On the Use of Potential Vorticity Tendency Equations for Diagnosing Atmospheric Dynamics in Numerical Models

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ABSTRACT

This study critically assesses potential vorticity (PV) tendency equations used for analyzing atmospheric convective systems. A generic PV tendency format is presented to provide a framework for comparing PV tendency equations, which isolates the contributions to PV tendency from wind and mass field changes. These changes are separated into forcing terms (e.g., diabatic or friction) and flow adjustment and evolution terms (i.e., adiabatic motions).

One PV tendency formulation analyzed separates PV tendency into terms representing PV advection and diabatic and frictional PV sources. In this form the PV advection is shown to exhibit large cancellation with the diabatic forcing term when used to analyze deep convective systems, which compromises the dynamical insight that the PV tendency analysis should provide. The isentropic PV substance tendency formulation of Haynes and McIntyre does not suffer from this cancellation problem. However, while the Haynes and McIntyre formulation may be appropriate for many convective system applications, there are likely to be some applications in which the formulation is difficult to apply or is not ideal.

This study introduces a family of PV tendency equations in geometric coordinates that is free from the deficiencies of the above formulations. Simpler forms are complemented by more complex forms that expand the vorticity tendency term to offer additional insight into flow dynamics. The more complex forms provide insight similar to the influential Haynes and McIntyre isentropic formulation.

1. Introduction

Potential vorticity (PV) “thinking” (e.g., Hoskins et al. 1985, hereafter HMR85) describes a framework for understanding the relationship between the evolving mass and wind fields in the atmosphere. For a given background state (or boundary condition) and a balance condition, the mass and wind fields can be determined from the PV distribution. In this framework, changes in the mass and wind can be inferred from local and remote changes in PV distribution, thereby providing simple, valuable conceptual tools for understanding many atmospheric phenomena. However, the assumption of balance introduced by the balance condition means that the diagnosed flow will be most accurate for systems that evolve approximately in that state of balance. This includes synoptic scales and above (HMR85) and some mesoscale phenomena [e.g., tropical cyclones (TCs)].

Because PV is conserved in adiabatic flow and not conserved where there is friction or diabatic heating or cooling, diabatic and frictional processes can be viewed as isolated PV sources or sinks, and PV tendency equations can be constructed to measure the contribution to the PV distribution from these isolated sources or sinks. Thus, in theory, the change in mass and wind fields associated with
the PV sources or sinks can be determined from a PV inversion. In practice this may only be true for processes in which the scale of the diabatic and frictional “forcing” is large compared to the Rossby radius (the adjustment length scale) because, as we discuss in this paper, smaller-scale forcing can induce circulations that advect the PV far from the source location.

Diabatic and frictional forcing respectively act directly on the mass and wind fields alone. Such one-sided changes will disrupt any existing balance. However, the atmospheric response to such forcing will often restore balance (i.e., an adjustment process will occur). In the adjustment process some fraction of the forcing energy is radiated away as inertia–gravity waves and some fraction is responsible for the acceleration of a transverse (secondary) circulation. Together these processes adjust the mass and wind fields of the primary circulation toward a new state of balance. While inertia–gravity waves cannot transport PV, the transverse circulation can. Thus, for flows in which the transverse circulation is spatially large compared to the PV source or sink region (e.g., the in–up–out circulation from diabatic heating in a thunderstorm), PV advection away from the source or toward the sink could be significant. In this paper we demonstrate in a convective burst example that the diabatic PV sink term can be large and mostly opposed by the adiabatic PV advection term, with only a small residual net change in PV. While the residual between these two terms describes the nonfrictional change in PV, which in theory could be inverted to diagnose the mass and wind field, we argue that the potential for significant truncation error is high. This adiabatic/diabatic cancellation problem is described in section 2.

An alternative PV tendency formulation proposed by Haynes and McIntyre (1987, hereafter HM87) does not suffer from the adiabatic/diabatic cancellation problem because, as we show in section 2, the vertical components of the adiabatic and diabatic terms exactly cancel in this formulation, and the remaining horizontal components offer useful insight into the changing primary circulation without the need for a PV inversion. Thus, the HM87 formulation is ideal for diagnosing mesoscale and smaller convective systems. HM87’s groundbreaking study demonstrated that a PV quantity is conserved between isentropic layers even when diabatic and friction processes are active, provided the surfaces do not intersect a boundary. This framework introduced the concept of pseudohorizontal conservation, and diabatic and friction processes could no longer be regarded as isolated sources or sinks; instead, they induced horizontal neighboring sources and sinks that sum to zero when integrated over the source/sink region. Mathematically the source/sink is represented as an isentrope-parallel flux that redistributes the PV quantity between isentropic layers from sink to source (although there is no physical transfer from sink to source).

The flux formulation of HM87 requires a volume conserved tendency quantity, which is the product of PV (conserved per unit mass) and density, termed PV substance (PVS; Haynes and McIntyre 1990). In isentropic coordinates, the isentropic PVS is simply the vertical component of isentropic absolute vorticity $\eta_\nu$, which reduces the more complex concept of PV tendency to a simpler and arguably more intuitive vorticity tendency.

While HM87’s isentropic PV tendency formulation enables powerful, physical insights, it is subject to a number of potential limitations: (i) PVS is an approximation to PV; (ii) the isentropic PVS tendency equations provide information on wind field changes but not mass field changes that contribute to PV tendency (demonstrated herein); (iii) the hydrostatic approximation is necessarily applied when calculating the tendency terms in isentropic coordinates; and (iv) the physical insight gained from the concept of pseudohorizontal PV conservation could be lessened in flows in which isentropes are far from horizontal (e.g., frontal systems). While these limitations may not cause problems for many atmospheric applications, we consider whether an alternative set of PV tendency equations can be constructed free from these limitations and free from the adiabatic/diabatic cancellation problem.

This paper provides a theoretical assessment of PV tendency equations applicable to a wide range of atmospheric phenomena. To compare various PV tendency formulations we propose a conceptual model for understanding PV tendency in which diabatic and frictional forcing disrupts some form of atmospheric balance that induces an adiabatic response (adjustment) that returns the atmosphere to a new state of balance. While in this conceptual picture the forcing precedes adjustment, we note that it is common for the forcing to be sustained such that forcing and adjustment overlap in time. In section 2a we derive a commonly used form of PV tendency equation, classify the terms with respect to wind and mass contributions, and separate these contributions into adiabatic/frictionless and diabatic or frictional components. In section 2b another dynamically insightful PV tendency equation is introduced along with the isentropic HM87 formulation, and these insights are discussed in section 2c. In section 2d we demonstrate the adiabatic/diabatic

\footnote{While the application of the HM87 PV tendency in isentropic coordinates necessitates the assumption of hydrostatic balance, HM87 demonstrated that the concept of PV substance conservation on isentropic surfaces was not limited to hydrostatic flows.}
cancellation problem of the source/sink PV tendency formulation when applied to deep convective systems. In section 3 a class of PV tendency equations in geometric coordinates is introduced that retains the insight of the influential isentropic HM87 formulation, while free from the adiabatic/diabatic cancellation problem and free from the PV approximations inherent in the HM87 formulation. Included in this new class of equations is a PVS tendency equation in flux form that represents a geometric coordinate equivalent equation to the isentropic PVS tendency equation of HM87. In section 4 the pros and cons of each tendency equation are discussed, and we argue that the new class of PV tendency equations should offer improved physical insight into flow dynamics when used to diagnose numerical model data.

2. PV tendency

a. PV and PVS tendency derivation

To compare and critique PV tendency formulations, it is useful to derive the tendency equations from first principles. Ertel’s PV can be expressed as

\[ \frac{\partial Q}{\partial t} = \frac{1}{\rho} \left( \nabla \cdot (\nabla \times \mathbf{u}) + \frac{f}{\rho} (\mathbf{u} \times \nabla) \right) - \frac{Q}{\rho} \frac{\partial \rho}{\partial t} \]

where \( \rho \) and \( \theta \) are the air density and potential temperature, respectively, \( \mathbf{V} = \mathbf{v} \times \mathbf{u} + f \mathbf{k} = (\omega_1, \omega_2, \eta)^2 \) is the three-dimensional absolute vorticity vector, \( \mathbf{V} \) is the three-dimensional gradient operator, \( \mathbf{V} \times \) is the curl operator, \( f \) is the Coriolis parameter, and \( \mathbf{k} \) is the unit vertical vector. The PV substance mentioned in the introduction behaves in a manner similar to a chemical tracer in a fluid (HM87; Haynes and McIntyre 1990). It represents “the amount of PV per unit volume of physical space” (HM87); that is, PVS is

\[ \rho \mathbf{Q} = \mathbf{V} \cdot \mathbf{\theta} = \mathbf{V} \cdot (\nabla \theta). \]

Note that because a unit volume differs between vertical coordinate systems, PVS is coordinate system dependent.

We consider PV and PVS to be the dot product of a wind parameter (\( \eta \)) and a mass parameter (\( \nabla \theta / \rho \) for PV and \( \mathbf{V} \theta / \rho \) for PVS). The chain rule can be used to separate the wind and mass tendency contributions to the PV or PVS tendency and arrive at equations that separate the tendency into wind change (WC) and mass change (MC) terms. Thus, the PV and PVS tendency in geometric coordinates can be expressed respectively as

\[ \frac{\partial \mathbf{Q}}{\partial t} = \frac{1}{\rho} \left( \mathbf{V} \theta \cdot \mathbf{V} \mathbf{F} \right) - \mathbf{u} \cdot \mathbf{V} \mathbf{Q} + \frac{1}{\rho} (\nabla \mathbf{Q}) \mathbf{M} \]

The concept of wind change and mass change terms is perhaps easiest to illustrate with Eq. (2b) or (2c), where the WC (MC) term represents the tendency in PVS due to a tendency in the wind (mass) field. In Eq. (2a) the mass change term has been expanded to separate the density and \( \mathbf{V} \theta \) tendencies. (For comparison purposes, tendency equations for PV and PVS, and PVS in flux form, are grouped together and labeled with \( a, b, \) and \( c \) suffixes, respectively whenever they appear in this manuscript. The flux formulation can be useful for performing PVS budget studies; e.g., HMR85; HM87.)

Equations (2a)–(2c) could in theory be used as they are to provide PV and PVS tendency information from the direct impact of changes in \( \theta, \eta, \) and \( \rho \). Replacing the time derivatives on the RHS of Eqs. (2a)–(2c) using their tendency equations can provide additional physically intuitive information as to the processes responsible for PV and PVS tendency. However, as we demonstrate later, care needs to be taken to ensure the diagnostic form is used with due attention to its advantages and disadvantages.

Substituting the following forms of the tendency equations

\[ \frac{\partial \mathbf{Q}}{\partial t} = -\mathbf{V} \cdot (\nabla \mathbf{\theta}) \]

into Eqs. (2a)–(2c), we get the familiar PV tendency form (e.g., HMR85) and the nonflux and flux forms of PVS tendency (e.g., HMR85; HM87):

\[ \frac{\partial \mathbf{Q}}{\partial t} = \frac{1}{\rho} \left( \mathbf{V} \theta \cdot \mathbf{V} \mathbf{F} \right) - \mathbf{u} \cdot \mathbf{V} \mathbf{Q} + \frac{1}{\rho} (\nabla \mathbf{Q}) \mathbf{M} \]

The Greek letter \( \omega \) is used exclusively to represent the horizontal vorticity components in this paper and should not be confused with the vertical pressure velocity.
\[
\frac{\partial \rho Q}{\partial t} = \mathbf{V} \cdot (\mathbf{F}_{WF} - \mathbf{u} \cdot \nabla Q) - \rho Q (\mathbf{V} \cdot \mathbf{u})_{\text{CAE}}
\]
\[+ \eta \cdot \nabla \bar{\theta}_{\text{MF}}. \tag{6b} \]
\[
\frac{\partial \rho \theta}{\partial t} = -\nabla \cdot (-\mathbf{F} \times \mathbf{V} + \rho \bar{Q} \mathbf{u}_{\text{CAE}} - \eta \bar{\theta}_{\text{MF}}). \tag{6c} \]

Here \( \mathbf{F} = (F, G, 0) \) represents the frictional force per unit mass exerted on the fluid and \( P \) is pressure. Also, note that the baroclinic term in Eq. (4) (last term on the RHS) vanishes when dotted with \( \mathbf{V} \bar{\theta} \). We label Eqs. (6a)–(6c) “simple diagnostic” PV tendency formulations because they incorporate the diagnostic forms of Eqs. (3)–(5). As noted in the introduction we consider the wind and mass forcing terms (WF and MF subscripts, respectively) to interrupt some form of balance, which results in mass and wind field adjustment. The combined adjustment and evolution (CAE) terms represent the combined wind and mass adjustments in addition to any mass and wind field evolution independent of forcing. Later we separate the CAE terms into wind and mass contributions (WAE and MAE, respectively). These forcing, flow adjustment, and flow evolution concepts are used below to compare tendency formulations.

Equation (6a) is arguably the simplest PV tendency form. It represents the material rate of change of the adiabatically conserved quantity \( Q \) plus the diabatic and frictional source/sink terms. The CAE term in Eq. (6a) appears as PV advection, and the WF and MF terms are PV production/dissipation from friction and diabatic heating/cooling, respectively. Equation (6b) describes the evolution of PVS in nonflux form wherein the CAE term is split into a PVS advection and a three-dimensional divergence term. Finally, Eq. (6c) [Eq. (4.3) of HM87] describes the evolution of PV substance in flux form. The CAE term there represents a three-dimensional advective flux, and the WF and MF terms have been labeled by HM87 as nonadvective fluxes.

### b. Two influential examples of PV and PVS tendency in the literature

Most PV tendency equations in the literature appear to be variants of Eq. (6a), and many are approximated to suit the fluid under investigation. Here we present equations that provide useful dynamical insight into moist convective flows, one an exact PV tendency equation (Raymond and Jiang 1990) that differs from Eq. (6) and the other the HM87 isotropic PVS tendency equation. The former is similar to Eq. (6a) except that the PV advection is included in the material derivative and the diabatic and frictional forcing terms are expressed in flux form:

\[
\frac{DQ}{Dt} = \frac{1}{\rho} \nabla \cdot (\bar{\theta} \bar{\theta}_{\text{MF}} + \theta \mathbf{V} \times \mathbf{F}_{WF}). \tag{7} \]

[We note that the divergence of \( \theta \mathbf{V} \times \mathbf{F} \) is equivalent to the divergence of \( \mathbf{F} \times \mathbf{V} \bar{\theta} \), the frictional flux term of Eq. (6c).] The MF and WF terms are, in the language of HM87, nonadvective fluxes. The flux formulation is particularly useful for PV budget studies and the conceptual ideas they inspire because the volume-integrated PV tendency reduces to a surface integral of the flux terms normal to the volume being investigated (Stokes’ divergence theorem). However, because the density term in Eq. (7) is not part of the flux quantities, it must be considered independently and may need to be approximated to represent a typical or mean density within the volume.

The PVS tendency equation in isentropic coordinates of HM87 is a simplified form of Eq. (6c). In isentropic coordinates \( d\bar{\theta}/dt \) is zero, \( \mathbf{V} \bar{\theta} = (0, 0, 1) \), and \( w, \bar{z}, \) and \( \rho \) are replaced by \( \bar{\theta}, d\bar{\theta}/dt, \) and \( \sigma \) (the isentropic density), respectively. As mentioned in the introduction, \( \sigma Q \) reduces to \( \bar{\theta} \rho \) because the horizontal components of \( \mathbf{V} \bar{\theta} \) are zero on \( \theta \) surfaces, and the vertical component is unity. Thus, in isentropic coordinates Eq. (6c) becomes

\[
\frac{\partial \bar{\theta}}{\partial t} = -\nabla \cdot \left[ \bar{\theta} (\mathbf{u} \cdot \mathbf{V} \bar{\theta})_{\text{CAE}} - d\bar{\theta}/dt \right]_{\text{MF}}
\]

\[+ \nabla \cdot \left[ -\mathbf{F} \times (G, F, 0)_{\text{WF}} \right]. \tag{8} \]

Equation (8) shows that in the isentropic formulation the vertical component of the CAE and MF tendency terms exactly cancel, yielding only horizontal vectors. Thus the isentropic PVS tendency can be described purely in terms of horizontal fluxes (parallel to isentropes), which led to the HM87 conclusions regarding PV conservation between isentropic layers or on isentropic surfaces. Note that in the isentropic coordinate system there is ambiguity as to what constitutes wind and mass forcing, adjustment, and evolution because in isentropic coordinates the diabatic heating \( \bar{\theta} \) is vertical velocity. After cancelling the vertical components of CAE and MF it is perhaps most useful to consider the remaining components to make up the WAE term (see also the discussion in section 2e).

Although the horizontal components of vorticity \( \omega_1 \) and \( \omega_2 \) do not contribute to the isentropic PVS, Eq. (8) shows that these terms are included in the PVS tendency. The horizontal component of the nonadvective-tilting flux in Eq. (8) (the MF term) is equivalent to HM87’s Eq. (2.6) because of the partial cancellation between the two horizontal vorticity components when the divergence operator is applied. Thus, Eq. (8) reduces to
\[
\frac{\partial \sigma Q}{\partial t} = -\nabla \cdot \left\{ \sigma Q(u, v, 0) + \dot{\theta} \left( \frac{\partial v}{\partial \theta} - \frac{\partial u}{\partial \theta} \right) \right\}_{\text{WAE}}
+ \left( -G, F, 0 \right)_{\text{WF}}. \tag{9}
\]

HM87 described the first term as the advective flux and the remaining terms as nonadvective fluxes. Hereafter, we add the labels “tilting” and “frictional” respectively to distinguish between the two nonadvective fluxes. For completeness and a discussion on tilting and vertical advection in section 3 we include the nonflux form of Eq. (9) by expanding the divergence operator:

\[
\frac{\partial \sigma Q}{\partial t} = -u_h \cdot \nabla (\sigma Q) - \sigma Q(\nabla_h \cdot u_h) - \frac{\partial}{\partial t} \frac{\partial \sigma Q}{\partial \theta} - \left( \frac{\partial \theta}{\partial x} \frac{\partial u}{\partial \theta} - \frac{\partial \theta}{\partial y} \frac{\partial u}{\partial \theta} \right) + \left( \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right). \tag{10}
\]

Here the terms on the RHS resemble the familiar horizontal advection, stretching (or convergence), vertical advection, tilting, and friction tendencies. We reiterate HM87’s warning against performing PV tendency budget studies of such quantities because of the inherent cancellation between the pairs of terms that emerge from the application of the chain rule. The physical significance of the vertical advection and tilting-like terms in relation to the purely horizontal nonadvective-tilting flux term of Eq. (9) is discussed next.

c. Dynamical insight gained from the above formulations

The PV tendency formulations of Eqs. (7) and (9) provide important physical insight into flow dynamics, which has helped to shape our conceptual understanding of atmospheric dynamics, and in particular the understanding of PV and vorticity dynamics in convective systems. The notion of PV fluxing downward in convective systems (e.g., Ritchie and Holland 1997) was no doubt inspired by work such as Raymond (1992), who used an equation similar to Eq. (7) to show that differential diabatic heating along the absolute vorticity vector introduces a PV source and sink couplet that represents a nonadvective PV flux down the absolute vorticity vector. While such equations provide a qualitative explanation for diabatically generated PV anomalies observed in convective systems, they provide little insight into the changing primary circulation and PV structure during and after the fluid adjusts to the PV source or sink. Indeed, the vertical component of the secondary circulation (induced by diabatically generated buoyancy gradients) may advect away [Eq. (6a)] the diabatically generated PV anomalies. In this case, more information is required to explain quantitatively the positive (negative) cyclonic PV anomalies observed in the lower (upper) regions of convective clouds.

The HM87 conclusion that PVS is conserved between isentropic surfaces suggests that PV tendency should be considered in terms of near-horizontal PV conservation, which broadens the conceptual model of PV tendency in convective systems to include the surrounding fluid. More specifically, the advective flux term demonstrates that diabatically induced convergent and divergent flow respectively “concentrate” and “dilate” PVS (HM87), and the nonadvective-tilting flux demonstrates the generation of PVS sources and sinks through a tilting-like process in diabatic heating regions [see, e.g., the appendix of Tory et al. (2006)]. These tendencies give rise to a conceptual model of PV converging into the base of a convective updraft and diverging outwards above, with the nonadvective processes also contributing to a near-horizontal PV redistribution through apparent sources and sinks.

The HM87 result revolutionized PV and vorticity thinking and helped identify a common misconception related to vertical transport of PV and \( \eta \) in convective or stratiform processes. We demonstrate now that while it is mathematically correct to consider vertical advection of these quantities, such advection should always be considered in conjunction with tilting because the combined effect yields zero net change of PVS on isentropic surfaces or between isentropic layers.

The isentropic PVS advective and nonadvective tilting tendencies are illustrated schematically in Fig. 1 for a Northern Hemisphere cyclonic circulation. To decompose the WAE tendency of Eq. (9) into the advective and nonadvective tilting tendencies we consider an instantaneous application of diabatic heating, followed by an adiabatic response that returns the system toward a new state of balance. In the isentropic coordinate system the isentropes in Fig. 1 are fixed horizontal lines. The initial absolute vorticity varies with height, indicated by the black vectors on vortex lines\(^4\) in Fig. 1a. The associated horizontal flow is depicted by crosses and circles, representing flow into and out of the page, respectively.

The diabatic heating is applied to the shaded region, which in isentropic space is analogous to upward flow (open arrows, Fig. 1b) although there is no actual upward flow in height coordinates yet. On the isentropic surface between the two layers the magnitude of the

\(^3\) HM87 also showed that the vertical component of absolute vorticity is similarly conserved on pressure surfaces.

\(^4\) Like streamlines do for the wind field, vortex lines point everywhere in the direction of the vorticity vector. Because vorticity is nondivergent the magnitude of the vorticity is represented by the separation distance between the vortex lines.
vorticity vectors increases compared to Fig. 1a because of the upward movement of the stronger vorticity relative to the isentropes (analogous to vertical advection of vorticity). However, the horizontal gradient of the diabatic heating (e.g., of the isentropic vertical velocity) tilts the outer two vorticity vectors downward relative to the central isentrope, which reduces the magnitude of the vertical vorticity (i.e., reduces $\mathbf{h}_v$). In this way the apparent vertical advection contributes to a positive tendency on the central isentrope, and the apparent tilting contributes to a negative tendency. [See also the description in the appendix of Tory et al. (2006).]

The combined tilting- and vertical advection-like terms of Eq. (10) (described above) are equivalent to the divergence of the nonadvective tilting flux of Eq. (9). This flux tendency is simple to visualize if Fig. 1 is considered to depict the $x$–$\theta$ plane. The nonadvective tilting flux then reduces to $-\partial(-\mathbf{h}_v)/\partial x$. Assuming that the diabatic heating increases smoothly from zero at the edges of the shaded region in Fig. 1b to a maximum at the center in Fig. 1a, a possible distribution of $-\mathbf{h}_v$ across the diabatic heating region is represented by the curved line in Fig. 1c. (Remember that $\mathbf{h}_v$ is negative to the left, positive to the right, and zero at the center of the image.) It follows that the nonadvective tilting tendency is equivalent to $-\partial(-\mathbf{h}_v)/\partial x$, which is positive between the two dashed lines and negative on either side (consistent with the vertical advection and tilting description above). An interesting implication of this result, which offers additional physical insight, becomes apparent when considering the total area-integrated nonadvective tilting PVS tendency on the central isentrope, [i.e., the intersection of the isentropic surface (horizontal plane) with the shaded region in Fig. 1c]. Using Stokes’ divergence theorem and Eq. (9), the area integral reduces to a line integral of the nonadvective tilting flux around the boundary of the diabatic heating region (on the isentropic surface), which by definition is zero if the boundary is defined by $\theta = 0$. Thus, the total nonadvective tilting tendency (or combined tilting and vertical advection tendency) is zero integrated across the diabatic heating region, despite the locally positive and negative tendencies within. The above argument can be extended to show the volume-integrated tendencies are zero through isentropic layers.

The advective flux tendency becomes apparent in the advective response phase. In geometric space the response is an in–up–out secondary circulation, which in isentropic space reduces to an inflow in the lower layer and an outflow in the upper layer (Fig. 1d) because by
definition there can be no adiabatic flow across an isentrope. The associated convergence and divergence respectively contribute to the PVS concentration and dilution described by HM87. Recognizing PVS as the vertical component of the isentropic absolute vorticity, it is apparent that the converging (diverging) flow brings the vortex lines into closer (farther) proximity, which represents increasing (decreasing) vorticity magnitude.

The above arguments are equally valid for $h$ tendencies on isobaric surfaces (HM87). With the isentropes of Fig. 1 replaced with isobars, and the open arrows in Fig. 1b representing the pressure coordinate vertical velocity, the $\eta$ tendency is illustrated. [See also Tory and Montgomery (2008), Davis and Galarneau (2009), and Tory and Frank (2010) for schematic representations of $\eta$ tendency.]

The above insight into flow dynamics is a significant improvement on the qualitative explanation of diabatic PV sources and sinks available from Eq. (7), and the insight carries over to PV dynamics in height coordinates for the many atmospheric systems in which local density changes are relatively small and isentropic surfaces remain close to horizontal. For fluids in which the isentropic framework is problematic (e.g., cold fronts, in which isentropes deviate substantially from horizontal and intersect the surface), a geometric coordinate equivalent equation should offer improved accuracy and retain the insight of HM87. A family of such equations is proposed in section 3.

d. The adiabatic/diabatic cancellation problem

As mentioned in the introduction, some PV tendency formulations [e.g., Eqs. (6) and (7)] can exhibit large cancellation between the adiabatic and diabatic tendency terms (e.g., Lackmann 2002). This includes the PVS tendency formulation of HM87 posed in geometric coordinates [their Eq. (4.3), our Eq. (6c)],\(^5\) or for that matter any PV or PVS tendency formulation in which vertical advection or the vertical flux of PV or PVS can transport the diabatic PV or PVS source or sink away from the source or sink region.

The adiabatic/diabatic (a/d) cancellation is illustrated schematically in Fig. 2, which depicts a vertical cross section of a stably stratified fluid in geometric coordinates. The thick lines represent isentropes that bound two isentropic layers. A horizontal circulation, indicated by crosses and dots that represent flow into and out of the page, is in balance with a pressure minimum that is reflected in the upper sloping isentrope. In this scenario the vertical components of the $V\theta$ and $\eta$ vectors dominate and, assuming density changes are small, PV is approximated by

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\(^5\) Equation (6c) includes a vertical advective PVS flux $\rho \overline{Q} v$ in the CAE term, and a potentially opposing vertical nonadvective diabatic PVS flux $-\eta \theta$ in the MF term.
where \( \rho_0 \) is a reference density. Figure 2a represents an initial state in which the relative vorticity and hence \( \eta \) is constant with height. The instantaneous application of diabatic heating to the center region results in a significant dipping of the middle isentrope (shaded, Fig. 2b). The associated increase and decrease of \( \partial \theta / \partial z \) in the layers below and above respectively describe increasing and decreasing PV magnitudes [e.g., see Eq. (11)] and represent the MF tendencies of Eqs. (6) and (7). In response to the diabatic heating an in–up–out secondary circulation develops (i.e., the adiabatic response), which has been split into vertical and horizontal components in Figs. 2c and 2d (open arrows) in order to isolate the mass (MAE) and wind (WAE) contributions to CAE. The vertical flow returns the middle isentrope toward its original position (Fig. 2c, shaded) and the horizontal flow enhances and weakens \( \eta \) in the lower and upper layers, respectively, through \( \eta \) convergence and divergence, respectively (Fig. 2d). The \( a/d \) cancellation is thus the difference between MF and MAE, which is represented by the return of the middle isentrope toward its original position (shaded, Fig. 2c). The final position is somewhat lower than the original (dashed line), which reflects the stronger pressure gradient force required to balance the now more intense lower-layer \( \eta \). The final position of the upper isentrope is slightly raised to reflect the weaker pressure gradient required to balance the now weaker upper-layer circulation.

If the diabatic forcing was continuous rather than instantaneous, the processes depicted in Figs. 2b–d would act simultaneously and potentially in a state of near balance, causing the middle isentrope to move gradually downward in balance with the changing \( \eta \) profile. Thus, for continuous diabatic forcing the opposing MF and MAE (or CAE when MAE is combined with WAE) processes may not be physically apparent, yet they remain dominant terms in the PV tendency analyses of Eqs. (6) and (7). Indeed, tropical cyclone formation studies (e.g., Montgomery et al. 2006; Tory et al. 2006) suggest that the majority of diabatic heating is cancelled by adiabatic cooling in updrafts, and interestingly tropical cyclone spinup may typically occur in a near-balanced state. The former point suggests that during TC formation one might expect significant cancellation between the MF and MAE terms, and that it is physically more intuitive to consider the mass change as a whole [i.e., the MC terms of Eq. (2)]. The adoption of MC PV tendencies is explored in section 3.

To illustrate the \( a/d \) cancellation in a real-world case, Fig. 3 shows the PV tendency for a young convective burst from a simulation\(^6\) early in the development of TC Yasi (South Pacific, 2011). The panels show horizontal slices at a height of 3380 m with vertical velocity contoured, and the horizontal wind represented by vectors. PV is shaded in Fig. 3a, the Eq. (6a) CAE and MF terms are shaded in Figs. 3b and 3c respectively, and the total tendency excluding friction (sum of CAE and MF) is shaded in Fig. 3d. (The friction term is the same for all PV tendency formulations and is minimal at this level.) Hourly averaged fields were used to provide ample time to eliminate temporal noise from diabatic forcing. To remove the trivial tendency associated with the convective burst horizontal motion, the mean horizontal wind, calculated over a \( 1^\circ \times 1^\circ \) box centered on the updraft, was subtracted from the horizontal wind before the tendency terms were calculated (i.e., the tendency terms are in a coordinate system moving with that wind). Thus, Fig. 3b is dominated by vertical advection of PV. The maximum cyclonic tendency (negative in the Southern Hemisphere, shaded blue) is about 6 PV units (PVU) per day (the shading scale was chosen for comparison with PV tendencies introduced later in section 3), which is opposed by a maximum anticyclonic tendency (shaded red) of the same magnitude from the diabatic heating term, Fig. 3c. Because of small frictional PV tendency at this level, Fig. 3d effectively represents the total PV tendency, of which the maximum magnitude is about 1 PVU day\(^{-1}\). This maximum total tendency magnitude represents a relatively small residual between the two opposing tendencies of 6 times greater magnitude. The errors associated with calculating these terms, using imperfect model data and numerical methods, could conceivably be of similar magnitude to the residual. This is explored further in section 3 when the total tendency is compared with an alternative PV tendency equation.

At the level shown in Fig. 3 (3380 m) the diabatic heating decreases with height (not shown) and the updraft is decelerating (horizontal divergence), similar to the upper layer of Fig. 2. The strong anticyclonic diabatic tendency (Fig. 3c) is associated with the spreading apart of isentropes as illustrated in Fig. 2b. Because the PV advection term represents the combined mass and wind adjustment and evolution, some cancellation would be expected from the opposing MF and MAE component of CAE (as illustrated in Figs. 2b and 2c). However, this does not necessarily mean that the Eq. (6a) diabatic (MF)

\begin{equation}
Q \approx \frac{1}{\rho_0} \frac{\partial \theta}{\partial z} \eta. \quad (11)
\end{equation}

\(^6\) The forecast is run using the atmospheric component of the Australian Community Climate and Earth Systems Simulator (ACCESS) model (Puri 2005). The atmospheric model is based on the U.K. Meteorological Office Unified Model version 6.4 (Martin et al. 2006).
and advection (CAE) terms should always be of opposite sign, because the sign of the PV advection term is determined purely by the PV gradient and wind direction. The a/d cancellation will not be problematic where the two terms are of the same sign, but the terms still offer limited dynamical insight.

**e. The a/d cancellation and the HM87 isentropic formulation**

The a/d cancellation in the HM87 isentropic formulation is exact because the MAE and MF terms exactly cancel, leaving only the WAE and WF tendencies.
demonstrate we examine the derivation of Eq. (6c) from Eq. (2c) in a little more detail. Remembering that the baroclinic term in Eq. (4) disappears when dotted with \( \mathbf{V} \), the \( \theta (\partial \mathbf{V}/\partial \mathbf{t}) \) (WC) term in Eq. (2c) can be expressed as \( \theta (\mathbf{V} \times \mathbf{C}) \), where from Eq. (4) \( \mathbf{C} = -\mathbf{u} \times \mathbf{F} \). Using \( \mathbf{V} \cdot [\theta (\mathbf{V} \times \mathbf{C})] = \mathbf{V} \cdot (\mathbf{V} \times \mathbf{C}) \), and Eq. (5), Eq. (2c) becomes

\[
\frac{\partial \rho Q}{\partial t} = -\mathbf{V} \cdot [-(\mathbf{F} \times \mathbf{V})_{\text{WF}} + (\mathbf{u} \times \mathbf{V}) \theta_{\text{WAE}} + (\mathbf{u} \cdot \mathbf{V}) \eta_{\text{MAE}} - \hat{\theta} \eta_{\text{MF}}].
\]  

Here the wind change term has been separated into a WF (i.e., friction) term and a wind adjustment and evolution (WAE) term. Similarly, the mass change term has been separated into a mass adjustment and evolution (MAE) term and an MF (i.e., diabatic) term. The sum of the WAE and MAE terms is equivalent to the CAE terms of Eqs. (6) and describes PVS tendency purely from adiabatic, nonfrictional flow.

While not necessary for demonstrating the a/d cancellation, we continue the derivation of Eq. (6c) for completeness. Using the triple cross product vector identity,

\[
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C},
\]

we can show that the sum of the WAE and MAE terms equals the three-dimensional advective PVS flux of Eq. (6c),

\[
(\mathbf{u} \times \mathbf{V}) \theta_{\text{WAE}} + (\mathbf{u} \cdot \mathbf{V}) \eta_{\text{MAE}} = (\mathbf{u} \cdot \mathbf{V}) \mathbf{u} = \rho \mathbf{Q} u_{\text{CAE}},
\]

and thus Eq. (12) reduces to Eq. (6c).

Transforming Eq. (12) to isentropic coordinates [i.e., the transformations described prior to the introduction of Eq. (8)], one can see that

\[
\frac{\partial \rho Q}{\partial t} = -\mathbf{V} \cdot \left[ -(G, F, 0)_{\text{WF}} + \eta (u, v, 0) - \hat{\theta} (\omega_1, \omega_2, 0)_{\text{WAE}} + \hat{\theta} \eta_{\text{MAE}} - \hat{\theta} \eta_{\text{MF}} \right],
\]

which clearly demonstrates the exact cancellation between the MF and MAE terms. Here the WAE term contains the HM87 advective flux and the nonadvective tilting flux.  

To compare the isentropic formulation of HM87 and Eq. (6a), the PVS and PVS tendency terms are presented in Fig. 4 for the same convective burst depicted in Fig. 3. PVS is shaded in Fig. 4a and the first two terms of Eq. (9) are shaded in Figs. 4b and 4c. The total tendency excluding friction is shaded in Fig. 4d. The terms are presented on an isentropic surface close to the 3380-m level (315 K). Diabatic heating is contoured. As in Fig. 3 the horizontal advection contribution was minimized by subtracting the mean horizontal wind (calculated over a 1° × 1° box centered on the updraft) from the horizontal wind field before interpolation to isentropic coordinates and calculation of the tendency terms. For easy comparison with Fig. 3 the shading has been scaled by the mean isentropic density (\( \rho = 210 \text{ kg m}^{-3} \text{ K}^{-1} \)) averaged over the same 1° × 1° box centered on the updraft. The blank patch in the top-left corner of the panels is due to complications with the interpolation to isentropic coordinates. At some lower level the interpolation scheme failed where \( \partial \theta / \partial z \) was negative. This illustrates one of the practical issues of employing the HM87 formulation in realistic data.

Unlike the Eq. (6a) terms there are no large and opposing tendency terms in the HM87 formulation. The scaled tendency terms of Fig. 4 have maximum PV tendency about 6 times smaller than the two opposing tendency terms of Figs. 3b and 3c and of similar magnitude to the total tendency of Eq. (6a) (Fig. 3d), which suggests that the potential for truncation error is greater in the Eq. (6a) formulation. (Truncation error would be expected in both formulations due to multiple applications of finite differencing to generate the derivative terms.) The physical insight offered by the HM87 terms is also greater. The anticyclonic tendency in the vicinity of the diabatic heating in Fig. 4b represents the divergence of the advective flux, or PVS dilution in the language of HM87. This term incorporates horizontal advection and horizontal divergence of \( \eta \) [cf. Eqs. (8), (9), and (10)]. Similarly Fig. 4c illustrates the combined effect of tilting and vertical advection of \( \eta \). An approximate ring of anticyclonic tendency surrounds a small region of weak cyclonic tendency in the southern half of the contoured diabatic heating region (just evident with this shading regime). This is consistent with vertical advection of cyclonic PV substance from below, and tilting of \( \eta \) into the horizontal around the outside. Tilting is likely to also be

\[ 7 \] For NWP applications the subgrid-scale wind and mass processes will be incorporated in the wind and mass forcing terms, respectively, which means temperature changes associated with turbulent mixing will be included in MF.

\[ 8 \] Recall that when the partial cancellation between the two horizontal vorticity components \( \omega_1 \) and \( \omega_2 \) is taken into account, and the cancelling MAE and MF terms are ignored, Eq. (12) is identical to the HM87 formulation presented in Eq. (10).
responsible for the positive and negative tendencies surrounding the central diabatic heating region. The sum of the two tendencies (Fig. 4d) shows mostly anticyclonic tendency in the vicinity of the central diabatic heating region, with relatively strong cyclonic tendencies to the east and southwest and weaker cyclonic tendencies to the southeast. This pattern of anticyclonic and cyclonic tendencies is qualitatively consistent with the concept of PVS conservation on isentropic surfaces.

As mentioned above, despite the positive qualities of the HM87 formulation, there will likely be some flows in which the formulation is not ideal, including flows in which the changing mass field contribution to PV tendency is important. For example, the gradual downward evolution of the middle isentrope in Fig. 2 is invisible in isentropic coordinates [although it can be easily inferred if some other height variable (e.g., \( p \) or \( z \)) is also plotted]. Additionally, the implementation of isentropic...
coordinates can be problematic where \( \theta \) is not everywhere monotonically increasing with height (e.g., the top-left corner of the panels in Fig. 4). The pros and cons of all PV and PVS tendency formulations presented in this paper are summarized in Table 1.

### 3. An alternative PV and PVS formulation for NWP applications

With the above concerns in mind, we set out to design an alternative PV formulation that provides useful physical insight into the system flow dynamics, is free from potential cancellation problems, and is exact for nonhydrostatic flows. For comparison with HM87 and symmetry, we include equivalent equations for PVS in flux and nonflux form. We argued above that the a/d cancellation occurs when there is large opposition between the MAE and MF terms, and that this cancellation is more problematic in formulations with MF and CAE tendency terms [Eqs. (6) and (7)] for flows in which the residual tendency is relatively small and comprises much of the total tendency. When applied to real data the largely cancelling terms may introduce nontrivial truncation error, and the combined adjustment and evolution terms make it impossible to tease out from the residual tendency the individual mass and wind field contributions. Thus, in designing an alternative family of PV and PVS tendency equations we ensure the MAE and WAE terms are separate, and to address the truncation error issue we combine MF and MAE by retaining \( \partial \theta / \partial t \) from Eqs. (2a)–(2c).

Before rederiving Eqs. (6a)–(6c), it is worth investigating the potential for cancellation in the other diagnostic equations [Eqs. (3) and (4)]. HM87 warned of potential cancellation between terms such as those on the RHS of Eq. (3) and noted that the cancellation can be avoided by using the less expanded form; for instance, for Eq. (3),

\[
\frac{\partial \theta}{\partial t} = -\nabla \cdot \left( \rho \mathbf{u} \right).
\]

In the vorticity tendency equation [Eq. (4)], significant cancellation might be expected between the friction and adiabatic tendency terms in the boundary layer where there may be strong opposition between adiabatic and frictional forcing. Above the boundary layer, however, the diagnostic form of the vorticity tendency equation should be largely free from such cancellation. In this paper we do not attempt to investigate a formulation that avoids the boundary layer cancellation. However, an approach analogous to the a/d cancellation problem would be to retain the \( \left( \partial \eta / \partial t \right) \) term in Eqs. (2a)–(2c).

Alternative PV and PVS tendency equations corresponding to Eqs. (2a)–(2c) for use outside the boundary layer in which the explicit \( \theta \) tendency is retained could be of the following form:

\[
\frac{\partial Q}{\partial t} = \frac{1}{\rho} \left( \nabla \theta \cdot \nabla \times F \right)_{WF} - \frac{1}{\rho} \left( \nabla \theta \cdot [\nabla \times (\mathbf{u} \times \mathbf{u})] \right)_{WAE} + \frac{Q}{\rho} \nabla \cdot (\rho \mathbf{u}) + \frac{1}{\rho} \left( \eta \cdot \frac{\partial \theta}{\partial t} \right)_{MC},
\]

### Table 1. PV and PVS tendency formulations: summary of equation types and their pros and cons. The comparative terms “more” and “less” are relative to other formulations in that class of quantity or equation type.

<table>
<thead>
<tr>
<th>Type</th>
<th>Quantity</th>
<th>PV</th>
<th>PVS</th>
<th>PVS (flux formulation)</th>
<th>Pros</th>
<th>Cons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple tendency</td>
<td>Eq. (2a)</td>
<td>Eq. (2b)</td>
<td>Eq. (2c)</td>
<td></td>
<td>Conceptually simple, no a/d cancellation</td>
<td>Less physical insight</td>
</tr>
<tr>
<td>Simple diagnostic</td>
<td>Eq. (6a)</td>
<td>Eq. (6b)</td>
<td>Eq. (6c)</td>
<td></td>
<td>Relatively simple</td>
<td>a/d cancellation</td>
</tr>
<tr>
<td>Material diagnostic</td>
<td>Eq. (7)</td>
<td>Eq. (10)</td>
<td>Eq. (9)</td>
<td></td>
<td>Simple equations, horizontal conservation, no a/d cancellation</td>
<td>a/d cancellation, less physical insight</td>
</tr>
<tr>
<td>Isentropic PVS</td>
<td>Eq. (10)</td>
<td>Eq. (9)</td>
<td></td>
<td></td>
<td></td>
<td>Hydrostatic assumption, no mass change PV tendency information</td>
</tr>
<tr>
<td>Simple alternative</td>
<td>Eq. (15a)</td>
<td>Eq. (15b)</td>
<td>Eq. (15c)</td>
<td></td>
<td>No a/d cancellation</td>
<td>No insight from expanded vorticity tendency terms</td>
</tr>
<tr>
<td>Expanded alternative</td>
<td>Eq. (21a)</td>
<td>Eq. (21b)</td>
<td>Eq. (21c)</td>
<td></td>
<td>No a/d cancellation, more physical insight</td>
<td>More complex</td>
</tr>
</tbody>
</table>

**Pros**
- True PV
- Good for small \( \partial \rho / \partial t \)
- Useful budget properties, good for small \( \partial \rho / \partial t \)

**Cons**
- More complex
- Not PV
- Not PV
\[ \frac{\partial \rho}{\partial t} = (\mathbf{V} \cdot \nabla)_{\text{WF}} \left[ \mathbf{V} \cdot (\mathbf{u} \times \mathbf{V}) \right]_{\text{WAE}} - \{ \mathbf{V} \cdot (\mathbf{V} \times (\nabla \times \mathbf{u})) \} \text{WAE} + \left( \eta \cdot \frac{\partial \mathbf{V}}{\partial t} \right) \text{MC}, \]  

(15b)

and

\[ \frac{\partial \rho}{\partial t} = -\mathbf{V} \cdot \left[ -\mathbf{F} \times \mathbf{V} \right]_{\text{WF}} + (\eta \cdot \mathbf{u}) \times \mathbf{V} \theta_{\text{WAE}} - \frac{\partial \mathbf{V}}{\partial t} \eta \text{MC}. \]  

(15c)

As in Eq. (12) the PV and PVS tendency associated with friction is indicated by the WF subscript, and the tendency associated with adiabatic frictionless flow evolution and/or adjustment to mass or wind forcing is indicated by the WAE subscript. The combined mass terms (MC subscript) are identical to the mass change term of Eq. (2). Each MC term incorporates PV or PVS tendency associated with net warming/cooling, and in Eq. (15a), the additional mass convergence term.

While it might be tempting to consider further expansion of terms in order to deliver additional physical insight or to search for more mathematical cancellation, care needs to be taken to avoid generating more opposing terms. For example, one can expand mass convergence and WAE terms on the RHS of Eq. (15a) to produce

\[ \frac{Q(\mathbf{V} \cdot \mathbf{u})}{\rho} = \frac{Q}{\rho} (\mathbf{u} \cdot \nabla \rho) + Q(\mathbf{V} \cdot \mathbf{u}) \quad \text{and} \quad (16) \]

\[ \frac{1}{\rho} \{ \nabla \cdot (\mathbf{V} \times (\nabla \times \mathbf{u})) \} = \frac{1}{\rho} \{ \nabla \theta \cdot [(\mathbf{u} \cdot \mathbf{V}) \eta - (\mathbf{V} \cdot \mathbf{u}) \eta] \}

\[ + \eta (\mathbf{V} \cdot \mathbf{u}) - \mathbf{u}(\mathbf{V} \cdot \eta) \}. \]  

(17)

Here \(Q(\mathbf{V} \cdot \mathbf{u})\) in Eq. (16) can be cancelled with the third term on the RHS of Eq. (17), and the fourth term vanishes because of the nondivergence of vorticity. The remaining two terms inside the square brackets are the three-dimensional vorticity advection \((\mathbf{u} \cdot \mathbf{V})\eta\) and a combination of tilting and stretching respectively; that is,

\[ -(\eta \cdot \mathbf{V}) \mathbf{u} = -\left[ \omega \frac{\partial u}{\partial y} + \eta \frac{\partial u}{\partial z} + \eta \frac{\partial v}{\partial z} + \omega \frac{\partial w}{\partial y} \right] - \left[ (\omega_1 \frac{\partial u}{\partial x} + \omega_2 \frac{\partial v}{\partial y} + \omega_3 \frac{\partial w}{\partial z}) \right]. \]  

(18)

The problem with this expansion and cancellation is that the remaining term in Eq. (16), the density advection term, does not offer useful physical insight because most density changes associated with vertical density advection will be largely opposed by three-dimensional divergence associated with adiabatic expansion/compression. Additionally, this expansion is counter to the PV tendency framework we present in this paper, in which PV tendency is viewed as resulting from wind and mass forcing and wind and mass adjustment and evolution, because it partially combines the WAE and MC terms. In this case, an expansion of the WAE term alone may be more physically insightful.

The \( \mathbf{V} \times (\eta \times \mathbf{u}) \) vector represents vorticity tendency associated with the WAE term. A possible expansion is presented in the square brackets on the RHS of Eq. (17), in which the first term represents vorticity advection, the fourth term is zero (as mentioned above), and the second and third terms can be combined to make the same tilting term as Eq. (18) plus a convergence term [which serves the same purpose as the stretching term of Eq. (18)]. Thus, \( \mathbf{V} \times (\eta \times \mathbf{u}) \) is expanded to yield the familiar vorticity tendency terms of advection, convergence, and tilting:

\[ \mathbf{V} \times (\eta \times \mathbf{u}) = (\mathbf{u} \cdot \mathbf{V})\eta_{\text{ADV}} + \left[ \omega_1 \frac{\partial u}{\partial y} + \omega_2 \frac{\partial u}{\partial z} + \omega_3 \frac{\partial w}{\partial y} \right]_{\text{CONV}} - \left[ \omega_1 \frac{\partial u}{\partial x} + \omega_2 \frac{\partial u}{\partial y} + \omega_3 \frac{\partial w}{\partial z} \right]_{\text{TILT}}. \]  

(19)

However, in this expansion, like the expansion of Eq. (9) to yield Eq. (10), the chain rule has been used to express gradients of vorticity flux-like quantities, which introduces cancellation between the resulting pairs of terms (HM87).

An alternative expansion of \( \mathbf{V} \times (\eta \times \mathbf{u}) \) yields two terms that resemble the advective and nonadvective-tilting terms of HM87 when dotted with \( \partial \theta \), in which the gradients of the vorticity flux-like quantities remain intact.
\[ \mathbf{\nabla} \times (\eta \times \mathbf{u}) = \left[ \frac{\partial}{\partial y} (\nu \omega_1) + \frac{\partial}{\partial z} (w \omega_1), \frac{\partial}{\partial x} (\nu \omega_2) + \frac{\partial}{\partial z} (w \omega_2), \frac{\partial}{\partial x} (\nu \eta) + \frac{\partial}{\partial y} (w \eta) \right] - \left[ \frac{\partial}{\partial y} (w \omega_2) + \frac{\partial}{\partial x} (\nu \eta), \frac{\partial}{\partial x} (w \omega_1) + \frac{\partial}{\partial z} (\nu \eta), \frac{\partial}{\partial x} (w \omega_1) + \frac{\partial}{\partial y} (w \omega_2) \right]. \] (20)

The similarity to HM87 is demonstrated below. The reduced cancellation in the above terms will need to be weighed up against any potential loss of physical insight provided by the more familiar vorticity tendency terms when applied to Eqs. (15a) and (15b). If one is fluent in the HM87 language of vorticity tendency, the Eq. (20) formulation is perhaps more physically intuitive. Here we choose to expand Eqs. (15) using Eq. (20), but note that the tendency triplet of advection, convergence, and tilting [Eq. (19)] could equally be applied.

Substituting Eq. (20) into Eqs. (15a) and (15b),

\[ \frac{\partial Q}{\partial t} = \frac{1}{\rho} (\mathbf{\nabla} \times (\eta \times \mathbf{F}))_{WF} - \mathbf{\nabla} \cdot \left[ \frac{\partial}{\partial y} (\nu \omega_1) + \frac{\partial}{\partial z} (w \omega_1), \frac{\partial}{\partial x} (\nu \omega_2) + \frac{\partial}{\partial z} (w \omega_2), \frac{\partial}{\partial x} (\nu \eta) + \frac{\partial}{\partial y} (w \eta) \right] \]

\[ + \mathbf{\nabla} \cdot \left[ \frac{\partial}{\partial y} (w \omega_2) + \frac{\partial}{\partial x} (\nu \eta), \frac{\partial}{\partial x} (w \omega_1) + \frac{\partial}{\partial z} (\nu \eta), \frac{\partial}{\partial x} (w \omega_1) + \frac{\partial}{\partial y} (w \omega_2) \right]_{WAE} + \frac{Q}{\rho} \mathbf{\nabla} \cdot (\rho \mathbf{u}) + \frac{1}{\rho} (\eta \cdot \mathbf{\nabla} \frac{\partial \theta}{\partial t})_{MC}. \] (21a)

And, in flux form using Eq. (12) to expand the WAE term,

\[ \frac{\partial \rho Q}{\partial t} = \mathbf{\nabla} \cdot \{ \mathbf{F} \times (\mathbf{\nabla} \theta) \}_{WF} - \mathbf{\nabla} \cdot \left[ \mu \left( \frac{\partial \theta}{\partial y} \omega_1 + \frac{\partial \theta}{\partial z} \eta \right), \nu \left( \frac{\partial \theta}{\partial x} \omega_1 + \frac{\partial \theta}{\partial z} \eta \right), \omega_2 \left( \frac{\partial \theta}{\partial x} \omega_1 + \frac{\partial \theta}{\partial z} \eta \right) \right]_{WAE} + \mathbf{\nabla} \cdot \left\{ \frac{\partial \theta}{\partial t} \right\}_{MC}. \] (21b)

To demonstrate the similarity of the WAE terms to the HM87 isentropic PVS formulation, we rotate the coordinate system so that the coordinate direction \( \mathbf{k} \) is parallel to \( \mathbf{\nabla} \theta \) at some point of interest in the fluid. Then both the WAE terms of Eqs. (21b) and (21c) reduce to

\[ -\frac{\partial \theta}{\partial z} \mathbf{\nabla} \cdot [(u, v, 0) \eta - w (\omega_1, \omega_2, 0)], \]

and a division by \( \rho \) yields the equivalent term for Eq. (21a). This equation closely resembles the advective and nonadvective tilting terms of Eq. (8). The two flux terms can be considered to represent the divergence of (i) the horizontal flux of vertical vorticity and (ii) the vertical flux of horizontal vorticity.

The WAE tendency terms on the RHS of Eqs. (21a) and (21b) are by definition associated with changes in the component of the vorticity vector parallel to \( \mathbf{\nabla} \theta \). Thus, the vorticity tendencies reduce to the two-dimensional (the plane parallel to the isentropes) divergence of (i) an isentrope-parallel flux of isentrope-perpendicular vorticity (depicted in Fig. 1d) and (ii) an isentrope-perpendicular flux of isentrope-parallel vorticity (depicted in Fig. 1c), both scaled by the parallel component of \( \mathbf{\nabla} \theta \).

Similar conclusions can be drawn for Eq. (21c) because the components of \( \mathbf{\nabla} \theta \) effectively scale the vorticity components in (i) and the wind components in (ii) according to their orientation relative to \( \mathbf{\nabla} \theta \) (with greater weighting in the direction of \( \mathbf{\nabla} \theta \)). Given the close similarity to the HM87 advective and nonadvective tilting fluxes, we consider Eq. (21c) to be the geometric
coordinate equivalent of HM87’s isentropic PVS tendency equation, and we note that the WAE terms of all three equations offer similar physical insight to PV and PVS tendency to the HM87 formulation.

Hereafter, we give Eqs. (15) and (21) the label “alternative” and distinguish between the two families with the labels “simple alternative” [Eq. (15)] and “expanded alternative” [Eq. (21)] tendency formulations, respectively. See Table 1 for a summary of the tendency equations pros and cons.

To assess the performance of the alternative tendency equations, a comparison with Eq. (6a) is made in Fig. 5, which shows terms from Eqs. (15a) and (21a) for the same convective burst as that illustrated in Fig. 3. Figure 5a shows the Eq. (15a) WAE term, which is essentially the PV tendency associated with the changing vorticity. The expanded forms of this term in Eq. (21a) (the advective flux–like term and the nonadvective tilting–like term) are represented in Figs. 5b and 5c. The two components of MC common to Eqs. (15a) and (21a) are represented in Figs. 5d and 5e (density- and temperature-change terms, respectively), and the total tendency common to Eqs. (15a) and (21a) (excluding friction) is represented in Fig. 5f. As for the HM87 tendency terms (Fig. 4), all of these six panels show maximum PV tendency magnitudes similar to or less than the total tendency of Eq. (6a) (Fig. 3d), which suggests the potential for truncation error using the alternative formulations is significantly less than using Eq. (6a).

The results in Fig. 5 yield greater dynamical insight than do those in Figs. 3 and 4. In this example the majority of the PV tendency is associated with changes in vorticity (Fig. 5a), with density changes contributing least to PV tendency (Fig. 5d) and temperature changes contributing minimally (Fig. 5e). The advective flux–like term in the vicinity of the updraft shows an anticyclonic PV tendency (Fig. 5b). In this example, it is dominated by horizontal vorticity divergence (the horizontal advection component has been minimized by the subtraction of the mean local horizontal wind, as described in section 2d). The nonadvective tilting–like term shows cyclonic tendency at the center of the updraft, partly enclosed by a ring of anticyclonic tendency. This is a good depiction of the competing processes of vertical advection and tilting in geometric coordinates, as described in section 2c for isentropic coordinates and illustrated in Figs. 1b and 1c.

The motivation for splitting the Eq. (15a) WAE term into terms similar to the HM87 advective and nonadvective tilting flux terms was to avoid the unnecessary cancellation that is encountered when incorporating the Eq. (19) expansion. The Eq. (19) terms expressed as PV tendencies are presented in the last three panels of Fig. 5 [η advection (Fig. 5g), η convergence (Fig. 5h), and η tilting (Fig. 5i)]. In this convective burst example the η advection is dominated by the cyclonic tendency of vertical advection, which is largely cancelled on the eastern edge of the updraft by tilting. Cancelling advection and tilting tendencies are also present on the southwestern side of the updraft. While it is advisable to avoid terms that cancel when quantitatively diagnosing PV tendency, such as depicted in Figs. 5g and 5i, these terms do offer complementary physical insight into the interpretation of the tendencies. The convergence term (Fig. 5h) is nearly identical to the advective flux–like term of Eq. (21a) (Fig. 5b), which confirms our earlier speculation that this anticyclonic tendency was dominated by vorticity divergence (i.e., horizontal advection was minimal).

4. Choosing an appropriate PV tendency formulation

The most appropriate PV tendency formulation will vary from application to application. The cancellation issues discussed above suggest that problematic cancellation is case specific. In the following discussion of the dynamics of PV tendency we return to the concept of PV tendency as the sum of wind and mass forcing and an adiabatic response that incorporates wind and mass adjustment and evolution [e.g., Eq. (12)]. The WF and MF terms respectively act directly on the wind and mass alone to disrupt balance between the mass and wind fields. The adiabatic response will oppose the forcing and thus tend to be of opposite sign (i.e., WF and WAE, and MAE and MF, will tend to be of opposite sign). Although the forcing and response are distinct in this conceptual model, in reality the forcing is not instantaneous and the two will act simultaneously for some period of time.

a. Potential WAE/MAE cancellation issues

In section 2d we noted problematic a/d cancellation between MF and MAE (or terms containing MAE). For completeness we should consider the potential for problematic cancellation between WAE and MAE. Because the adjustment process (e.g., geostrophic adjustment; Gill 1982) involves both the conversion of potential energy to kinetic energy (or vice versa) and the loss of some energy to infinity through inertia–gravity waves, it is almost inevitable that PV budget equations that separate WAE and MAE will show some cancellation between these terms. In the linear geostrophic adjustment problem (e.g., Gill 1982, sections 7.2 and 7.3), the only
change to the PV field is that which is initially imparted by an instantaneous forcing event. The subsequent adjustment process described by the CAE term [e.g., the PV advection term of Eq. (6a)] is zero, which means that WAE and MAE must exactly cancel. Thus, any changes to the mass and wind fields are a result of linear processes responsible for the conversion between potential and kinetic energy. To aid the following discussion we illustrate this simple relationship for a frictionless example with the following equation:
Here $\alpha$ and $\beta$ are of the same sign, $\alpha$ represents the PV forcing and $\beta$ the adjustment, and we note that for realistic solutions $\alpha > \beta$. It is worth noting that for tendency equations that combine MAE and WAE [e.g., Eqs. (6) and (7)], the equations provide the trivial information that the PV tendency is equal to the tendency of the PV source. On the other hand, the tendency equations that combine MF and MAE [e.g., Eqs. (15) and (21)] offer greater insight because they show that the change in PV associated with mass forcing results in specific changes to the mass and wind fields after a period of adjustment.

Meanwhile, the validity of applying linear ideas to realistic convective systems is questionable. Nonlinear systems with nonimpulsive, nonstationary forcing relative to the flow behave very differently to linear systems, as is illustrated in Figs. 3. Here it is clear that the sum of MAE and WAE (CAE in Fig. 3b) is far from zero and largely in opposition to MF (Fig. 3c). One of the main differences between Gill’s linear example and the nonlinear convective burst example is the tendency reference frame. In the linear example the reference frame is relative to the mean fluid motion, whereas the vertically stationary reference frame of the convective burst differs greatly from one that follows the strong vertical motions at the center of the PV tendency analysis. In effect the diabatic PV source and sink in the convective updraft moves downward relative to the upward flow, which introduces a strong advective component to the PV tendency that is not present in the linear framework.

b. WAE/MAE cancellation in formulations using MC

Nevertheless, Eq. (22) and subsequent assumptions about the signs and relative magnitudes of $\alpha$ and $\beta$ are valid in a deep convective system with a vortex that amplifies with time (e.g., a developing TC). In this situation the increase in low-level vorticity reflects a positive WAE contribution, and MAE would be expected to largely but not completely oppose the mass forcing MF (some remnant mass change is necessary to balance the changed wind field). This means the cancellation between WAE and MAE is not problematic when the MF and MAE terms are combined because the combined term has the same sign as WAE. Thus, the combined mass term of Eqs. (15) and (21) will not be subject to adjustment and evolution cancellation issues with WAE and will also be free from the a/d cancellation problem. However, this does not rule out the potential for cancellation within WAE subterms, especially if $\nabla \times (\eta \times u)$ is expanded in to the familiar vorticity tendency terms of advection, convergence, and tilting, as illustrated in Fig. 5.

Nontrivial cancellation might be expected between MC and WAE for evolving flows without mass forcing when MC reduces to MAE. In this case Eqs. (6) might offer greater accuracy with regard to the PV tendency (because there is no cancellation between terms), but they cannot offer the insight into the potentially significant mass and wind field changes that could be taking place.\(^{10}\)

c. Comparisons with HM87

While the HM87 isentropic formulation in theory fulfills the recommended tendency style in which WAE is separate from MAE, and MAE and MF can be combined, the exact cancellation between MAE and MF in the isentropic coordinate system [Eq. (13)] yields a tendency equation with terms that only describe wind processes. For example, the change in wind and mass field associated with the transition from a tropical low to a tropical cyclone includes an increase in vorticity (change in wind field), and the dipping of isentropic surfaces (change in mass field). Because of the isentropic coordinate system the mass changes are not apparent. This is perhaps not surprising given that the isentropic PV tendency associated with changes in mass can only be inferred from consideration of the changing isentropic surfaces with respect to another coordinate system.

d. Recommendations

Based on the above discussion we would recommend the use of the HM87 formulation [Eq. (9)] where possible because of its simplicity. But Eqs. (21) will provide better alternatives when a geometric coordinate system is desirable, when isentropes deviate substantially from horizontal, or when information about the mass change contribution to PV is important. When the local mass convergence contribution to PV tendency is nontrivial, the full PV tendency formulation of Eq. (21a) would be most appropriate.

A preliminary assessment of PV tendency using Eq. (15a) could reveal which tendency equation is most appropriate. If the first part of the MC term is small (e.g., Fig. 5d) then a PVS tendency analysis will be a good

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\(^{10}\) For example, consider the trivial case of a stationary PV anomaly in convergent flow [for simplicity let the PV be described by Eq. (10) and the convergence be associated with vertical stretching]. The stretching both increases the vorticity (wind change) and decreases the \(\theta\) gradient (mass change) with zero change in the PV.
approximation for PV tendency, and the useful properties of the flux formulation [Eqs. (15c) and (21c)] can be utilized. If the second part of the MC term is also small (e.g., Fig. 5e), changes in thermal structure have limited influence on PV tendency, and sufficient insight into the system dynamics may be obtained purely through a vorticity tendency analysis. If the WAE term is small, then Eqs. (15a)–(15c) may be sufficient. On the other hand, if the WAE term is nontrivial then an understanding of PV or PVS tendency associated with the various vorticity tendency contributions of Eqs. (21a)–(21c) could offer useful physical insight into the system dynamics. Comparisons with the Eq. (15a) tendency terms and the more expanded forms should identify any problematic cancellation between the expanded terms if present.

5. Summary

The inversion of PV to determine a balanced mass and wind field distribution provides a useful framework for understanding a variety of atmospheric phenomena (e.g., HMR85). This “PV thinking” describes an intimate connection between mass and wind fields in which the mass and wind forcings are balanced. Any flow evolution or forcing that disrupts the force balance initiates a somewhat predictable adjustment toward a new state of mass and wind balance. In theory, an understanding of the changing PV distribution could provide insight into the changing balanced mass and wind fields through successive PV inversions. Greater insight can be gained from PV and PVS tendency equations that separate the mass (MF) and wind (WF) forcing terms and the mass (MAE) and wind (WAE) adjustment and evolution terms because they provide not only information on changing PV, but useful information about (i) how forcing or evolving flow can disrupt the balance and (ii) what processes contribute to restoring the balance.

A variety of PV and PVS tendency formulations exist in the literature, many combining and/or representing the abovementioned terms in a variety of ways, and each providing varying insights into the flow evolution. Each formulation has its pros and cons (e.g., Table 1), and the appropriateness of particular formulations will vary with the problem at hand. It is thus important that the limitations of each formulation are understood.

For example, the simple formulation [Eq. (6a)] identifies the location of PV sources and sinks (WF and MF) as well as tendencies associated with the advection of PV anomalies relative to a particular reference frame. The PV advection term combines WAE and MAE and thus cannot provide insight into the relative contributions to PV tendency from the wind and mass adjustment and evolution processes. In deep convective systems there can be large opposition between the diabatic forcing contribution (MF) and the mass adjustment contribution (MAE) as local buoyancy gradients introduced by the diabatic heating initiate an adiabatic response that largely “undoes” the mass change associated with MF.

The above situation describes the a/d cancellation problem. For Eqs. (6) and (7), which use a combined adjustment and evolution term (CAE), the large opposition is between MF and CAE. This can be problematic above the boundary layer where the two terms essentially constitute the entire tendency. Such formulations offer little insight into the changing near-balanced mass and wind fields because they only identify that a large disruption to the mass field has induced a strong transverse circulation, with often only a relatively small PV or PVS change that will be subject to nontrivial truncation error.

The a/d cancellation problem can be avoided by using formulations that combine MF and MAE by retaining \( \partial \theta / \partial t \) in the mass change tendency terms of Eqs. (2). Useful physical insight into the contribution of mass and wind changes to PV and PVS tendency can be obtained from formulations that separate WAE and MAE.

In the HM87 isentropic PVS tendency formulation the a/d cancellation problem is avoided because the MF and MAE terms exactly cancel. The remaining WF and WAE tendency terms demonstrate PVS conservation on isentropic surfaces or between isentropic layers that do not intersect a boundary. The physical insight offered by this formulation is profound. PVS is effectively redistributed within isentropic layers via PVS concentration (dilution) from diabatically induced convergence (divergence) and the generation of balancing PVS sources and sinks from tilting-like processes (induced by diabatic heating or cooling) and frictional processes. On the other hand, the MF and MAE cancellation means no direct information is available regarding PVS tendency due to the changing mass field. While acknowledging the exceptional advancement of PV tendency conceptual understanding provided by the HM87 isentropic formulation, there may be applications where it is less ideal. Some examples include situations when PV substance is not a good approximation for PV, when geometric coordinates are desired, or when isentropes deviate substantially from horizontal.

In this paper a new class of PV tendency equations was introduced, which combines the MF and MAE terms and thus avoids the a/d cancellation problem. These equations include relatively simple terms that describe PV tendency associated with the changing wind (friction and absolute vorticity) and mass (mass divergence and warming/cooling) fields [Eqs. (15a)–(15c)]. In the more complex forms the absolute vorticity tendency was expanded into two or more terms, and equivalent equations
for PVS tendency in both flux and nonflux forms are included [Eqs. (21a)–(21c)]. The equations are formulated in geometric coordinates and are valid for non-hydrostatic flows. Equation (21c) is shown to be the geometric coordinate equivalent equation to the influential isentropic PVS tendency equation of HM87, and the WAE tendency terms of Eqs. (21a) and (21b) provide similar physical insight into HM87’s advective and non-advective tilting tendencies. The WAE contribution can also be arranged, using the more familiar expansion of $V \times (\eta \times u)$, into vorticity advection, convergence, and tilting tendencies. However, this expansion could potentially introduce cancellation between the resulting terms (Fig. 5). The new class of equations should be ideal for the analysis of diabatic processes in fluids in which the diabatic forcing induces relatively large vertical circulations because the tendency terms provide important insight into the evolving primary circulation mass and wind fields.

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