2(\beta y)^2\right]\hat{v} = 0. \qquad (2.14)$$

An equation similar to (2.14) was derived by Gerkema and Shrira (2005b). Their Eq. (A1) is more general in that it applies also to the midlatitude β plane and does not neglect the vertical acceleration term in the vertical momentum equation (since they were concerned with higher frequency motions than in the present paper). Equation (2.14) can be derived from their Eq. (A1) by neglecting the squared oscillation frequency ω^2 when it is added to either N^2 or γ^2 but retaining it when added to $(\beta y)^2$. Guided by the form of the Kelvin wave solution, we substitute

$$\tilde{v}(y) \equiv \hat{v} \exp\left(-\frac{i\nu\lambda^{-1}}{2} y^2\right), \qquad (2.15)$$

giving

$$\frac{d^2\tilde{v}}{dy^2} + \left(\frac{1}{N^2 + \gamma^2}\right)\left[\nu^2\omega^2 - k^2N^2 - \frac{\beta k N^2}{\omega} - \left(\frac{\nu^2 N^2}{N^2 + \gamma^2}\right)(\beta y)^2\right]\tilde{v} = 0. \qquad (2.16)$$

A transformation equivalent to (2.15) was used by Hua et al. (1997) in studying inertial instability on the nontraditional β plane.

Equation (2.16) can be transformed into a Hermite differential equation (see, e.g., Arfken 1985) by introducing the substitutions

$$y \equiv \sqrt{\frac{c_K}{\alpha\beta}}\xi, \quad \tilde{v}(y) \equiv \tilde{V}(\xi)\exp\left(\frac{\xi^2}{2}\right), \qquad (2.17)$$

where $c_K \equiv \sqrt{N^2/\nu^2}$ as before, giving

$$\frac{d^2\tilde{V}}{d\xi^2} - 2\xi\frac{d\tilde{V}}{d\xi} + \left(\frac{\omega^2}{\beta c_K} - \frac{k^2 c_K}{\beta} - \frac{k c_K}{\omega} - 1\right)\tilde{V} = 0. \qquad (2.18)$$

Equation (2.18) has polynomial solutions if

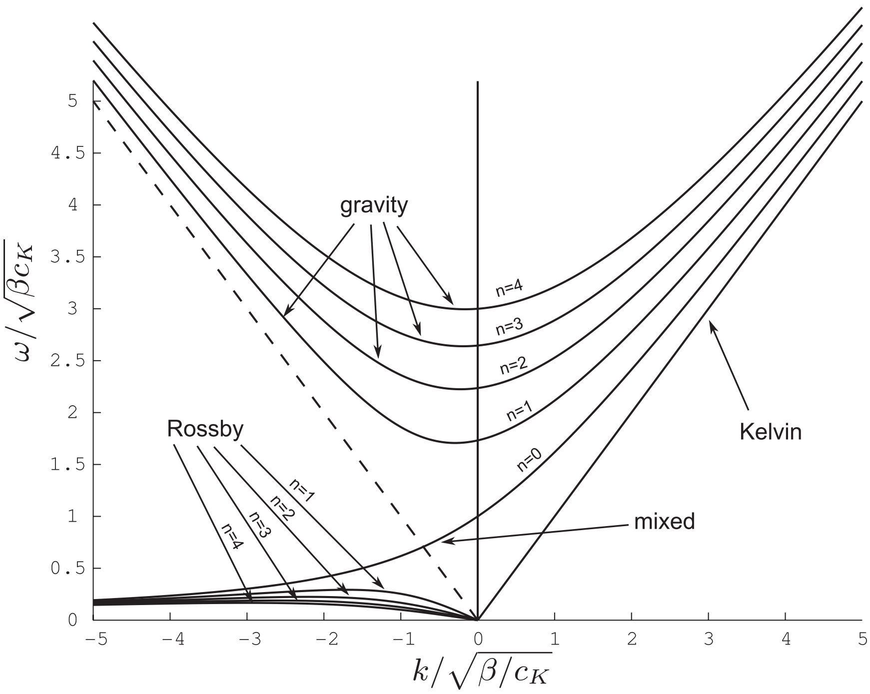

FIG. 2. Dispersion diagram for equatorially confined waves, showing the Kelvin wave, mixed Rossby–gravity wave, and the lowest four meridional-mode Rossby and inertia–gravity waves. The dashed line is the line $\omega = -c_K k$.

$$\frac{\omega^2}{\beta c_K} - \frac{c_K k^2}{\beta} - \frac{c_K k}{\omega} - 1 = 2n, \tag{2.19}$$

where $n = 0, 1, 2, \ldots$, namely, the Hermite polynomials

$$H_0(\xi) = 1, \quad H_1(\xi) = 2\xi,$$
$$H_2(\xi) = 4\xi^2 - 2, \quad H_3(\xi) = 8\xi^3 - 12\xi, \ldots. \tag{2.20}$$

If (2.19) is not satisfied, the solutions to (2.18) grow like $\exp(\xi^2/2)$, implying $\hat{v}$ is $O(1)$ as $y \to \pm\infty$. The solutions to (2.19) are plotted in Fig. 2. The inertia–gravity waves all have periods shorter than $2\pi/\sqrt{\beta c_K}$. The highest frequency waves in the system (short inertia–gravity and Kelvin waves) have periods of approximately $2\pi/(kc_K)$. Our neglect of the vertical acceleration terms is thus consistent with the solution for all Rossby waves and for Kelvin, inertia–gravity, and mixed Rossby–gravity waves with aspect ratio $\nu/k \gg \sqrt{N/4\pi\Omega}$, on the order of 10 for the atmosphere and unity for the ocean.

Reversing the substitutions and using (2.13), we find the solution

$$v(x, y, z, t) = V_0 H_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\exp\left\{i\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\}, \tag{2.21}$$

$$p(x, y, z, t) = V_0\left(\frac{c_K^2}{\omega^2 - k^2c_K^2}\right)\left\{-\omega\sqrt{\frac{\beta}{\alpha c_K}}H_n'\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right) + (\omega + kc_K)\frac{\beta}{c_K}yH_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\right\}\exp\left(-\frac{\alpha\beta}{2c_K}y^2\right) \times \exp\left\{i\left[\frac{\pi}{2} + kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\}, \tag{2.22}$$

and

$$u(x, y, z, t) = V_0\left(\frac{kc_K^2}{\omega^2 - k^2c_K^2}\right)\left\{\left[\sqrt{\frac{\alpha\beta}{c_K}}H_n'\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right) - \left(\frac{\omega + kc_K}{kc_K}\right)\frac{\alpha\beta}{c_K}yH_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\right]\exp\left(-i\frac{\pi}{2}\right)\right.$$
$$\left. + \frac{\gamma}{N}\left(\frac{N}{\nu c_K}\right)\left[\frac{\omega}{kc_K}\sqrt{\frac{\alpha\beta}{c_K}}H_n'\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right) - \left(\frac{\omega + kc_K}{kc_K}\right)\frac{\alpha\beta}{c_K}yH_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\right]\right\}$$
$$\times \exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\exp\left\{i\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\}, \quad (2.23)$$

where V_0 is a constant.

The dispersion relation (2.19) is identical to that in the traditional system, and setting $\gamma = 0$ exactly recovers the solution from the traditional system.

In the meridional velocity, the effect of the γ terms is to increase the width of the equatorial trapping by $O(\gamma/N)$ and to change the curves of constant phase (at fixed $kx - \omega t$) from lines of constant z to the parabolas $z - (1/2)\lambda^{-1}y^2 = \text{const}$. In the limit as N becomes small, these curves approach the curves of constant planetary angular momentum $M^{(p)} \equiv -(1/2)\beta y^2 + \gamma z$, becoming everywhere parallel to the rotation vector $\boldsymbol{\Omega}_0$.

The zonal velocity is altered in a similar way, with the additional effect of a modification of $O(\gamma/N)$ to its spatial structure, including a phase shift upsetting the phase quadrature between u and v. The phase shift is in the opposite sense for upward propagating ($N/\nu c_K = 1$) and downward propagating ($N/\nu c_K = -1$) waves.

Using the boundary condition $w = 0$ on the flat bottom surface, the vertical velocity can be calculated from (2.1e):

$$w(x, y, z, t) = V_0\left(\frac{c_K}{N}\right)\left(\frac{\omega^2}{\omega^2 - k^2c_K^2}\right)\left\{\left(\frac{N}{\nu c_K}\right)\left[\sqrt{\frac{\alpha\beta}{c_K}}H_n'\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right) - \left(\frac{\omega + kc_K}{\omega}\right)\frac{\alpha\beta}{c_K}yH_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\right]\exp\left(i\frac{\pi}{2}\right)\right.$$
$$\left. - \frac{\gamma}{N}\left[\frac{kc_K}{\omega}\sqrt{\frac{\alpha\beta}{c_K}}H_n'\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right) - \left(\frac{\omega + kc_K}{\omega}\right)\frac{\alpha\beta}{c_K}yH_n\left(\sqrt{\frac{\alpha\beta}{c_K}}y\right)\right]\right\}$$
$$\times \exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\left(\exp\left\{i\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\} - \exp\left\{i\left[kx - \nu\left(\frac{\lambda^{-1}}{2}y^2\right) - \omega t\right]\right\}\right). \quad (2.24)$$

The changes to w are similar to those in u, primarily a phase change of $O(\gamma/N)$.

Except for the mixed Rossby–gravity mode ($n = 0$), there is a change in the meridional shape of the u and w components that depends on the nondimensional phase speed $\omega/(kc_K)$. For long inertia–gravity waves ($k \to 0$, $\omega \to \sqrt{\beta c_K}$), the nontraditional term proportional to $(\lambda/N)H_n'$ is amplified in u and reduced in w compared to the corresponding traditional terms. The opposite is true for short Rossby waves ($k \to -\infty$, $\omega \to 0$). For short inertia–gravity waves, ($\omega \to kc_K$), there is very little structural change. The phase and structural changes in u and w are a consequence of the coupling of u and w in (2.1a), (2.1c), and (2.1d) due to the inclusion of the nontraditional terms.

c. Example: Mixed Rossby–gravity wave

To illustrate the changes, we look at the simplest case, the mixed Rossby–gravity wave ($n = 0$). The dispersion relation (2.19) becomes $[\omega(\omega - kc_K) - \beta c_K](\omega + kc_K) = 0$. The root $\omega = -kc_K$ is excluded because it along with (2.13) implies a solution for pressure that diverges as $y \to \pm\infty$, as pointed out for the traditional case by Matsuno (1966). Hence, for equatorially confined mixed Rossby–gravity waves, $\omega(\omega - kc_K) - \beta c_K = 0$.

The vertically propagating solution can be written

$$v_0(x, y, z, t) = V_0 \exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\cos\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t\right], \quad (2.25a)$$

$$u_0(x, y, z, t) = -V_0\frac{\sqrt{\alpha}\omega}{c_K}y\exp\left(-\frac{\alpha\beta}{2c_K}y^2\right)\sin\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2}y^2\right) - \omega t + \left(\frac{N}{\nu c_K}\right)\tan^{-1}\frac{\gamma}{N}\right], \quad (2.25b)$$

$$w_0(x, y, z, t) = \left(\frac{N}{\nu c_K}\right) V_0 \frac{\sqrt{\alpha}\omega^2}{N c_K} y \exp\left(-\frac{\alpha\beta}{2c_K} y^2\right) \left\{ \sin\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2} y^2\right) - \omega t - \left(\frac{N}{\nu c_K}\right) \tan^{-1}\frac{\gamma}{N}\right] - \sin\left[kx - \nu \frac{\lambda^{-1}}{2} y^2 - \omega t - \left(\frac{N}{\nu c_K}\right) \tan^{-1}\frac{\gamma}{N}\right]\right\}, \quad (2.25c)$$

and

$$p_0(x, y, z, t) = -V_0 \omega y \exp\left(-\frac{\alpha\beta}{2c_K} y^2\right) \sin\left[kx + \nu\left(z - \frac{\lambda^{-1}}{2} y^2\right) - \omega t\right]. \quad (2.25d)$$

In (2.25b) and (2.25c) the identity

$$\cos\theta + \delta \sin\theta = \sqrt{1 + \delta^2} \cos(\theta - \tan^{-1}\delta) \quad (2.26)$$

has been used. Owing to the nontraditional terms, the latitudinal confinement width of the solutions is larger and u and w have phases shifted by $O(\gamma/N)$ (for $\gamma \ll N$). The maximum amplitudes of u and w are apparently reduced by a factor $\sqrt{\alpha}$ but, in fact, are the same as in the traditional system because the maxima occur at larger y.

Contours of horizontal divergence $-\partial w_0/\partial z$ for a vertically standing mixed Rossby–gravity wave are plotted in Fig. 3 using the same vertical wavelength and stratification as for Fig. 1. For comparison, the nontraditional solution is plotted in the Southern Hemisphere and the corresponding traditional solution in the Northern Hemisphere. The meridional velocity component of the mixed Rossby–gravity wave resembles the zonal velocity component of the Kelvin wave (Fig. 1) but is $\pi/2$ out of phase in the vertical.

3. Summary and discussion

Meridionally confined zonally propagating wave solutions to the linear hydrostatic Boussinesq equations on a generalized equatorial β plane that includes the ‘‘nontraditional’’ Coriolis force terms associated with the poleward component of planetary rotation have been calculated. Kelvin, Rossby, inertia–gravity, and mixed Rossby–gravity modes are generalized from the traditional case.

That the dispersion relation is identical to that in the traditional system is consistent with earlier results for midlatitude inertia–gravity waves for which the nontraditional terms only affect the frequencies of *meridionally* propagating modes, whereas here the waves do not propagate in latitude. It is also consistent with the correlation between the OLR variability observed by Wheeler and Kiladis (1999) and the dispersion diagram of equatorially confined waves.

The main effect of the nontraditional terms is the curving upward with latitude of the surfaces of constant phase and the increase in the equatorial trapping width of all solutions by $O(\Omega/N)^2$ compared to the traditional case. In addition, for the Rossby, inertia–gravity, and mixed Rossby–gravity modes, the structures of the zonal and vertical velocity components are altered. For the mixed Rossby–gravity mode ($n = 0$) the change is a simple phase shift in the traditional case, the zonal and vertical velocity components are in phase quadrature with the meridional component, and here there is an additional phase difference $O(\Omega/N)$. For long inertia–gravity and short Rossby waves, the meridional structure of the phase-shifted parts of u and w is changed depending on the nondimensional phase speed ω/kc_K.

The differences between the traditional and nontraditional systems with respect to small oscillations about a state of rest depend on the relative strengths of the two restoring forces involved, namely buoyancy, measured by N, and the centrifugal force due to the solid-body rotation of the atmosphere at rest in the absolute reference frame, measured by Ω. For $N \gg \Omega$ ($\alpha \to 1$), buoyancy dominates: the lines of constant phase are almost orthogonal to gravity, and the nontraditional terms do not have much effect. For $N \ll \Omega$ ($\alpha \to 0$), the centrifugal force dominates and lines of constant phase become almost tangent to the rotation vector. Viewed in the original spherical geometry of which this system is an approximation, the (here parabolic) phase lines tend toward lines of constant distance from the axis of rotation, perpendicular to the centrifugal force. Figure 4 illustrates how in both limits the phase lines are perpendicular to the dominant restoring force. Harlander and Maas (2007) study the $N \ll \Omega$ limit in great detail, concentrating on zonally symmetric solutions.

That the deformation radius is larger in the nontraditional system, indicating that rotational effects become important at larger meridional scales than they do in the traditional system, is perhaps surprising given that there are additional force terms associated with rotation. However, since the centrifugal force acts parallel to gravity at the equator, the total vertical restoring force is

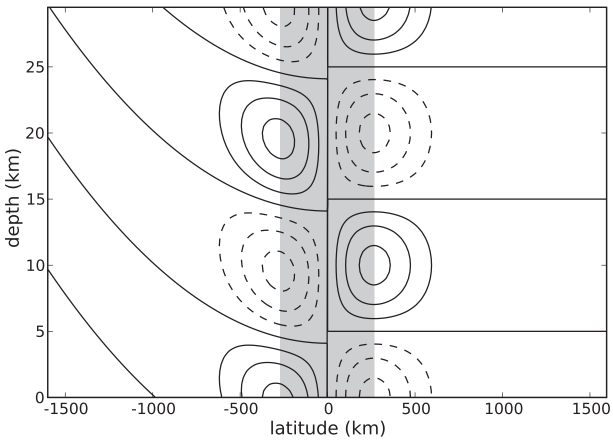

FIG. 3. Contours of horizontal divergence amplitude for the standing mixed Rossby–gravity wave. The nontraditional solution is plotted in the Southern Hemisphere, and the corresponding traditional solution in the Northern Hemisphere. The shaded region is within one deformation radius of the equator.

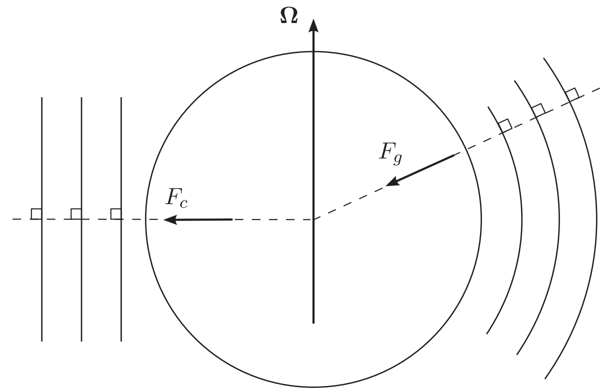

FIG. 4. Schematic of lines of constant phase in spherical geometry for the two limits (left) $N \ll \Omega$ and (right) $N \gg \Omega$. In both cases, the phase lines are perpendicular to the direction of the dominant restoring force—respectively, the centrifugal force and gravity.

greater than in the traditional system, while the total meridional restoring force is unchanged.

Finally, while for typical atmospheric and oceanic scales and average values of N, the effect of the nontraditional terms on these waves is small—Ω/N is about 1% in the atmosphere and 10% in the ocean—the cumulative effect of the nontraditional terms on complicated wave processes may not be. Recent numerical simulations by the author and colleagues have shown that the destabilization of zonally short mixed Rossby–gravity waves leads to final states with different structure when the nontraditional terms are and are not included, in particular in terms of the vertical symmetry of the final state (Fruman et al. 2009). Future work might focus on the vertical flux of zonal angular momentum associated with equatorial waves in the nontraditional system and their interaction with the mean flow.

It would be useful if the result could be extended to the case of a nonuniform stratification $N(z)$. This, unfortunately, is not trivial as it is in the traditional system because, when the Coriolis force is not in the plane perpendicular to the direction of stratification, solutions do not have a common z dependence for u, v, and p that naturally separates. Also, it is not clear what effect a rigid upper surface, not locally perpendicular to $\mathbf{\Omega}$, would have. Perhaps it would lead to new oscillation modes, as was found by Kasahara (2003) to be the case on a nontraditional f plane.

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