Effects of Varying the Shape of the Convective Heating Profile on Convectively Coupled Gravity Waves and Moisture Modes

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ABSTRACT

The analytical model of convectively coupled gravity waves and moisture modes of Raymond and Fuchs is extended to the case of top-heavy and bottom-heavy convective heating profiles. Top-heavy heating profiles favor gravity waves, while bottom-heavy profiles support moisture modes. The latter behavior results from the sensitivity of moisture modes to the gross moist stability, which is more negative with bottom-heavy heating.

A numerical implementation of the analytical model allows calculations in the two-dimensional non-rotating case as well as on a three-dimensional equatorial beta plane. In the two-dimensional case the analytical and numerical models are mostly in agreement, although minor discrepancies occur. In three dimensions the gravity modes become equatorial Kelvin waves whereas the moisture modes are more complex and require further investigation.

1. Introduction

A generally accepted theory that models the interactions between large-scale motions and deep convection in the tropics is still missing. We are closer to understanding the convectively coupled equatorial waves that correspond in the dry atmosphere to Matsuno’s (1966) solutions, but tropical disturbances such as the Madden–Julian oscillation (MJO) and easterly waves are still incompletely understood. Historically it was thought that quasi-equilibrium theory, which assumes that disturbances have a first baroclinic mode form in the vertical, can explain the main features of the convectively coupled equatorial waves. Although quasi-equilibrium theory is useful for many applications, today we know that the equatorial waves cannot be characterized only by the first baroclinic mode; we also need the higher-order vertical structure. The open questions are the dynamical role of the vertical structure and to what extent it is important.

Emanuel (1987) introduced a highly simplified form of the convective quasi-equilibrium hypothesis of Arakawa and Schubert (1974) in a linearized model for the MJO. Deep convection is assumed to be near equilibrium with mechanisms that create convective available potential energy (CAPE), while additional processes such as wind-induced surface heat exchange (WISHE) act to destabilize the modes. This model is characterized as implementing strict quasi-equilibrium, since the convection drives the atmosphere instantly to a moist adiabat consistent with the equivalent potential temperature of the boundary layer, which is determined by the interactions of surface fluxes, radiation, and vertical motion.

Various models proposed by Yano and Emanuel (1991), Emanuel (1993), Emanuel et al. (1994), and Neelin and Yu (1994) follow from a relaxation of strict quasi-equilibrium theory; convective effects are assumed to lag the forcing by the typical convective time scale of a few hours rather than occurring instantaneously. The phase speeds of disturbances with first baroclinic mode vertical structure in quasi-equilibrium with convection are reduced compared with corresponding free adiabatic modes and the modes are damped rather than neutral as
in strict quasi-equilibrium. However, the first baroclinic mode structure of simple quasi-equilibrium models, such as those noted above, does not capture the observed complex vertical structure of the temperature, heating, and vertical velocity for convectively coupled equatorial Kelvin waves from Straub and Kiladis (2003) and Kiladis et al. (2009).

To solve this problem, Mapes (2000) proposed a two-vertical-mode model where the vertical heating profile is a superposition of deep convective and stratiform parts. The deep convective profile has the structure of the first baroclinic mode, while the stratiform has vertical wavelength half that of the deep mode. The stratiform mode lags the deep convective mode as a result of assumed cloud microphysical processes. Particular significance in the Mapes model is given to the convective inhibition (CIN) closure, without which there are no wave disturbances. When CIN is turned on, the unstable convectively coupled gravity waves have the observed phase speed of convectively coupled Kelvin waves.

Subsequent two-mode models were developed by Majda and Shefter (2001a,b) and Majda et al. (2004). Those models used a lower-tropospheric CAPE, which bears more relationship to a CIN closure than to the full-tropospheric CAPE closure of quasi-equilibrium models. The models developed by Khouider and Majda (2006, 2008) and Kuang (2008) differ from earlier models in that they rely on the effect of midtropospheric humidity to control the depth of convection. This so-called moisture–stratiform mode derives its lag among shallow convection, deep convection, and stratiform conditions from this humidity dependence rather than from cloud microphysical delays in the production of stratiform rain. The moisture–stratiform instability appears to be the main instability mechanism for convectively coupled wave development in the cloud system–resolving model simulations.

To avoid the quasi-equilibrium and imposed two-vertical-mode assumptions in the above models, Fuchs and Raymond (2007) and Raymond and Fuchs (2007, hereafter RF07) developed a vertically resolved model that produces the observed complex vertical structure of convectively coupled Kelvin waves assuming a simple, sinusoidal vertical heating profile. The RF07 model is a linearized, two-dimensional nonrotating model of the tropical atmosphere that incorporates three crucial factors into its convective closure. These factors, based on the results from observations and numerical models, are the importance of convective inhibition (Raymond et al. 2003; Firestone and Albrecht 1986), the control of precipitation by the saturation fraction or column relative humidity of the troposphere (Bretherton et al. 2004; Sobel et al. 2004; Lucas et al. 2000; Derbyshire et al. 2004; Raymond and Zeng 2005), and the effects of surface moist entropy fluxes (Raymond et al. 2003; Back and Bretherton 2005; Maloney and Esbensen 2005). The resulting modes are fast gravity waves, convectively coupled gravity waves, and the so-called moisture mode. The moisture mode arises as a consequence of the moisture prognostic equation and is consistent with the results obtained using the weak temperature gradient approximation (WTG) of Sobel et al. (2001) in the limit of zero meridional moisture gradient. Unstable moisture modes arise when two conditions are satisfied: 1) precipitation increases with tropospheric humidity and 2) convection itself, possibly working with convectively coupled surface flux and radiation anomalies, tends to increase the humidity (Sugiyama 2009a,b). Although not emphasized by these authors, Neelin and Yu (1994) obtained moisture modes in their linearized quasi-equilibrium model.

RF07 show that two large-scale modes predicted by their model—a slowly propagating moisture mode that is driven primarily by saturation fraction anomalies and negative effective gross moist stability (GMS), and a convectively coupled gravity mode that is governed by anomalies in convective inhibition caused by buoyancy variations just above the top of the planetary boundary layer—are unstable. The gravity mode is assumed to map onto the equatorial Kelvin wave in the earth’s atmosphere and its propagation speed and vertical structure are in agreement with observation (Straub and Kiladis 2002). Since a CIN closure differs from a lower-tropospheric CAPE closure only by the depth range over which parcel buoyancy is taken to control convection, this model is clearly similar to the Khouider and Majda (2006, 2008) and Kuang (2008) models. However, it differs in that it shows that the essential results of these models can be obtained without assuming variation in the vertical heating profile with wave phase.

The disturbances predicted by the two-vertical-mode models appear to have the characteristics of convectively coupled Kelvin waves. On the other hand, there is increasing evidence that the MJO is in essence a moisture mode (Maloney and Esbensen 2005; Raymond and Fuchs 2009; Sugiyama 2009a,b).

In this paper we use the same thermodynamics as in RF07 but consider different heating and moisture profiles. In particular, we examine the effects of top- and bottom-heavy vertical heating profiles. Moisture profiles are also varied to produce different values of GMS. Additionally we develop a numerical version of the model which allows us to expand from two dimensions into a three-dimensional equatorial beta plane, thus allowing rotation to play a role. This enables us to explore the behavior of the unstable modes in a more realistic
environment. We find that convectively coupled Kelvin waves are favored by top-heavy heating profiles with positive GMS. In contrast, the unstable moisture modes are produced whenever the GMS is negative. Section 2 develops the modified analytical model equations and section 3 presents the results of the analytical model. The numerical model is developed in section 4 and the results from the model for two- and three-dimensional cases are given in section 5. Conclusions are drawn in section 6.

2. Analytical model

The linearized, slab-symmetric governing equations (horizontal momentum, hydrostatic, mass continuity, and thermodynamic equations) for buoyancy $b$, mixing ratio $q$, and moist entropy $e$ in a nonrotating two-dimensional atmosphere at rest under the Boussinesq approximation are as follows:

$$\frac{\partial u}{\partial t} + \frac{\partial \Pi}{\partial x} = 0, \quad (1)$$

$$\frac{\partial \Pi}{\partial z} - b = 0, \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \quad (3)$$

$$\frac{\partial b}{\partial t} + \Gamma_{Bw} = S_B, \quad (4)$$

$$\frac{\partial q}{\partial t} + \Gamma_{Qw} = S_Q, \quad (5)$$

$$\frac{\partial e}{\partial t} + \Gamma_{Ew} = S_E. \quad (6)$$

In the above equations $u$ is the horizontal wind perturbation, $\Pi$ is the mean potential temperature times the Exner function perturbation, and $w$ is the vertical velocity perturbation. The buoyancy perturbation is defined as $b = g s_d^i/T_R$, where $s_d^i$ is the perturbation dry entropy that comes from dry entropy being split into mean and perturbation parts $s_d = s_d^m(z) + s_d^i$, $g$ is the acceleration of gravity, and $T_R = 300$ K is a constant reference temperature. Also, $\Gamma_B = (g/C_p)\frac{ds_d^m}{dz}$ is the constant square of the Brunt–Väisälä frequency, where $C_p$ is the specific heat of air at constant pressure. The scaled buoyancy source term is $S_B = (g/C_p)\frac{ds_d^m}{dz}$. The moist entropy perturbation is scaled by $g/C_p$, $e = (g/C_p)s^i$, where $s^i$ is the perturbation moist entropy; $s = s_d^m(z) + s^i$. The scaled moist entropy gradient is $\Gamma_E = (g/C_p)\frac{ds_d^m}{dz}$ and the scaled moist entropy source term is $S_E = (g/C_p)\frac{ds_d^m}{dz}$. The scaled mixing ratio anomaly is given by $q = e - b$, $q = (gL/C_pT_R)r'$, where the mixing ratio $r = r_0(z) + r'$, and $L$ is the latent heat of condensation. The scaled mixing ratio gradient is $\Gamma_Q = (gL/C_pT_R)\frac{dr_0}{dz}$ and the scaled mixing ratio source term is $S_Q = (gL/C_pT_R)\frac{dr_0}{dt}$.

Equations (1)–(6) lead to an equation for the vertical velocity $w$

$$\frac{d^2w(z)}{dz^2} + m^2w(z) = \frac{k^2}{\nu^2}S_B(z), \quad (7)$$

and the heating profile is assumed to take the form

$$S_B = Bm_0X\exp(m_1\nu z)\sin(m_0\nu z), \quad (8)$$

where $B$ is the vertically integrated heating anomaly, the quantity into which the thermodynamics of the model will be incorporated. The quantity $\nu$ is a dimensionless parameter, $m_0 = \pi/h$, $h$ is the height of the tropopause, and $X = (1 + \nu^2)/[1 + \exp(\pi\nu)]$. Varying $\nu$ from negative to positive values allows us to change the vertical heating profile from bottom heavy to top heavy as can be seen from Fig. 1; $\nu = 0$ brings us back to the neutral or unperturbed heating profile and the results of RF07.

The polarization relations for the buoyancy and the scaled moist entropy perturbation obtained from Eqs. (4) and (6) respectively are

$$b = (i\omega)(S_B - \Gamma_{Bw}), \quad (9)$$
\[ e = (i \omega)(S_E - \Gamma_E w), \]  

(10)  

where the \( x \) and \( t \) dependence take the form \( \exp[i(kx - \omega t)] \), with \( k \) and \( \omega \) being the zonal wavenumber and frequency. The vertical wavenumber is \( m = k \Gamma_B^{1/2}/\omega \).

The solution to the vertical velocity equation (7) using an upper radiation boundary condition is

\[ w(z) = \frac{m_0 B X}{\Gamma_B M_+ M_-} \left\{ (1 + \Phi^2 \nu^2 - \Phi^2) \exp(m_0 \nu z) \sin(m_0 z) - 2 \nu \Phi^2 \exp(m_0 \nu z) \cos(m_0 z) 
+ 2 \nu \Phi^2 \exp(-im_0 z/\Phi) - M_+ \Phi \exp(\pi \nu - i \pi / \Phi) \sin(mz) \right\}, \]  

(11)  

where \( M_+ = \Phi^2 + (\nu \Phi + i)^2 \) and \( M_- = \Phi^2 + (\nu \Phi - i)^2 \). Substitution of Eq. (11) into Eq. (9) results in

\[ b(z) = \frac{im_0 B X}{\alpha \nu \kappa M_+ M_-} \left\{ (M_+ M_- - 1 - \Phi^2 \nu^2 + \Phi^2) \exp(m_0 \nu z) \sin(m_0 z) + 2 \nu \Phi^2 \exp(m_0 \nu z) \cos(m_0 z) 
- 2 \nu \Phi^2 \exp(-im_0 z/\Phi) + M_+ \Phi \exp(\pi \nu - i \pi / \Phi) \sin(mz) \right\}, \]  

(12)  

where \( \kappa = h \Gamma_B^{1/2} k/(\pi \alpha) \) is the dimensionless wavenumber and where \( \Phi = \omega/(\alpha \kappa) = m_0 m \) is the dimensionless phase speed.

The thermodynamics of the model come into the equations via the vertically integrated heating anomaly which is assumed to depend on the scaled precipitation rate anomaly \( P \) and the radiative cooling rate anomaly \( R \):

\[ \int_0^h S_B \, dz = B = P - R \]

\[ = \alpha (1 + \epsilon) \int_0^h q(z) \, dz + \mu_{\text{CIN}}(e_s - e_i). \]  

(13)  

The quantity \( \alpha \) is a moisture adjustment rate. The variable \( \epsilon \) incorporates the effect of cloud–radiation interactions, which are assumed to cause a radiative heating anomaly in phase with precipitation (see Fuchs and Raymond 2002).

The first term on the right-hand side of Eq. (13) is proportional to the precipitable water anomaly. The second term represents the effect of convective inhibition on heating. The quantity \( e_s \) is the scaled perturbation in boundary layer moist entropy, while \( e_i \) is a scaled threshold value of the perturbation moist entropy. The constant \( \mu_{\text{CIN}} \) governs the sensitivity of precipitation rate to deep convective inhibition. We set \( e_i \) equal to the saturated moist entropy perturbation at elevation \( Dh \), where \( D \) is this elevation expressed as a fraction of the tropopause height. In the simplified thermodynamic scheme \( e_i \) is related to the buoyancy anomaly \( b(D) \) at \( Dh \):

\[ e_i = \lambda_b(D). \]  

(14)  

The boundary layer moist entropy is subject to a balance primarily between a positive tendency due to surface moist entropy fluxes and a negative tendency due to convective downdrafts and turbulent entrainment of dry air into the boundary layer. In RF07 it was expressed through the simplified assumption that stronger surface wind speeds cause increased surface evaporation, which results in enhanced boundary layer moist entropy anomaly \( e_s \). In this paper we choose to show the results in the absence of WISHE, which means that we can ignore this term. We show the results without WISHE to simplify the model equations; besides giving an eastward propagation speed to the moisture mode when mean easterlies are imposed, WISHE mechanism does not alter the model results significantly.

Combining Eqs. (9), (10), and (13) results in an equation for the vertically integrated heating \( B \):

\[ B = \frac{i z \kappa \Phi + \epsilon}{1 - i \kappa \Phi} \mu_{\text{CIN}} f_i b(D) \left( 1 + \frac{\epsilon}{1 - i \kappa \Phi} \right) 
\int_0^h B w w dz, \]  

(15)  

where \( \Gamma_M \) is a version of the gross moist stability of Neelin and Held (1987), which we call the normalized gross moist stability (NGMS; Raymond and Sessions 2007; Raymond et al. 2009):

\[ \Gamma_M = \int_0^h \Gamma_E w w dz \left/ \int_0^h \Gamma_B w w dz. \right. \]  

(16)  

The integral of the vertical velocity from Eq. (11) is

\[ \int_0^h w(z) \, dz = \frac{B}{\Gamma_B M_+ M_-} F(\Phi), \]  

(17)  

where
The NGMS can be expressed as

\[
F(\Phi) = \Phi^2 - 3\Phi^2\chi^2 - 1 + 2i\nu X\Phi^3[1 - \exp(-i\pi/\Phi)] \\
+ XM_+ \Phi^2 \exp(\pi \nu - i\pi/\Phi)[1 - \cos(\pi/\Phi)]
\]

(18)

and the integral that includes the variations in the scaled ambient moist entropy \(e_0(z)\) that is assumed to take a piecewise linear form in the troposphere as in Fuchs and Raymond (2007) is

\[
\int_0^h \Gamma_{ew} dz = \frac{B\Delta e X}{M_+ M_-} J(H, \Phi),
\]

(19)

where

\[
J(H, \Phi) = [H + H \exp(\pi \nu) - 1](1 + 3\Phi^2\nu^2 - \Phi^2)(1 + \nu^2) - \nu(1 + \Phi^2\nu^2 - 3\Phi^2) \exp(\pi H \nu) \sin(\pi H)/(1 + \nu^2) \\
- (\Phi^2 - 3\Phi^2\nu^2 - 1) \exp(\pi H \nu) \cos(\pi H)/(1 + \nu^2) + 2i\nu \Phi^3[H \exp(-i\pi/\Phi) - H + 1 - \exp(-i\pi H/\Phi)] \\
- \Phi^2 M_+ \exp(\pi \nu - i\pi/\Phi)[H - 1 + \cos(\pi H/\Phi) - H \cos(\pi/\Phi)].
\]

(20)

The NGMS can be expressed as

\[
\Gamma_M = \int_0^h \Gamma_{ew} dz / \int_0^h \Gamma_{pw} dz = \frac{\Delta e I(H, \Phi)}{XH(1 - H)F(\Phi)}.
\]

(21)

When \(|\Phi|^2 \ll 1\), as is true for all of the interesting modes studied here, it takes the form

\[
\Gamma_M \approx \frac{\Delta e}{H(1 - H)} \left\{ H - \frac{1}{1 + \exp(\chi \nu)} \right. \\
- \exp(\pi H \nu) \left[ \frac{\nu \sin(\pi H) - \cos(\pi H)}{1 + \exp(\pi \nu)} \right] \right\}.
\]

(22)

The parameter \(H\) is the fractional height relative to the tropopause of the minimum in the ambient moist entropy profile and \(\Delta e = \Delta e_0/\Gamma_\mu\) is the scaled difference between the surface and tropopause values of moist entropy (assumed to be the same) and the minimum value at height \(Hh\). Note that this approximate form [Eq. (22)] is real and independent of \(\Phi\), which means that \(\Gamma_M\) can be treated as a constant external parameter under these conditions.

Applying the heating closure [Eq. (13)] and combining Eqs. (11), (18), (15), and (14) results in the dispersion relation for the phase speed \(v\):

\[
M_+ M_- (\kappa \Phi + i) + 2\chi_x(\chi + i\kappa \Phi)L(D, \Phi)/\kappa \Phi \\
+ i(1 + \varepsilon)(1 - \Gamma_M)F(\Phi) = 0.
\]

(23)

The dimensionless parameter \(\chi_x = \lambda_\mu_\text{CIN} m_0/(2\alpha)\) represents the sensitivity of precipitation to the degree of convective inhibition while the auxiliary function \(L(D, \Phi)\) is defined as

\[
L(D, \Phi) = (M_+ M_- + 1 - \Phi^2\nu^2 + \Phi^2) \exp(\pi \nu D) \sin(\pi D) + 2\nu \Phi^2 \exp(\pi \nu D) \cos(\pi D) \\
- 2i\nu \Phi^2 \exp(-i\pi D/\Phi) + M_+ \Phi \exp(\pi \nu - i\pi/\Phi) \sin(\pi D/\Phi).
\]

(24)

The dispersion relation given by Eq. (23) is transcendental and has to be solved numerically using Newton’s method.

3. Analytical model results

a. Control case

The parameters used to calculate the dispersion curves in Fig. 2 were discussed at length in RF07 and their values are given in Table 1. These parameters are \(\varepsilon, H, \Delta e, D, \chi_x, \) and NGMS. The sensitivity to the vertical heating profile parameter is expressed via \(\nu\). The control case has \(\nu = 0\) as in RF07, which means that the heating profile is unperturbed (i.e., neither top nor bottom heavy).

Figure 2a shows the phase speed \(\text{Re}(\omega/k)\) in units of meters per second while Fig. 2b shows the growth rate \(\text{Im}(\omega)\) in units of change per day. The phase speed and the growth rate are shown as a function of planetary wavenumber \(l\) defined as the circumference of the earth divided by the zonal wavelength. There are three types of modes, one that corresponds to a fast gravity wave, eastward and westward propagating with a phase speed of 48 m s\(^{-1}\) and decaying; a convectively coupled gravity wave propagating eastward and westward with a phase speed of about 18 m s\(^{-1}\) and growing in time; and the moisture mode that is also growing in time. The convectively coupled gravity wave has a maximum growth rate for \(l = 7\), which together with the phase speed of 18 m s\(^{-1}\) agrees well with the observations.
The moisture mode is stationary when there are no mean easterlies present and slowly propagates eastward in the presence of surface mean easterlies under the influence of WISHE (not shown here). From RF07 we know that the convectively coupled Kelvin wave is unstable because of variations in CIN correlated with convection, while the moisture mode is unstable because of cloud–radiation interactions and negative gross moist stability (i.e., when the effective gross moist stability is negative).

b. Variations in vertical heating and moisture profile

We now explore how changes in the heating and moisture profiles affect the unstable modes from Fig. 2 (i.e., the convectively coupled gravity wave and the moisture mode). The heating profile is varied from top heavy to bottom heavy by varying $n$. Positive $n$ corresponds to a top-heavy heating profile and negative to a bottom-heavy one, while $n = 0$ corresponds to the unmodified heating profile of RF07. The values used are given in Fig. 1. The moisture profile is varied to yield different values of the gross moist stability for a fixed value of $n$.

Figure 3 shows convectively coupled gravity wave dispersion curves for different heating profiles. It is clear that the growth rate is highly dependent on the heating profile. The gravity mode is strongly unstable for the top-heavy heating profile while decaying for the bottom-heavy one. The wavelength of the maximum growth rate shifts as well; for the top-heavy heating profile the maximum growth rate occurs at larger planetary wave-number $l = 12$, whereas in the control case it occurs at $l = 7$. The phase speed increases from 17 m s$^{-1}$ for the bottom-heavy to 21 m s$^{-1}$ for the top-heavy heating profile. As the physical mechanism responsible for destabilizing the convectively coupled gravity wave is variations in CIN, we now vary $\chi_t$ while keeping the heating profile fixed. Figure 4 shows the dispersion curves for the top-heavy heating profile. Each curve corresponds to a different $\chi_t$ value; $\chi_t = 12$ was the value taken in a control case. The mode is unstable for a range

![Figure 2](image1.png)

**FIG. 2.** (a) Real part of phase speed and (b) imaginary part of frequency for control case (see Table 1). The solid lines show convectively coupled gravity waves, dotted lines show fast gravity waves, and dashed lines show the moisture mode.

![Figure 3](image2.png)

**FIG. 3.** Convectively coupled gravity wave for top-heavy, unmodified, and bottom-heavy heating profile.
of $\chi_t$ values (i.e., for $\chi_t \geq 1$). The shift of the maximum growth rate to larger planetary wavenumbers is notable as $\chi_t$ values become larger. Varying the moisture profile via different values of NGMS shows that the unstable convectively coupled gravity waves are not sensitive to NGMS per se. However, they are more likely to occur when NGMS is positive since positive NGMS is generally correlated with the top-heavy heating profiles (see Table 2).

Figure 5 shows the growth rate of the moisture mode when the heating profile is varied and NGMS is fixed to $G_M = -0.1$. We ignore CIN as unimportant to the development of the moisture mode. The moisture modes with the same NGMS develop regardless of the shape of the heating profile. This can be understood if we take the limit of nondimensional phase speed $|\Phi|^2 \ll 1$, which is true for the moisture modes, from Eq. (23). The dispersion relation in Eq. (23) then approximately reduces to $\Phi = i(\varepsilon - \Gamma_M)/k$ and there is no direct dependence on $\nu$ (i.e., on the vertical heating profile). This is also consistent with the results obtained using the WTG approximation (Sobel et al. 2001); in the context of our model, WTG is equivalent to setting $b = 0$ in the governing equations. We conclude that when the effective NGMS, $\Gamma_M - \varepsilon$, is less than zero, the moisture mode is unstable. Figure 6 shows the growth rate of the moisture mode when the moisture profile is varied to change the NGMS and the heating profile is bottom-heavy. As expected, more negative NGMS values result in more unstable modes. To summarize, in the real atmosphere, unstable moisture modes are expected when effective NGMS is negative, with their growth rates becoming larger when effective NGMS is more negative (see Table 2).

c. Vertical structure

Figure 7 shows the vertical structure of the eastward-moving convectively coupled gravity wave in the $x$–$z$ plane. Figures 7a and 7b show respectively the buoyancy

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$H = 1/3$</th>
<th>$H = 1/2$</th>
<th>$H = 2/3$</th>
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<tr>
<td>0.5</td>
<td>0.21</td>
<td>0.14</td>
<td>0.04</td>
</tr>
<tr>
<td>0</td>
<td>0.1</td>
<td>0</td>
<td>-0.1</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.04</td>
<td>-0.14</td>
<td>-0.21</td>
</tr>
</tbody>
</table>

FIG. 5. Growth rate of the moisture mode for fixed NGMS = $-0.1$ for different heating profiles, with $\chi_t = 0$.

FIG. 6. Growth rate of the moisture mode for different NGMS values for bottom-heavy heating profile.

FIG. 4. Convectively coupled gravity wave for different $\chi_t$ values for top-heavy vertical heating profile.

FIG. 7. Growth rate of the moisture mode for different NGMS values for bottom-heavy heating profile.
anomalies at the planetary wavenumber of maximum growth rate for the unmodified heating profile and the top-heavy vertical heating profile. For both vertical heating profiles the characteristic boomerang structure with westward-tilting contours of positive buoyancy anomaly in the low to middle troposphere and eastward tilt above is seen. This matches the findings from observations (Wheeler et al. 2000; Straub and Kiladis 2002) and from cloud-resolving numerical simulations (Peters and Bretherton 2006; Tulich et al. 2007). The westward-tilting contours of buoyancy anomaly reach a bit higher in the case of top-heavy heating profile.

Figure 8 shows the vertical structure of the moisture mode for $l = 2$ in the $x-z$ plane. It shows the buoyancy anomalies and the convective heating for the unmodified and bottom-heavy heating profile. The buoyancy anomalies are in phase with heating as the mode is stationary and there is no tilted structure.

The vertical structure of the moisture mode is very different from that of the convectively coupled gravity wave. The buoyancy anomaly contours are arbitrary in Figs. 7 and 8, but their ratio to the heating is fixed. We can use that ratio to evaluate the relative magnitude of the temperature anomaly between the two unstable

![Figure 7](image1.png)

**FIG. 7.** Heating anomaly and buoyancy perturbation for the convectively coupled gravity wave, showing (a) the unmodified heating profile and (b) the top-heavy vertical heating profile. Shading represents the heating anomaly: light shading indicates positive values and dark shading negative. Contours represent the buoyancy anomaly: solid lines indicate positive values and dashed lines negative.

![Figure 8](image2.png)

**FIG. 8.** As in Fig. 7, but for the moisture mode.
modes. It turns out that the ratio between the buoyancy and heating for the convectively coupled Kelvin waves is an order of magnitude larger than for the moisture mode, confirming that in its essence the moisture mode is a weak temperature gradient mode (Sobel et al. 2001; Sobel and Bretherton 2003).

4. Numerical model

The dynamical core of the numerical model follows the model of Raymond and Fuchs (2009). It is cast in sigma isentropic coordinates (i.e., the vertical coordinate is defined in terms of the potential temperature \( \theta \)):

\[
\sigma = (\theta - \theta_B)/(\theta_T - \theta_B),
\]

where \( \theta_T \) is the (constant) potential temperature at the top of the domain and \( \theta_B \) is the (possibly variable) potential temperature at the bottom of the domain. When \( \theta_B \) is taken to be constant, the vertical coordinate mimics isentropic coordinates. The model can be run on an \( f \) plane or on a \( \beta \) plane. It is periodic in the longitudinal direction, while in the latitudinal direction it can be either periodic or bounded by rigid walls (channel model). The top layer of the domain is a sponge layer. Its purpose is to absorb upward-moving waves, thus emulating an upward radiation boundary condition. Though the dynamics of the model are fully nonlinear we only analyze results with small amplitude flow perturbations, making the model effectively linear. For further information on the dynamics, see Raymond and Fuchs (2009).

The thermodynamics of the model is similar to that of the analytical model described above. Since the implementation of the thermodynamics is done in the context of a non-Boussinesq numerical calculation, we repeat the presentation in the new context. The perturbation precipitation is calculated from

\[
P = \alpha(W - W_m) - \mu_{\text{CIN}} \left( \frac{\rho_R C_p}{\sigma} \right) \left[ \theta'_x(d) - \theta'_x(0) \right],
\]

where \( W \) is the precipitable water and \( W_m \) is the vertical mean value of this quantity. If precipitable water is bigger than the mean precipitable water, the result is a positive rainfall perturbation and vice versa. The quantity \( \alpha \) is the moisture relaxation constant that regulates the strength of the precipitable water control on the rainfall as in the analytical model. Also, \( \theta'_x(d) \) is the perturbation saturated equivalent potential temperature above the boundary layer at height \( d \) and \( \theta'_x(0) \) is perturbation of equivalent potential temperature at the surface. Note that in the analytical model we use the moist entropy perturbation while in the numerical model we use the equivalent potential temperature perturbation. In all other aspects the precipitation closure is the same. The parameter \( \mu_{\text{CIN}} \) governs the sensitivity of precipitation rate to CIN and \( \rho_R \) is the air density of boundary layer air.

The vertical profile of the heating (i.e., the potential temperature perturbation source) takes the form

\[
S_y = B \sin(m_0 z) \exp(vz) \int_0^{\sigma_f} \eta(\sigma) \sin(m_0 z) \exp(vz) \, d\sigma,
\]

where the vertically integrated heating is

\[
B = L(P - R)/C_p,
\]

with \( R \) being the radiative cooling perturbation. As in the analytical model, it is assumed to be proportional to minus the moisture-induced precipitation rate:

\[
R = -\varepsilon\alpha(W - W_m).
\]

For our numerical simulations, \( \varepsilon \) takes the value of 0.2 as is the case in the analytical model. The parameter \( m_0 = \pi h/\Delta \), where \( h \) is the depth of the troposphere, \( \eta \) is the density in the sigma isentropic coordinate system and \( z \) is the geometrical height. The skew parameter \( n \) determines the height at which the maximum heating perturbation will occur. For \( n = 0 \) it takes form of the first baroclinic mode, as in the analytical model.

The moisture perturbation source is

\[
S_r = (E - P) \exp(-z/z_q) \int_0^1 \eta \exp(-z/z_q) \, d\sigma,
\]

where \( z_q \) is the scale height of water vapor; \( E \) stands for evaporation rate anomaly and it is parameterized to be proportional to the background zonal wind in the boundary layer. However, the simulation results presented in this paper are done with zero background wind and therefore the surface fluxes are shut off (\( E = 0 \)). As a consequence, \( \theta'_x(0) \) in Eq. (26) is equal to zero and the CIN variations are entirely due to variations in the saturated equivalent potential temperature above the boundary layer.

As noted above, there are two distinct mechanisms that are responsible for the two unstable modes of RF07. One is associated with variations of CIN, mainly variations of the saturated equivalent potential temperature or the saturated moist entropy just above the boundary layer. The other mechanism is related to small or negative GMS. For the analysis in the numerical model we
use a normalized form of the gross moist stability (NGMS) from Raymond et al. (2009) defined in the sigma isentropic coordinates as

$$\Gamma_R = -\frac{C_p}{L} \int_0^1 \eta S_\sigma \frac{\partial \theta_e}{\partial \sigma} d\sigma,$$  \hspace{1cm} (31)

where $r$ is the total mixing ratio and $S_\sigma = d\sigma/dt$ is the vertical velocity in the vertical coordinate of the numerical model. As Raymond et al. (2009) show, this form of the NGMS is essentially equivalent to Eq. (16). Furthermore, both are equivalent to the NGMS used by Raymond and Fuchs (2009) when the horizontal advection of equivalent potential temperature and moisture are neglected, as is justified in the linearized models used in this paper.

From Eq. (31) it is evident that a change in either the heating or the moisture profile will change the value of NGMS. Figures 9a and 9b show respectively the variety of heating perturbation profiles and moisture perturbation profiles used for the simulations. The values are scaled but show the height at which the maximum heating perturbations occur and how slowly the mixing ratio decreases with height. The height of the troposphere is taken to be 15 km. Note that some of the moisture profiles that are used are idealized, in the sense that such slow decreases of the water vapor with height are impossible in the earth’s atmosphere, as supersaturation would occur in the upper troposphere. We use them only to obtain a wide range of NGMS values.

As we compare the results of the numerical model with the results from the analytical model, two things should be noted about their differences. First, the dynamics of the two models are different; the analytical model uses the Boussinesq approximation, while the numerical model represents the full set of primitive equations. Second, the solution methods are different. In particular, for given values of the CIN parameter and NGMS, the analytical model calculates a dispersion relation. Thus, the growth rates and the phase speeds are calculated for a range of wavenumbers. In the numerical model the wavenumber cannot be externally specified, and a dispersion curve cannot be constructed. A numerical solution for given heating and moisture profiles gives only the most unstable mode (i.e., the peak of the dispersion curve). We find that the wavelength and the period of the resulting disturbances are sensitive to the spatial resolution of the model and are therefore not very robust. However, the phase speeds and the growth rates of modes are not sensitive to model details, and this gives us more confidence in the results for these parameters. Thus, for the purpose of comparing the numerical and analytical model results we use the phase speed and the growth rate.

Each numerical simulation is initiated by applying random density perturbations. The model is then run for 3.5 months. The initial perturbations are very small (fractional density $10^{-6}$) in order to allow the most unstable modes to emerge while maintaining linearity. For this reason, disturbances take a long time to develop and the simulations have very long run times. The analysis is restricted to periods with small amplitudes of the resulting disturbances.

5. Numerical model results

a. Two-dimensional cases

In this section we make comparisons between the analytical and numerical model results. The numerical
model is run in two-dimensional, nonrotating mode with zero background wind. The computational domain is 12,000 km in the horizontal and 22 km in the vertical, with the tropopause set to 15 km. The horizontal grid size is 100 km and 40 levels equally spaced in $\sigma$ are defined in the vertical. A time step of 100 s is used. The vertical gradient of the potential temperature $d\theta/dz$ is the same in both the troposphere and stratosphere and is taken to be 3.3 K m$^{-1}$, which corresponds to a Brunt–Väisälä frequency of $10^{-2}$ s$^{-1}$. The rest of the parameters are the same as in the analytical model, shown in Table 1. A series of simulations is performed where each simulation has a prescribed unique combination of the heating and moisture profiles shown in Fig. 9.

The first set of simulations corresponds to the case in which only CIN variations are allowed (i.e., $\alpha = 0$). In the analytical model this set of parameters produces only convectively coupled gravity waves. The same holds true for the numerical model results. Figure 10 shows the phase speed and growth rate of the most unstable convectively coupled gravity wave as a function of the height of the level of maximum heating rate. The lines indicate least squares fits to the numerical results. As in the case of the analytical model, the phase speed and growth rate both increase with the level of maximum heating. This demonstrates that convectively coupled gravity waves are favored by top-heavy convective heating profiles. The results presented in this case are insensitive to changes in the moisture profiles. This demonstrates that the convectively coupled gravity waves are insensitive to changes in the NGMS, since this quantity depends on both the environmental moisture profile (via its effect on the moist entropy profile) and on the heating profile. This invariance is explained by Eq. (26). Provided that $\alpha = 0$, only changes in the saturated equivalent potential temperature above the boundary layer should affect the characteristics of the resulting modes. Since $\theta_e$, at this level is only a function of temperature, varying the mixing ratio does not change the result.

Another set of simulations was done to determine how sensitive the wave characteristics are to changes in the parameter $\mu_{\text{CIN}}$ (results not shown). All the runs had the same top-heavy heating profile, but different values for $\mu_{\text{CIN}}$. For larger values of $\mu_{\text{CIN}}$, the resulting gravity waves had larger growth rates but the same phase speed, which agrees with the results from the analytical model.

An analogous series of simulations was performed with $\alpha \neq 0$ and $\mu_{\text{CIN}} = 0$. In cases where the NGMS is very small or negative, moisture modes develop. They are always stationary, provided that there is no background flow. The growth rates of these modes depend only on the NGMS. Thus, in two simulations initiated with different heating and moisture profiles but the same NGMS, moisture modes with the same growth rates develop. Generally, for physically realistic moisture profiles, bottom-heavy heating profiles must be invoked to produce negative NGMS.

Figure 11 gives the growth rate of the moisture mode as a function of NGMS. As the figure shows, simulations with more negative NGMS result in moisture modes with faster growth rates. There are no developing moisture modes in simulations with NGMS larger than 0.04.

Simulations initiated with a top-heavy heating profile in this case sometimes produce weakly intensifying convectively coupled gravity waves even though $\mu_{\text{CIN}} = 0$. The destabilizing mechanism for these modes is rather mysterious. Top-heavy heating profiles exhibit large
positive NGMS values and therefore the mechanism that drives moisture modes is absent. In addition the CIN control is explicitly turned off. Furthermore, when CIN control is gradually introduced, these modes evolve into normal convectively coupled gravity modes. We suspect that these modes are destabilized as a result of inaccuracies in their numerical representation, but we have been unable to track down the exact mechanism. In any rate, their growth rates are significantly smaller than the convectively coupled modes which develop when $\mu_{\text{CIN}} \neq 0$.

Simulations where both the precipitable water anomaly and the CIN parameter are allowed to control the precipitation perturbation were performed (results not shown). Bottom-heavy heating profiles resulted in unstable moisture modes while top-heavy heating profiles led to unstable convectively coupled gravity waves. More precisely, gravity modes occur when the NGMS is positive and moisture modes prevail when it is negative.

**b. Three-dimensional cases**

Three-dimensional simulations were investigated to extend the reach of our analytical model to the case of a rotating environment. We are interested in whether any fundamentally new types of unstable modes appear in three dimensions, and how the third dimension changes the characteristics of the modes we see in two dimensions.

The model is run in channel mode on an equatorial beta plane. The domain size is 12,000 km in longitudinal direction, 6000 km in the latitudinal direction, centered on the equator, and 22 km in the vertical with the tropopause at 15 km. The horizontal resolution is 100 km in both directions, there are 40 levels in the vertical and the time step is 0.1 ks. Throughout the troposphere the Brunt–Väisälä frequency is $10^{-2}$ s$^{-1}$ and $1.4 \times 10^{-2}$ s$^{-1}$ in the stratosphere. There are sponge layers on the north–south boundaries and a cyclic boundary condition is applied in the east–west direction. A sponge layer also exists at the top of the domain as in the two-dimensional simulations. For all the simulations the sea surface temperature is constant in space and time. We examine linear perturbations on a base state at rest.

To analyze the behavior of convectively coupled gravity waves in three dimensions, we perform a series of simulations with $\mu_{\text{CIN}} \neq 0$ and $\alpha = 0$. The parameters used for the three-dimensional calculation are the same as in two-dimensional case. The convectively coupled waves take on the latitudinal structure of equatorial Kelvin waves moving to the east as Fig. 12 shows. As
expected, there is no westward-moving wave component. The phase speeds and the growth rates are similar to those in the two-dimensional case. As in the two-dimensional case, the waves are most unstable and propagate most rapidly with top-heavy heating profiles. The Kelvin wave in the illustrated case has a wavelength of about 7000 km, a latitudinal scale of about 1000 km, a propagation speed of roughly $22 \text{ m s}^{-1}$, and a growth rate of approximately 0.45 day$^{-1}$. The vertical structure is similar to that obtained from the analytical model (Fig. 13).

The results for the moisture mode in three-dimensional calculations are more complex than in the two-dimensional case (see Fig. 14). The two-dimensional results are similar to those from the analytical model, with the heating and potential temperature anomalies in phase. This does not occur in the three-dimensional calculations, most likely due to the beta effect. The vertical structure of the temperature anomaly differs somewhat between the analytical and numerical results. However, consistent with WTG, the magnitude of the temperature anomaly is very small in both cases. The three-dimensional case requires further investigation.

6. Conclusions

In RF07 we demonstrated that two types of large-scale tropical modes, convectively coupled gravity waves and moisture modes, arise from two different convective forcing mechanisms, namely coherent variations in CIN correlated with convection and a moisture feedback mechanism associated with small or negative values of gross moist stability. This work is highly idealized in the sense that the vertical heating profile is specified to have a sinusoidal, first baroclinic mode shape, suggesting that the essential dynamics of these modes are determined more by the gross atmospheric response to heating rather than the detailed structure of the heating pattern.

In the present work we extend these results to the case in which the heating profile ranges from top heavy to bottom heavy, with maximum heating in the upper and lower troposphere, respectively. We demonstrate that
top-heavy heating profiles favor convectively coupled gravity waves whereas bottom-heavy profiles favor moisture modes. The results regarding convectively coupled waves are in agreement with those of Kuang (2010) and Tulich and Mapes (2010). These authors characterize their convective control as a “shallow CAPE,” but this is physically very similar to our “deep CIN.”

The moisture modes respond more to the value of the NGMS than to the shape of the heating profile per se. Bottom-heavy heating profiles favor the moisture mode simply because those profiles tend to produce small values of NGMS. However, the imposition of unrealistic thermodynamic structures that result in small NGMS even with top-heavy heating profiles produces moisture mode instability, demonstrating that NGMS is the real controlling factor.

In addition to the new analytical results, we have also made two- and three-dimensional numerical calculations on an equatorial beta plane using a numerical implementation of our analytical convective heating model. The two-dimensional calculations are mostly in agreement with the analytical results. Minor discrepancies are probably attributable to differences in model dynamics (full primitive equations versus Boussinesq approximation) and the inevitable errors that arise from modeling a continuous process on a finite grid. In three dimensions the convectively coupled gravity waves become convectively coupled equatorial Kelvin modes with their characteristic structure. The moisture modes take on a complex structure in the three-dimensional case, which we do not investigate further here.

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