Origin of Non-Gaussian Regimes and Predictability in an Atmospheric Model

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ABSTRACT

This study details properties of non-Gaussian regimes and state-dependent ensemble spreads of trajectories in a reduced phase space of an idealized three-level quasigeostrophic (QG3) dynamical model. Methodologically, experiments using two empirical stochastic models of the QG3 time series disentangle the causes of state-dependent persistence properties and nonuniform self-forecast skill of the QG3 model. One reduced model is a standard linear inverse model (LIM) forced by state-independent, additive noise. This model has a linear deterministic operator resulting in a phase-space velocity field with uniform divergence. The other, more general nonlinear stochastic model (NSM) includes a nonlinear propagator and is driven by state-dependent, multiplicative noise. This NSM is found to capture well the full QG3 model trajectory behavior in the reduced phase space, including the non-Gaussian features of the QG3 probability density function and phase-space distribution of the trajectory spreading rates.

Two versions of the NSM—one with a LIM-based drift tensor and QG3-derived multiplicative noise and another with the QG3-derived drift tensor and additive noise—allow the authors to determine relative contributions of the mean drift and multiplicative noise to non-Gaussian regimes and predictability in the QG3 model. In particular, while the regimes arise predominantly because of the nonlinear component of the mean phase-space tendencies, relative predictability of the regimes depends on both the phase-space structure of multiplicative noise and the degree of local convergence of mean phase-space tendencies.

1. Introduction

At the leading order, one may describe atmospheric low-frequency variability in terms of a damped linear stochastic system:

$$dx = Lx dt + RdW.$$  \hspace{1cm} (1)

Here the tendency $dx$ of the resolved modes of atmospheric state vector $x$ over the time interval $dt$ is given by the sum of a linear deterministic component involving a constant matrix $L$ and the additive noise term $RdW$ (with state-independent amplitude $R$, which parameterizes the effect of spatially correlated unresolved modes of the underlying atmospheric system, and $W$ denotes a Wiener process). The linear propagator and the noise covariance matrix in (1) can be derived empirically from the observed evolution of an appropriate reduced climate state vector. The resulting empirical models, known as linear inverse models (LIMs), are widely used for analysis of atmospheric behavior because of their relative simplicity and ability to emulate atmospheric low-frequency variability quite well (Penland 1989, 1996; Penland and Ghil 1993; Penland and Matrosova 1994; Berner 2005; Branstator and Berner 2005, among others). LIMs driven by Gaussian noise cannot, by construction, reproduce non-Gaussian distributions of atmospheric states; however, this property allows them to be used as a statistical benchmark to differentiate between statistically linear versus statistically nonlinear behavior (Branstator and Berner 2005; Kravtsov et al. 2009; Kondrashov et al. 2010; Peters et al. 2012). The non-Gaussian distributions of atmospheric states due to the latter nonlinear processes are commonly referred to as atmospheric regimes.

Among the numerous previous studies that have addressed the origin of non-Gaussian regimes within the phase space occupied by the atmospheric state vector, two predominant conceptual theories have been formulated

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to explain this behavior. In one of these theories, the
regimes arise because of the nonlinear deterministic
dynamics of the atmosphere (Legras and Ghil 1985;
Marshall and Molteni 1993; Ghil and Robertson 2002,
among others); this situation is akin to the motion of
a particle agitated by additive Gaussian white noise in
a nonlinear system with multiple attractors, and is
described by the Ito\' stochastic differential equation
(SDE) of the form analogous to (1),

\[ dx = h(x, t) dt + R dW, \]

but in which the deterministic part of the model\'s opera-
tor \( h(x, t) \) is nonlinear. The latter nonlinearity in this case
is the only cause of deviations from Gaussianity in the
distribution of the resolved variable \( x \).

Another view of atmospheric regime behavior prin-
cipally involves interaction of fast processes, such as those
associated with synoptic eddies, with low-frequency
modes of the atmospheric system (Robinson 1996, 2000;
Feldstein and Lee 1998; Lorenz and Hartmann 2001,
2003; Berner 2005; Sura et al. 2005; Rennert and Wallace
2009; Peters et al. 2012). This behavior is conceptualized
by the Ito\' SDE of the form

\[ dx = h(x, t) dt + G(x, t) dW, \]

which is analogous to (2) except for the dependence of the
noise term \( G(x, t) dW \) on the model state vector \( x \); this
so-called multiplicative noise term simulates interactions
between the explicitly resolved state variable \( x \) and faster
processes approximated by Gaussian-distributed noise
\( dW \). Equation (3) illustrates the possibility of non-Gauss-
ian regimes arising from a combination of deterministic
dynamics and multiplicative noise, or even exclusively from
the multiplicative noise in case the deterministic part of
the model\'s operator is linear; that is, \( h(x, t) = Lx \) (Sura
et al. 2005).

The regimes in observations and model simulations
often exhibit anomalous persistence characteristics
(Kondrashov et al. 2004, 2006; Kravtsov et al. 2009;
Peters et al. 2012). Thus it has been long theorized that
regimes may be a source for improvements in medium-
range atmospheric predictability; this underscores their
practical significance. Peters et al. (2012) showed that
persistent non-Gaussian regimes of a realistic three-
layer quasigeostrophic model (Marshall and Molteni
1993)—hereafter, the QG3 model—are connected to
phase-space regions of enhanced predictability. These
authors also provided preliminary evidence that the degree
to which the QG3 regimes are predictable is dependent
on a combination of nonlinear deterministic dynamics and
the phase-space topology of the multiplicative noise; thus,
both terms on the right-hand side of (3) are important.

The goal of the present study is to analyze the be-
havior of a low-order statistical model of the form (3),
empirically derived from the output of a long QG3 simula-
tion in order to develop a better understanding of
the degree to which deterministic dynamics and stochastic
multiplicative noise contribute to the state dependence of
predictability in this reduced model. Note that the time
series produced by the model considered in this paper are
statistically stationary, so that \( h \) and \( G \) from (3) only de-
depend on time implicitly, via time dependence of \( x \). Our
results provide a refinement on the conceptual un-
derstanding of the reduced phase-space behavior of the
parent dynamical model established by Peters et al.
(2012). In section 2, we outline our methodology for
constructing statistical models based on the QG3 out-
put, review statistical properties of trajectories in the
reduced phase space of the dynamical QG3 model
(Peters et al. 2012), and evaluate how well the empiri-
cal models reproduce the QG3 statistics. Section 3
describes the experiments with our statistical models
designed to pinpoint the origin of the QG3 regimes, as
well as the sources of their predictability; the results
are discussed in section 4. We summarize our findings,
put them into the context of past studies, and outline
avenues for future work in section 5.

2. Statistics of QG3 trajectories and their
simulation using empirical models

We analyzed daily output of the 1 000 000-day-long
simulation of the QG3 model described by Peters et al.
(2012). The QG3 model has been widely used for studies
of midlatitude atmospheric dynamics because of its fairly
realistic low-frequency variability (Marshall and Molteni
1993; D\'Andrea and Vautard 2001; D\'Andrea 2002;
Kondrashov et al. 2004, 2006; Selten and Branstator
2004, among others). The following analysis considers QG3
trajectories in the phase space of the leading two empiri-
cal orthogonal functions (EOFs) of the model\’s 200-hPa
streamfunction; the patterns of these EOFs (Peters et al.
2012) resemble observed teleconnection patterns.

a. Empirical model construction

We constructed two types of empirical stochastic
models. The coefficients of a simpler LIM [see (1)] were
computed, in our two-dimensional subspace, via multiple
linear regression of the QG3 model\’s daily princi-
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computed, in our two-dimensional subspace, via multiple
linear regression of the QG3 model\’s daily principal
component (PC) tendencies onto these PCs them-
sevles. Hence, \( L = (x_{t-8}^T, x_{t-8}) / (x_{t-8}^T, x_{t-8}) \), and \( R^T R = [1/(N - 1)]
[x_t - Lx_{t-8}]^T [x_t - Lx_{t-8}], \) where \( N \) is the length of the
time series, and the time step \( \Delta t = 1 \) day.
In subsequent simulations using LIM, the stochastic forcing was represented by additive spatially correlated Gaussian white noise, whose covariance structure was derived from that of the regression residuals. Note that linearity (in \(x\)) of the deterministic operator in the LIM equation (1) guarantees that (i) the resulting \(x\) distributions are also Gaussian and (ii) the LIM-simulated mean phase-space tendencies have constant divergence throughout the EOF 1–EOF 2 phase space. We will use property (i) to highlight non-Gaussian regime regions, as well as signatures of nonlinear dynamics within the trajectory distribution produced by the QG3 model (Branstator and Berner 2005; Kravtsov et al. 2009; Kondrashov et al. 2010). Furthermore, Peters et al. (2012) suggested that nonuniform divergence of mean phase-space tendencies plays an important role in the state-dependence of QG3 predictability; property (ii) of the LIM-based phase-space tendencies will allow us to determine quantitative ramifications of this effect (see section 4).

To introduce a more complete nonlinear stochastic model (NSM) [see (3)] of the QG3 trajectories, we first consider a general form of the Fokker–Planck equation (FPE; Gardiner 1985; Branstator and Berner 2005; Sura et al. 2005; Peters et al. 2012):

\[
\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x_i} [A_i(x, t) P(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} [B_{ij}(x, t) P(x, t)].
\]  

(4)

The FPE (4) describes the evolution of the probability density function (PDF) \(P(x, t)\) in the chosen phase space of the QG3 model. Here \(A_i(x, t)\) represents the \(i\)th component of the effective drift, and \(B_{ij}(x, t)\) are the components of the so-called diffusion matrix \((i = 1, 2; j = 1, 2)\); the repeating indices within each term in (4) imply summation. For a statistically steady system, \(A\) and \(B\) are temporally invariant and may be estimated from a long sample of model data (see, e.g., Berner 2005) as

\[
A_i(x) = \lim_{\tau \to 0} \frac{1}{N} \frac{1}{\tau} \sum_{n=1}^{N} \left[ \bar{x}_i^{(n)}(t + \tau) - \bar{x}_i^{(n)}(t) \right] |_{\bar{x}_i^{(n)}(t) = x_i},
\]

(5)

\[
B_{ij}(x) = \lim_{\tau \to 0} \frac{1}{2} \frac{1}{N} \sum_{\mu=1}^{N} \left[ \bar{x}_j^{(n)}(t + \tau) - \bar{x}_j^{(n)}(t) \right] \left[ \bar{x}_i^{(n)}(t + \tau) - \bar{x}_i^{(n)}(t) \right] |_{\bar{x}_i^{(n)}(t) = x_i, \bar{x}_j^{(n)}(t) = x_j}.
\]

(6)

Here \(x = (x_1, x_2)\) is the position vector in our two-dimensional phase space, and \(\bar{x}_i^{(n)}(t)\) \((i = 1, 2)\) denotes one of \(N\) QG3 model’s trajectories passing, at time \(t\), through a given point \(x\); (5) and (6) thus contain ensemble averaging over all such trajectories. We normalized the leading pair of QG3 PCs to unit variance, partitioned the resulting EOF 1–EOF 2 subspace into square grid boxes with the size of \(\frac{1}{4} \times \frac{1}{4}\) and computed the values of drift vectors and diffusion tensors for each grid box from (5) and (6) using \(\tau = 1\) day. In doing so, we did not correct for the finite-difference error of the order \(\tau\) (Sura and Barsugli 2002; Berner 2003), assuming this error to be small. Berner (2005) did confirm relative smallness of such an error in her study of nonlinear behavior simulated in an atmospheric general circulation model.

Note also that the estimates of FPE coefficients do depend on the value of \(\tau\) selected for their computation. In particular, as \(\tau\) increases, these coefficients tend toward those of the LIM; see Branstator and Berner (2005) for a thorough discussion of this issue. Since we are concerned with investigating nonlinear statistics that influence model trajectory behavior and resulting non-Gaussian distributions at intermediate time scales of 3–10 days, we have chosen our sampling interval to be \(\tau = 1\) day. The rationale for this choice was that smaller values of \(\tau\) emphasize shorter-term nonlinear behavior, while larger values bias the empirical model toward statistical linearity.

The topology of the QG3 drift vectors and diffusion tensor in EOF 1–EOF 2 plane is shown in Fig. 1. Figure 2 displays these quantities computed in the same manner for a 1 000 000-day LIM simulation. While the LIM captures the general structure of the QG3 drift vectors, there are notable quantitative differences between the two fields, indicating substantial nonlinear component in QG3 drift field. Furthermore, diagonal components of the QG3 diffusion tensor exhibit considerable variability throughout the phase space. In contrast, the LIM, with its state-independent additive noise, produces behavior characterized by the spatially constant diffusion field in each direction; the nontrivial spatial structure of the off-diagonal component of the diffusion tensor is due to our usage of spatially correlated white noise in the LIM. Note that nonuniformities in the distribution of LIM’s \(B_{11}\) and \(B_{22}\) give an idea of the finite-difference error introduced by our computation procedure (see previous paragraph); these nonuniformities are, indeed, small.
The set of coefficients for the NSM in (3) can be computed from the coefficients of the FPE using the following relationships (Gardiner 1985; Berner 2005):

\[ A(x) = h(x), \]
\[ B(x) = G(x)G(x)^T. \]

Note that since \( B \) is invariant to orthogonal transformations of \( G, G \) cannot be uniquely determined from \( B \). It is, however, sufficient for the purposes of this experiment to select any \( G \) that satisfies the above relationship; see Berner (2005) for further discussion.

b. Simulation of QG3 properties by empirical models

1) NON-GAUSSIAN REGIMES

Figures 3a and 3b show the PDFs of 1 000 000-day-long trajectories from QG3 and NSM (section 2a) simulations, respectively. The two PDFs are nearly identical and both exhibit non-Gaussian features, which is immediately obvious from the apparent skewness of these distributions. To further classify the PDF’s non-Gaussianity, we used the procedure developed by Kravtsov et al. (2009). This procedure identifies regions of the model phase space—the regimes—where the QG3 states exhibit higher probability of persistence relative to the Gaussian-distributed LIM-simulated states; this procedure was also used in Peters et al. (2012). In particular, we computed the ratio of the PDF based on low-pass filtered QG3 trajectories to the PDF based on raw QG3 trajectories in the EOF 1–EOF 2 subspace. We also computed the same quantity for 100 realizations of surrogate trajectories produced by the LIM and computed the 95th percentile of the resulting synthetic PDF ratios. The regime regions were defined as those in which the QG3 PDF ratio exceeds the 95th percentile of the LIM-based PDF ratios. These regions thus arise from statistically nonlinear dynamics that cannot be replicated by LIM. The distribution of positive values of the difference between the QG3 PDF ratio and the 95th percentile of the LIM PDF ratios is shown by the shading in Figs. 3c and 3d for the QG3 model and NSM, respectively. Three distinct regimes are identified; the locations of the corresponding regimes produced by QG3 and NSM evolution are nearly identical. Peters et al. (2012) labeled these regimes as the AO\(^+\) (regime 1), AO\(^-\) (regime 2),
and NAO (regime 3) and showed that two of these regimes—regimes 1 and 2—exhibit enhanced predictability evident in lower spreading rates of the QG3 trajectories near regime regions.

2) TEMPORAL CHARACTERISTICS OF THE NSM

The match between the QG3 and NSM temporal characteristics is less impressive than that in the distributions of QG3 and NSM-simulated quantities. In particular, the NSM-simulated synthetic PC 1 and PC 2 both decorrelate more slowly than their QG3 counterparts (see Fig. 4, left). Berner (2005) noted that the shape of the synthetic PCs autocorrelation function is sensitive to the value of $\tau$ used in computing the coefficients of NSM via (5)–(8), with larger values of $\tau$ approximating long-time behavior better than the small value of 1 day used here. In the latter case, however, the nonlinear properties of the trajectories turn out to be diluted by excessive averaging (Berner 2005; Branstator and Berner 2005). Since we are primarily interested in investigating nonlinear phenomena, we choose to retain the small value of $\tau$ to achieve accuracy in representation of the nonlinear effects in the NSM model at the expense of the positive bias in the time scale.

To quantify the relative predictability of regions within the model’s reduced phase space, we computed the root-mean-square distance (RMSD) between the states within an evolving ensemble of trajectories from the long model sample as follows:

$$\text{RMSD}(\mathbf{x}, \tau) = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}^{(n)}(t + \tau) - \bar{x}(t + \tau))^2}$$

Here the notations and ensemble averaging procedure are analogous to those used in (5) and (6) for the estimation of the drift and diffusion coefficients, while the mean position $\bar{x}(t + \tau)$ of the model states originally emanating from point $\mathbf{x}$ of the phase space in $\tau$ days is given by

$$\bar{x}(t + \tau) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}^{(n)}(t + \tau) \bigg|_{\mathbf{x}^{(n)}(t) = \mathbf{x}}.$$  

The RMSD quantity defined in (9) and (10) measures the degree of trajectory spreading at different lead times. Similarly to the decay rate of the autocorrelation function, the spreading rates of NSM trajectories averaged

![Fig. 2. As in Fig. 1, but for the simulation using empirical LIM fit to the QG3 output.](image-url)
over the entire EOF 1–EOF 2 phase space for each $\tau$ (gray solid line in Fig. 4b) are also slower than the corresponding QG3 quantities (black solid line). The standard deviation—over the model’s phase space—of the NSM spreading rates (gray dashed line) is comparable, albeit slightly less than that of the QG3-based spreading rates (black dashed line). This quantity is a measure of how distinguishable predictable (low RMSD) model
states are from unpredictable (high RMSD) model states.

Figure 5 compares the phase-space distribution of RMSD for the NSM and the QG3 model at lead times of 1, 5, and 10 days. Despite the fact that spreading rates of the NSM trajectories are slower than those of the QG3 trajectories, the phase-space distribution of the NSM-based RMSD is qualitatively similar to that of the QG3 RMSD for each lead time. In particular, both distributions are characterized by the local RMSD minima neighboring the regions of regimes 1 and 2 (see Fig. 3); we call these regions of RMSD minima the regime precursor regions. Peters et al. (2012) showed that after the ensemble mean of trajectories emanating from the regime precursor regions enters the corresponding regime 1–5 days later, the QG3 trajectories stay within this regime for

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**Fig. 5.** Phase-space distribution of the RMSD (shading) at lead times of 1, 5, and 10 days for (a),(c),(e) the QG3 model and (b),(d),(f) the NSM. Arrows show the direction of the mean drift computed as finite differences over the intervals corresponding to each lead time.
longer periods of time and exhibit lower spreading rates than analogous ensembles of LIM trajectories. The QG3 slowing down of QG3 trajectories in these regions relative to the LIM is thus responsible for the existence of the non-Gaussian anomalies themselves, and the low spreading rates are responsible for their anomalous predictability. Note that there is no low-RMSD precursor region corresponding to regime 3 in either the QG3 model or NSM (see section 4). Trajectories entering this regime do slow down relative to those of the LIM; however, they do not exhibit low spreading rates, and are thus not anomalously predictable.

Given the apparent success of the NSM in simulating nonlinear features of QG3 distribution, as well as the phase-space structure of the QG3 model RMSD, we design and perform, in section 3, two experiments with modified versions of the NSM to elucidate the relative contributions of various processes in defining the nonlinear regimes and their predictability.

3. Experiments with the NSM

Following Berner (2005), we now introduce systematic changes to the NSM constructed in section 2 and study the effect these changes have on the behavior of trajectories in the model phase space.

First, to isolate the effects of deterministic nonlinearity, we removed the dependence of the noise term in (3) on the model state vector. The NSM so modified is essentially reduced to the form (2), with the noise covariance matrix \( C = R^T R \) obtained via phase-space averaging of the matrix \( G \):

\[
G_{ij}(x) p(x) dx_1 dx_2,
\]

where \( p(x) \) is the stationary PDF of the model states in our phase subspace.

In an alternate modified version of the NSM, we leave the state dependence of the multiplicative noise in (3) intact, but replace the drift term \( h(x, t) \) with the linear operator \( L x \) of the LIM in (1). The resulting drift field is linear in \( x \) and is characterized by a constant divergence (not shown); however, the modified NSM in this case remains driven by the state-dependent multiplicative noise.

The PDF based on the 1 000 000-day simulation using the nonlinear NSM version with additive noise is shown in Fig. 6a, while the PDF resulting from the NSM with linear deterministic operator and multiplicative noise is shown in Fig. 6b. The PDF of the nonlinear NSM with additive noise is nearly indistinguishable from that of the full NSM and the QG3-based PDF (cf. Fig. 3), while
the PDF of the linear version of NSM with additive noise is notably closer to bivariate Gaussian, and is very similar to that of the LIM (not shown). We also repeated the regime identification procedure outlined in section 3 for the simulations of both NSM versions. Figure 6c reveals that the regimes of the nonlinear NSM driven by additive noise are nearly identical to those of the full NSM and QG3 model (cf. Fig. 3). On the other hand, the regime regions of the NSM with a linear deterministic operator and multiplicative noise (Fig. 6d) occupy only small portions of the phase space and are not topologically aligned with the QG3 regimes. Hence, we conclude that multiplicative noise apparently plays a little role in the QG3 regime behavior. In summary, the existence of QG3 regime appears to be largely related to the nonlinearity in the mean phase-space tendencies.

The phase-space distribution of RMSD (Fig. 7) for the modified NSMs provides further clues to the existence
and predictability of the QG3 regimes. In particular, for the simulations using additive noise, the spreading rates at a lead time of 1 day (Fig. 7a) are much more uniform than those of the full NSM and QG3 model (cf. Figs. 5a,b). Furthermore the low-RMSD precursor region to regime 1 is much less pronounced for the NSM with additive noise at lead times of 5 and 10 days (Figs. 7b,c) than for the full NSM and QG3 model (cf. Figs. 5c–f). On the other hand, the NSM with multiplicative noise and a linear deterministic operator mimics well the full NSM and QG3 model RMSD at a short lead time of 1 day (cf. Figs. 7d and 5a). At lead times of 5 and 10 days (Figs. 7e,f) for this NSM modification, the low-RMSD precursor region to regime 2 is much less pronounced than for the full NSM and QG3 model (cf. Figs. 5c–f). It follows that multiplicative noise dominates the QG3 spreading rates at short lead times and substantially contributes to the predictability of regime 1, but not regime 2. Nonlinear mean phase space tendencies, on the other hand, have little effect on spreading rates at lead times of 1 day; however, at lead times of 5 and 10 days they substantially contribute to the predictability of regime 2.

4. Mechanisms of the regime existence and predictability

The results of the experiments described in section 3 highlight relative contributions of the mean drift and stochastic diffusion to the existence and predictability of the nonlinear regimes evident in the phase space of the QG3 model. In particular, the presence of the regimes is largely (but not completely; see below) related to the effects of deterministic nonlinearity: the regimes are replicated well in the modified nonlinear NSM with additive noise, and are absent from the phase space of the NSM with linear drift and multiplicative noise. The major mechanism leading to the formation of the regimes is the nonlinear slowdown of the mean tendency of the trajectories toward climatology predicted by the LIM, complemented by the local convergence of drift vectors. To quantify the former effect, Peters et al. (2012) defined the fractional slowdown $F$ of the QG3-based drift vectors relative to the LIM drift vectors:

$$F(x) = \frac{Lx \cdot (A(x) - Lx)}{Lx \cdot Lx} \times 100\%, \quad (11)$$

where the dot denotes the scalar product. If $A(x) = Lx$, $F(x) = 0; F$ is negative if $A(x)$ has a component in the direction opposite to the direction of the LIM drift vector $Lx$, and is positive otherwise. The magnitude of $F$ represents the length of the $A(x)$ component parallel to $Lx$, expressed as a percentage of the local LIM-based drift magnitude at a given point $x$. We computed the $F$ fields based on the QG3 and LIM drift vectors computed as in (5), with $\tau$ equal to 1 and 5 days; the results are plotted in Figs. 8a and 8b, respectively. The approximate locations of the regime regions are also plotted, for reference, as dotted ellipses.

Regions where QG3 tendencies tend to climatology at slower rates than those of the LIM correlate well with the regime regions, especially for $\tau = 5$ days; this is intuitively synonymous with the definition of the regime regions as those in which persistence of the QG3 model states is higher relative to the LIM (see section 2). Note that this property plays a role in the formation of all three QG3 regimes; however, it does not directly contribute to the relative predictability of a given regime, since the RMSD as defined in (9) is independent of the mean displacement of a trajectory ensemble. However, it may affect regime predictability indirectly by pausing trajectories in regions where other properties of trajectory behavior—namely local convergence of drift vectors and locally low multiplicative noise (see below)—promoting low spreading.

To illustrate the mechanism of drift-vector convergence, let us consider the evolution of the initially bell-shaped PDF of model states $P$ placed in a localized region of the phase space, governed by the Fokker–Planck equation (4). We may expand the first term on the right-hand side of (4) using the product rule and rewrite this equation in terms of the total time derivative of PDF $dP/dt = (\partial P/\partial t) + \left\{ A_i (\partial P/\partial x_i) \right\}$, which includes advection of the PDF by the mean drift field:

$$dP/dt = -P \frac{\partial A_i}{\partial x_i} + \frac{1}{2} \frac{\partial^2}{\partial x_i \partial x_j} (B_{ij} P). \quad (12)$$

The evolution of the PDF following the mean drift field in (12) depends on the divergence of the drift field $\text{V} \cdot \text{A} = \partial A_i / \partial x_i$ and the local rate of diffusion. If the drift field is nondivergent, the shape of the PDF of an evolving ensemble of model states is dictated by the local rate of diffusion along the path of the trajectories. However, in the general case, the shape of the PDF is additionally altered by the local divergence of mean drift vectors. In particular, if $\text{V} \cdot \text{A} < 0$, the rate of change of the PDF $dP/dt$ due to the divergence effect will be greater than 0, with the magnitude of $dP/dt$ being the largest at the mode of the initial distribution. Since the area integral of the PDF over the phase space must be equal to 1, the entire PDF field shrinks and becomes more localized near the center of the distribution relative to the nondivergent case. In terms of trajectory behavior, this shrinking corresponds to the decrease of the model’s RMSD in the regions where the drift-vector divergence is negative (in other words, drift vectors...
converge). By the same token, in the regions in which drift vectors diverge, the local RMSD tends to increase.

The divergence of the drift vectors of the LIM in (1) is constant and negative throughout the phase space, and is countered and overweighed by the uniform state-independent stochastic diffusion. On the other hand, the divergence of both 1- and 5-day mean tendencies of the QG3 model (Figs. 8c,d) is nonuniform and exhibits pronounced minima in the low-RMSD precursor regions of both predictable regimes 1 and 2 (see Fig. 5). For these regimes, the PDF of trajectory ensemble initialized in the corresponding regime precursor region expands at a slower rate than the analogous PDF of LIM trajectories because of enhanced drift-vector convergence and anomalously low multiplicative noise in the vicinity and throughout regime regions. Therefore, QG3 trajectories have a higher probability of persisting in the predictable regime regions than LIM trajectories. Note that the modified experiments of section 3 argue that the contribution, to the regime existence, of the locally reduced multiplicative noise is far less than the combined effect of the trajectories slowdown and enhanced convergence, since all of the regimes were still present in the NSM without multiplicative noise. Local drift-vector convergence also appears to be a secondary contributing factor to regime existence relative to the quantity $F$, since regime 3 exists despite relatively low local drift-vector convergence.

A comparison between Figs. 7 and 5 allows one to estimate contributions of deterministic nonlinearity and multiplicative noise to the regime predictability defined via RMSD measure. In particular, low multiplicative noise is the predominant factor in making regime 1 anomalously predictable at all lead times considered, while enhanced drift-vector convergence dominates the medium-range predictability of regime 2. Regime 3 arises from the fact that the QG3 trajectories slow down in this regime region relative to the LIM trajectories (Figs. 8a,b). However, since the phase space in the vicinity of regime 3 exhibits low drift-vector convergence and high multiplicative noise relative to the surrounding phase space, this regime is not anomalously predictable.

5. Summary and concluding remarks

In this study we considered the statistical sources of anomalously persistent non-Gaussian regimes and their
predictability in a three-level quasigeostrophic (QG3) model (Marshall and Molteni 1993) through the analysis of empirical nonlinear stochastic models (NSMs) fit to the output of a long QG3 simulation (Peters et al. 2012). The NSMs were constructed in the phase space of the leading EOF pair of the QG3’s upper-level stream-function. The most complete NSM in (3) introduced in section 2 included a nonlinear deterministic propagator and state-dependent multiplicative noise. The coefficients of the Fokker–Planck model estimated from the long NSM simulation replicate, by construction, the coefficients of the Fokker–Planck model derived from QG3 trajectories (Fig. 1). This NSM reproduces the non-Gaussian features of the QG3 PDF, including nonlinear regimes (Fig. 3) and phase-space distribution of the QG3 predictability measured via the root-mean-square distance (RMSD) of the 5- and 10-day forecasts of the QG3 model and NSM trajectories (Fig. 5). The major discrepancy between the NSM and QG3 model is uniform positive bias in the time scale of the processes simulated by the NSM relative to the time scales of the processes simulated by the QG3 model (Fig. 4). This bias, however, does not affect the phase-space PDF and RMSD patterns of the NSM, which are very similar to their QG3 model counterparts (Figs. 3 and 6).

In section 3, we conducted two experiments using modified versions of the full NSM—one with additive noise in place of multiplicative noise and one with the linear deterministic propagator from an equivalent linear inverse model (LIM) replacing the full QG3-based nonlinear effective drift field. The NSM with additive noise was found to reproduce the non-Gaussian PDF and persistent regimes of the QG3 model (Figs. 7a,b) but did not capture some attributes of the QG3 trajectory spreading rates (Fig. 7a,c,e)—namely, the low RMSD region associated with regime 1 (see Fig. 5 and Peters et al. 2012). On the other hand, the NSM with the LIM-based linear drift field and multiplicative noise failed to reproduce discernable regimes but did have a pronounced low RMSD precursor region in the vicinity of regime 1.

In section 4, we interpret these results by introducing the concepts of nonlinear slowdown of model trajectories (Fig. 8), as well as those of path-integrated divergence of mean drift vectors and path-integrated state-dependent stochastic diffusion. In the QG3 model, the existence of the regimes is dominated by the deterministic nonlinearity resulting in the slowing down of model’s trajectories within the regime region. On the other hand, the predictability of the regimes depends on a combination of factors. Regime 1 is predictable because of low stochastic diffusion of the trajectories originating in this regime’s precursor region; the low path-integrated diffusivity results in the low spread of trajectories in the regime region and thus contributes to the anomalous population and non-Gaussianity of this regime region. State-dependent stochastic diffusion plays a less important role in the predictability of regime 2, which is entirely dominated by the path-integrated drift-divergence effect. Finally, regime 3 is characterized by slower model trajectories within this regime region, but high path-integrated stochastic diffusion and low phase-velocity convergence, rendering this regime unpredictable.

Our findings are consistent with those of Berner (2005) and Sura et al. (2005) in that we argue, similarly to these authors, that accurate representation of nonlinear effects in reduced models of atmospheric low-frequency variability may require inclusion of both deterministic nonlinearity and multiplicative noise. The exact relative importance of these two processes will depend on the concrete problem at hand. Our study, along with that of Berner (2005), who considered reduced models of an atmospheric GCM, leans toward dominance of deterministic nonlinearity in the nonlinear behavior of the full dynamical model. This is also consistent with the analysis of the QG3 model by Kondrashov et al. (2006, 2010). On the other hand, Sura et al. (2005) seem to detect a larger relative influence of the multiplicative noise onto the observed low-frequency variability of the midlatitude atmosphere. The reasons behind these discrepancies are presently unclear and may involve a number of possible factors including dynamical incompleteness of the models, or the questionable statistical significance of the results obtained from their relatively short observational analysis. It must also be noted that these authors do not focus on the anomalous persistence of non-Gaussian anomalies, which may explain some differences between our results and theirs.

We must also include a cautionary note about interpreting the drift and diffusion parameters determined via finite-time differences of the full-model trajectories, since these quantities exhibit sensitivities to the averaging intervals (see Berner 2005; Branstator and Berner 2005). Furthermore, approximating the QG3 model behavior with a stochastic differential equation does not allow one to unambiguously regard the effective drift field as representing deterministic nonlinearity, due to the possibility of a noise-induced drift (see Gardiner 1985; Berner 2005; Sura et al. 2005 for additional discussion of this issue).

It would be a useful future endeavor to determine whether the predictability nonuniformities detected in the QG3 model phase space translate into enhancement of the forecast skill in a practical operational forecast setting. Another related avenue for future work may deal with the explicit dynamical identification and analysis of the interactions that we represented via “deterministic nonlinearity” (interaction between teleconnection patterns?) and “multiplicative noise” (synoptic eddies?) in our
reduced phase space. Finally, we are currently in the process of using the insights obtained from the present study for the analysis of observational data.

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