Potential Vorticity Attribution and Causality

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ABSTRACT

The electrostatic analogy provides a well-known paradigm for the concept of potential vorticity (PV) attribution. Just as electric fields can be attributed to electric charges, so are localized PV anomalies thought to induce far fields of flow and temperature, at least after geostrophic adjustment. Piecewise PV inversion (PPVI) exploits this concept. Idealized examples of PPVI are discussed by selecting isolated anomalies that are inverted to yield the far field “caused” by the PV anomaly. The causality of attribution is tested in this study by seeking an unbalanced initial state containing the same PV anomaly but without a far field from which the balanced state can be attained by geostrophic adjustment. It is shown that the far field of a balanced axisymmetric PV anomaly in shallow water, without mean PV gradients, may evolve from a localized anomaly without a far field. For the more general example of the electrostatics analogy, namely a three-dimensional spherical PV anomaly, the initial state has to be nonhydrostatic and needs to exhibit a mass deficit. As this mass deficit cannot be removed during hydrostatic and geostrophic adjustment, it follows that PV attribution does not imply a causal relationship between the far field of a PV anomaly and the anomaly itself.

1. Introduction

In this study we test the concept of a causal linkage between a potential vorticity (PV) anomaly and the balanced flow that is attributed to it via PV inversion. Attribution seeks to associate to a “feature on a weather chart, such as a vorticity anomaly, a unique influence on the rest of the atmosphere” (Bishop and Thorpe 1994, henceforth BT). The concept of this attribution is encapsulated in the so-called electrostatics analogy (BT; Hoskins et al. 1985, hereafter HMR; Thorpe and Bishop 1995, hereafter TB). According to this concept “the atmosphere must be considered to act as though it were composed of particles, or ‘charged’ of PV” (BT). Just as electric charges induce an electric field, PV anomalies are thought to induce circulation and temperature patterns at a distance, via geostrophic adjustment [HMR; BT; see also Thorpe (1997) for a careful discussion of these issues].

In line with the arguments above, Davis (1992) pointed out that “the usefulness of PV as a diagnostic involves treating portions of PV in isolation.” This leads to the commonly used technique of piecewise potential vorticity inversion (PPVI), where specific PV anomalies are separated and subsequently inverted (Davis and Emanuel 1991; Davis 1992). Wu et al. (2011) state that PPVI “allows the recovery of the subset of the wind and temperature fields attributable to a discrete PV anomaly. Thus the effect of a discrete PV anomaly during the development of a weather system can be clearly revealed from the PPVI results.” PPVI has also been used to understand the influence of PV anomalies on cyclone development (Davis and Emanuel 1991), the interaction of the stratosphere with the troposphere (Hartley et al. 1998), hurricane movement (Wu and Emanuel 1995; Wu et al. 2003), and the impact of upper-level PV structures on the evolution of polar lows (Bracegirdle and Gray 2009; Wu et al. 2011). Even vertical motion has been attributed to PV anomalies (Thorpe 1997).

However, it is important to note that there are fundamental differences between electrostatics and PV attribution. Electric charge is a property of elementary
particles while PV characterizes the flow state based on wind and temperature. The electric field exerts a force on charges, while PV is not exposed to forces. Boundary conditions also introduce a certain arbitrariness in the atmospheric case. Thorpe (1997), presumably, had these points in mind when he stated that attribution may be a useful concept even “without a firm cause-and-effect having been established.” Nevertheless, as shown above, the causality of PPVI appears to be generally accepted although corresponding tests have not been performed.

It is the purpose of this study to test the causality of attribution by analyzing idealized examples, where we stipulate a flow state consisting of two PV anomalies of opposite sign and perform PPVI by removing one of them. The three-dimensional balanced state resulting from solving the quasigeostrophic (QG) PV equation exhibits a cyclonic circulation around a positive anomaly where the isentropes are lowered (raised) above (below) the anomaly. Thus, there are far-field geostrophic winds and temperature perturbations without any PV anomalies outside the sphere. The claim that these fields are induced by the PV anomaly makes sense only if this solution for an isolated “charge” can be seen as the result of hydrostatic and geostrophic adjustment starting from an unbalanced initial state without a far field.

There is evidently an analogy to the electrodynamics situation, where the impact of an electric charge propagates with the speed of light, whereas inertia, gravity, and sound waves radiate away from an unbalanced PV anomaly. This process establishes a causal relationship between the PV anomaly and its associated far field due to the adjustment process related to these waves. It has yet to be demonstrated, though, that the result of geostrophic adjustment is identical to the balanced state obtained by PV inversion with respect to this PV anomaly. This mechanism was termed “virtual” (Egger 2009) because it may be dynamical possible but does not necessarily occur in the atmosphere.

It is suggested in Egger (2009) that virtual induction works for any PV anomalies in shallow water with constant background PV. We will briefly recapitulate this barotropic case in section 2 and turn to the three-dimensional setting in sections 3 and 4. We summarize and discuss our results in section 5.

2. Circular anomaly in shallow water

We assume shallow-water flow on an $f$ plane with constant Coriolis parameter $f$. The calculations will be restricted to the linear case where the shallow-water equations yield PV conservation:

$$ q' = \left( \zeta' - \frac{f}{H} h' \right)_r = 0, \quad (1) $$

with PV perturbation $q'$, relative vorticity $\zeta'$, mean depth $H$, and height deviation $h'$. Subscripts denote partial derivatives. Two axisymmetric PV anomalies of radius $r_1$ are assumed where $q' = q_0$ is constant within each circle and $q' = 0$ outside. The two perturbations of $q'$ are opposite in sign. The related balanced flow is assumed to be geostrophic [$v = \left( \tilde{g} f / H \right) h'$ with reduced gravity $\tilde{g}$]. PPVI removes, say, the anticyclonic anomaly and inverts the remaining cyclonic one. Note that such monopoles of PV are not acceptable on the sphere or within a closed domain without net fluxes of PV at the boundary (Haynes and McIntyre 1987) but can be admitted on an infinite $f$ plane in shallow water. Hence we can define the streamfunction perturbation $\psi' = \tilde{g} h'/f$ that solves

$$ \frac{1}{r} (r \psi')_r - \frac{f^2}{\tilde{g} H} \psi' = q'. \quad (2) $$

This solution has a far-field cyclonic circulation and a minimum of $\psi'$ at the center of the anomaly (see Fig. 1). The standard interpretation of PPVI attributes this far field to the cyclonic PV anomaly and concludes that the anticyclonic anomaly will be advected by the circulation.
induced by the positive anomaly. Let us test this proposition by looking for an unbalanced initial state with an identical PV distribution.

PV conservation (1) suggests the initial height perturbation for an unbalanced quiescent initial state \( (\zeta' = 0) \):

\[
h_0' = -\frac{H}{f} q_a, \quad \text{for } r \leq r_1, \tag{3}
\]

and \( h_0' = 0 \) outside the circle. Negative total heights are avoided by requiring \( q_a < f \). This initial state serves our purpose provided the mass balance is correct. This can be demonstrated by integrating (2) over the flow domain

\[
2\pi \frac{rgh'}{f} = 2\pi \int_0^r \frac{h'}{f} dr' = -\pi \frac{f}{H} h_0'r_1^2 \tag{4}
\]

to obtain a PV balance. This relation establishes also that the total mass of the initial state (3) is the same as that of the final balanced state provided there is no circulation at infinity \( [rv]' = (rgh'/f)' = 0 \).

Hence, integration over time with the initial state (3) results in the balanced flow shown in Fig. 1. This shows that indeed the initial state (3) without a far field evolves into a final state with the balanced far field, where the far field is caused by the specification of the initial height perturbation (3) in an initial state at rest. Virtual induction performs perfectly well in this case and demonstrates that the far field is induced by the height anomaly equivalent to a cyclonic PV anomaly.

The situation is less obvious if mean PV gradients exist (e.g., on the \( \beta \) plane). PV anomalies are generated in this case because of horizontal displacements during the process of geostrophic adjustment. That is, “charges” of PV are created and (1) is no longer valid (see also comments by BT on \( \beta \) effects). It is difficult if not impossible to find an unbalanced initial state without a far field in this case.

3. Spherical anomaly in 3D

Let us now consider the situation investigated by BT, who assumed a constant QG PV anomaly \( q_a \) in the sphere \( \tilde{r} \leq r_1 \) [where \( \tilde{r} = \sqrt{r^2 + \tilde{z}^2}, \tilde{z} = (N_0/\tilde{f}) \tilde{z}, \) and \( r_0 = \sqrt{x^2 + y^2} \)] with \( q_a = 0 \) for \( \tilde{r} > r_1 \). The flow domain is infinite. For PPVI we choose two such anomalies of opposite sign as before and invert the cyclonic anomaly while removing the anticyclonic counterpart. The QG PV is given by

\[
q^* = \left( \nabla^2 + \frac{\partial^2}{\partial \tilde{z}^2} \right) \psi', \tag{5}
\]

where the Brunt–Väisälä频率 \( N_0 \) and mean density are assumed constant. The anomaly is embedded in a stably stratified \( f \)-plane atmosphere at rest. These assumptions simplify the analysis but, as demonstrated by BT, are not necessary for the electrostatic analogy to be applicable.

The associated streamfunction is

\[
\psi' = -q_a \left( \frac{r_1^2}{2} - \frac{r^2}{6} \right) \quad \text{for } \tilde{r} \leq r_1 \tag{6}
\]

\[
\psi' = -q_a \frac{r_1^3}{2\tilde{f}} \quad \text{for } \tilde{r} > r_1. \tag{7}
\]

Note that a PV monopole is admissible in this case, because the perturbations extend all the way to infinity. As before, the standard view is to attribute the far field of the QG solution with its potential temperature perturbation and winds to the cyclonic PV anomaly. For example, Holton (2004) states with respect to this situation that “a potential vorticity anomaly at one level yields a nonzero geopotential anomaly (and hence nonzero geostrophic winds) at other levels.” Given these fields, causal attribution via PPVI asserts that the motion of the anticyclonic sphere is caused by the positive PV anomaly.

Several aspects of attribution can be clarified in this case even without turning to virtual induction. The horizontal integral over \( \psi' \)

\[
\int_0^{2\pi} \int_0^{r_1} \psi' r dr d\phi = -2\pi \frac{r_1^3}{3} q_a \quad \text{for } \tilde{z} > r_1 \tag{8}
\]

is negative and unbounded for \( r \to \infty \), where \( \psi' \) is equivalent to the hydrostatic pressure perturbation. It follows that there is no global mass anomaly because the hydrostatic pressure perturbation is symmetric with respect to the plane \( \tilde{z} = 0 \).

The horizontal integral of \( \psi'_z \) for \( \tilde{z} > r_1 \)

\[
\int_0^{2\pi} \int_0^{r_1} \psi'_z r dr d\phi = \sgn(\tilde{z}) \frac{N_0^2}{\tilde{f}^2} \frac{2\pi r_1^3 q_a}{\tilde{r}^3} \tag{9}
\]

does not depend on \( \tilde{z} \) except for the sign, where \( \psi'_z = (g/\tilde{f}) (\theta'/\tilde{B}) \). Attribution has to assume that the PV anomaly caused a mean warming (cooling) above (below) the anomaly. Diabatic contributions, yielding a mean warming (cooling), are excluded in the adjustment process and would in fact create PV. Hence, such a mean warming (cooling) can be caused only by mean descent (ascent) above (below) the anomaly. This would imply a concentration of mass in the domain. However, (8) does not show such a concentration of mass.

It is also helpful to establish a PV balance within the domain, in analogy to (4). Integrating (5) over the domain \( 0 \leq r < \infty, -D \leq \tilde{z} \leq D \), with \( D > r_1 \), we obtain
The first term on the left-hand side of (10) is zero if there is no circulation \( (r u' = r \psi') \) at \( r \to \infty \). In that case, the difference of the horizontal integrals of \( \psi' \) at \( \hat{z} = \pm D \) has to balance the integrated PV. Indeed, the QG solution (6) and (7) has no circulation at \( r \to \infty \) and the difference between the area-integrated potential temperature perturbations above and below the sphere balances the PV anomaly. Attribution claims that these mean temperature anomalies are induced by the PV anomaly.

It is of interest with respect to more realistic cases of PPVII to admit rigid lids above and below the anomaly. We can obtain a solution with vanishing potential temperature anomalies above and below the sphere balances the PV. The linear compressible equation (11)–(14) is sometimes called the (compressible) pseudo-PV, with \( N^2 = g/\bar{\alpha} \). The linear PV anomaly corresponding to (11)–(14) is

\[
q' = \xi' - \frac{f p' \bar{\alpha}}{p} + \frac{\theta' \bar{\alpha}}{\bar{\theta}}
\]

with

\[
q_i' + w' \bar{\theta}_z = 0,
\]

\[
\rho'_i + \frac{\bar{p}}{r} (ru')_r + (pw')_z = 0,
\]

and a thermodynamic equation

\[
\theta'_i + w' \bar{\theta}_z = 0,
\]

with

\[
\theta' = \frac{p' \bar{\alpha}}{\gamma \bar{p}} - \frac{\theta' \bar{\alpha}}{\bar{\theta}}
\]

where \( \gamma = c_p/c_v \) is the ratio of the specific heat at constant pressure and constant volume, respectively. By combining (11)–(14) we obtain the relation

\[
\tilde{q}'_i = 0,
\]

where

\[
\tilde{q}' = \xi' - \frac{f p' \bar{\alpha}}{p} + \frac{\theta' \bar{\alpha}}{\bar{\theta}}
\]

is sometimes called the (compressible) pseudo-PV, with \( N^2 = g/\bar{\alpha} \). The linear PV anomaly corresponding to (11)–(14) is

\[
q' = \xi' - \frac{f p' \bar{\alpha}}{p} + \frac{\theta' \bar{\alpha}}{\bar{\theta}}
\]

with

\[
q_i' + w' [-f(\ln \bar{p})_z + f(\ln \bar{\alpha})_z] = 0.
\]

Assuming hydrostatic and geostrophic balance, we are able to invert spherical anomalies both of \( q' \) and \( \tilde{q}' \). The related fields are presented in Figs. 2a,b and Figs. 2c,d, respectively (see the appendix for a more detailed model description). The pressure fields are quite similar with minima below the center of the anomaly. The vertical asymmetry is larger for the PV anomaly (Fig. 2a) compared with its pseudo-PV counterpart (Fig. 2c). Isentropes are raised below and lowered above the anomaly in both cases and the circulation is cyclonic of almost identical strength.

Generally, there is an equivalence between QG PV and pseudo-PV. It is easy to show that the density term in (16) and (17) drops out if we remove the tendency of density \( \rho'_i \) in (12). Using \( \psi' \bar{p} = p' \), we find

\[
\tilde{q}' = \frac{1}{r} (r \psi')_r + \frac{f^2}{\bar{p} \bar{\alpha}} (p \psi'_z) = 0
\]

so that \( \tilde{q}' \) equals the QG PV in the basic state related to (11)–(14).
It is evident from (15) and (18) that $q'$ is locally constant in time, whereas $q'$ varies because of vertical displacements across surfaces of constant basic state PV. Thus, if the balanced solution of a spherical PV anomaly $q'$ contains a far field of potential temperature, the relation

$$
\tilde{q}' = q' + \frac{f\theta'}{\bar{p}} \left( \frac{\bar{p}}{\bar{\sigma}_v} \right)_z
$$

(20) immediately shows that we cannot find an unbalanced initial perturbation for $q'$ without a far field in $\tilde{q}'$. However, a far field of $\tilde{q}'$ implies that a far field of $\theta'$ must exist in an initial quiescent state where $q'$ is to be confined to the spherical PV anomaly. It is thus impossible to obtain the balanced solution starting from a localized perturbation of $q'$ only. Hence, induction is not feasible for $q'$. Motivated by the equivalence of $\tilde{q}'$ and its QG counterpart, we circumvent this problem by replacing the spherical PV anomaly by a localized anomaly of pseudo-PV $\tilde{q}_0'$. Because of its local conservation in time, we are able to identify an unbalanced initial state that has the same pseudo-PV as the final balanced state in Figs. 2c and 2d.

The appropriate unbalanced initially quiescent ($\zeta'' = 0$) state without a far field for $\tilde{q}'$ must follow from (16):
\[
\dot{q}_0' = \frac{f}{\rho} \left( -\frac{p_0'}{\rho} - \frac{H \partial p_0'}{\kappa \partial z} + \frac{1}{\kappa g} \frac{\partial p_0'}{\partial z} \right), \quad (21)
\]

where subscript 0 indicates the initial state. Using the hydrostatic assumption, (21) reduces to

\[
\dot{q}_0' = \frac{fH}{g \kappa} \frac{\partial}{\partial z} \left( \frac{1}{\rho} \frac{\partial p_0'}{\partial z} \right), \quad (22)
\]

which has to be solved for a spherical anomaly of pseudo-PV in the absence of a far-field pressure perturbation.

It is obvious that (22) cannot have a consistent solution without a field. Therefore, we have to turn to a nonhydrostatic initial state with \( \theta' = 0 \) everywhere, yielding

\[
\dot{q}_0' = -\frac{f \rho_0'}{\rho}, \quad (23)
\]

\[
p_0' = \gamma \rho_0' R T \quad (24)
\]

for \( \hat{r} \leq r_1 \) and zero perturbations for \( \hat{r} > r_1 \). Similar to the barotropic case discussed in section 2, \( \dot{q}_0' < f \) is required. Thus, the initial state comprises a density deficit within the sphere, which has no far field, and both formulations of potential vorticity, \( \dot{q}' \) and \( q' \), are identical. This initial state corresponds to the barotropic initial state (3), which is, however, hydrostatic.

As mass has to be conserved during the adjustment process, it follows immediately from (23) that one cannot obtain the QG solution (6) and (7) via virtual induction. One has to prescribe fluxes of mass into the domain during geostrophic and hydrostatic adjustment in order to balance the initial mass deficit. However, even if we were able to formulate appropriate boundary conditions, this inflow would be a main cause for establishing the final balanced state. Thus, the balanced state cannot be attributed causally to the PV anomaly because the PV anomaly does not, per se, determine the mass fluxes. In fact, the mass supply for the QG case discussed in section 3 has to come from infinity, which is physically impossible and thus revokes the causality in PV attribution.

The QG case assumes an infinite domain. This situation is, of course, highly unusual in PPVI. Instead, there are nearly always vertical boundaries. Lateral boundaries are quite often assumed as well. Two examples of balanced solutions with vertical and lateral boundaries are shown in Fig. 3. For the case with a rigid lid at \( z = \pm D \), the hydrostatic pressure deficit within the domain is clearly discernible and the circulation is cyclonic throughout the entire domain (Figs. 3a,b). For an entirely closed domain, the difference of the hydrostatic pressures at \( z = \pm D \) balances the initial mass deficit (Figs. 3c,d).

However, one should be cautious about virtual induction in the latter case, except if the boundary conditions allow inertia, gravity, and sound waves to leave the domain. If the waves are not allowed to leave the domain, then they continuously reflect from the walls, and this prevents the flow from evolving into a balanced state. Including damping to eliminate these waves internally introduces a mechanism that acts to rearrange the PV within the domain and the final PV anomaly will thus no longer be constrained to the initial spherical anomaly in this case. Either way, causal attribution is not feasible in the presence of walls or without them.

5. Concluding remarks

The main concern of this note is to examine the causality of PV attribution in both shallow water and a three-dimensional atmosphere. The causality within attribution is the underlying principle of PPVI, which tries to determine the influence of one PV anomaly on another PV anomaly as well as on the rest of the atmosphere. The claim of causality is tested in a linear framework by performing virtual induction, in which one has to determine an unbalanced initial state without a far field that evolves, via adjustment, toward the balanced flow obtained by inversion.

Such an initial state is found for shallow-water flow in the absence of mean PV gradients. It is important to note that this initial state is specified via its mass and wind fields, PV results from that. Hence, although virtual induction is, in principle, feasible for the barotropic case, the initial state cannot be uniquely attributed to PV, rendering a causal linkage to PV alone equivocal.

In the three-dimensional case one has to consider nonhydrostatic initial conditions with potential temperature perturbations vanishing everywhere. This three-dimensional initial state has a mass deficit equivalent to an unbalanced (compressible) pseudo-PV perturbation. However, the balanced state related to the QG PV anomaly has no mass perturbation. Hence, virtual induction is impossible in this case and the QG PV anomaly cannot be seen as the cause of the balanced flow state. In fact, attribution for the QG case implies unphysical mass fluxes from infinity. The QG solution with rigid lids (\( \theta' = 0 \)) imposed at \( z = \pm D \) cannot be attributed to a finite PV anomaly either, because the solution has unbounded angular momentum. It follows, that causal attribution related to the three-dimensional PV anomaly alone must be rejected.

We would like to emphasize that it is not the intention of this study to generally undermine the usefulness of PV
thinking as a conceptualization of large-scale atmospheric flow. However, we strongly argue for a more careful usage of words and arguments in connection with PV attribution. Whereas many insights can be gained via a conceptual view of the atmosphere related to PV thinking, one has to be aware that there is no causal relationship between the PV anomaly and its associated far field. The connection between a PV anomaly and its far field is established via an inversion under the assumption of hydrostatic and geostrophic balance, implying that the far-field flow and the PV anomaly are inherently linked to each other, but without a causal relationship being established.

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APPENDIX

Model and Boundary Conditions

The numerical model solves the discretized versions of (A1) and (A2) on an array of 100 \times 100 grid points. The half depth $D$ and radial extent $r_D$ of the domain are 10 and 5000 km, respectively. The boundary conditions used for Fig. 2 are $p' = 0$ at $r = r_D$. Figure 3 shows...
the results for $\theta' = 0$ at $\pm D$ and $\rho' = 0$ (Figs. 3a,b) and $p'_r = 0$ (Figs. 3c,d), respectively. The pressure field for the balanced $q'$ and $\bar{q}'$ can be inferred from

\[
q' = \frac{1}{\rho f} (rp'_r)_r + \frac{f}{g \rho \kappa} \left[ \frac{p'}{g/\kappa} (\kappa + 1) + p'_x (\kappa + 2) + H p''_z \right],
\]

(A1)

\[
\bar{q}' = \frac{1}{\rho f} (rp'_r)_r + \frac{f H}{g \kappa} \left( \frac{p'_x}{\rho} \right)_x,
\]

(A2)

respectively. The constants used are outlined in Table A1.

### Table A1. Constants used in this study.

<table>
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<th>$T$</th>
<th>255 K</th>
<th>$g$</th>
<th>9.8 m s$^{-2}$</th>
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<td>$c_p$</td>
<td>1024 J kg$^{-1}$ K$^{-1}$</td>
</tr>
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<td>$c_v$</td>
<td>$c_p - R$</td>
<td>$\gamma$</td>
<td>$c_p$</td>
</tr>
<tr>
<td>$\kappa$</td>
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<tr>
<td>$r_D$</td>
<td>$5 \times 10^6$ m</td>
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</tbody>
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the references for $\theta' = 0$ at $\pm D$ and $\rho' = 0$ (Figs. 3a,b) and $p'_r = 0$ (Figs. 3c,d), respectively. The pressure field for the balanced $q'$ and $\bar{q}'$ can be inferred from

\[
q' = \frac{1}{\rho f} (rp'_r)_r + \frac{f}{g \rho \kappa} \left[ \frac{p'}{g/\kappa} (\kappa + 1) + p'_x (\kappa + 2) + H p''_z \right],
\]

(A1)

\[
\bar{q}' = \frac{1}{\rho f} (rp'_r)_r + \frac{f H}{g \kappa} \left( \frac{p'_x}{\rho} \right)_x,
\]

(A2)

respectively. The constants used are outlined in Table A1.

### References


