

The Vertical Structure of the Eddy Diffusivity and the Equilibration of the Extratropical Atmosphere

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ABSTRACT

Observations suggest that the time- and zonal-mean state of the extratropical atmosphere adjusts itself such that the so-called “criticality parameter” (which relates the vertical stratification to the horizontal temperature gradient) is close to one. T. Schneider has argued that the criticality parameter is kept near one by a constraint on the zonal momentum budget in primitive equations. The constraint relies on a diffusive closure for the eddy flux of potential vorticity (PV) with an eddy diffusivity that is approximately constant in the vertical.

The diffusive closure for the eddy PV flux, however, depends crucially on the definition of averages along isentropes that intersect the surface. It is argued that the definition favored by Schneider results in eddy PV fluxes whose physical interpretation is unclear and that do not satisfy the proposed closure in numerical simulations. An alternative definition, first proposed by T.-Y. Koh and R. A. Plumb, is preferred. A diffusive closure for the eddy PV flux under this definition is supported by analysis of the PV variance budget and can be used to close the near-surface zonal momentum budget in idealized numerical simulations. Following this approach, it is shown that $O(1)$ criticalities are obtained if the eddy diffusivity decays from its surface value to about zero over the depth of the troposphere, which is likely to be the case in Earth’s atmosphere. Large criticality parameters, however, are possible if the eddy diffusivity decays only weakly in the vertical, consistent with results from quasigeostrophic models. This provides theoretical support for recent numerical studies that have found supercritical mean states in primitive equation models.

1. Introduction

Observations suggest that the time- and zonal-mean state of the extratropical atmosphere adjusts itself such that isentropes leaving the surface in the subtropics reach the tropopause near the pole, and thus $\xi \equiv (a/H)s \sim O(1)$, where a is the planetary radius, H the height of the tropopause, and s the slope of the isentropes. The parameter ξ (in this or related formulations) is commonly referred to as the criticality parameter, since the condition that $\xi \sim O(1)$ bears resemblance to the marginal criticality condition in the two-layer quasigeostrophic (QG) model, where small criticality parameters denote a state that is stable to baroclinic instability, while large criticality parameters denote a strongly unstable state (Phillips 1954; Stone 1978). Numerical studies suggest that the

observed adjustment of the extratropical atmosphere to $O(1)$ criticalities holds over a wide range of forcings and parameters (e.g., Schneider and Walker 2006). Other studies, however, have shown that more supercritical states can also be obtained if external parameters are sufficiently changed (e.g., Zurita-Gotor 2008; Zurita-Gotor and Vallis 2010; Jansen and Ferrari 2012).

Various theories have been put forward to explain the observed equilibration of the midlatitude atmosphere to a state of marginal criticality. The original argument brought forward by Stone (1978) is based on the analogy to the two-layer QG model, where $\xi = 1$ denotes the transition from baroclinically stable to unstable states. The implication is that at a criticality of order 1, eddy fluxes transition from being inefficient to being highly efficient in reducing the isentropic slope. This argument, however, is not very compelling because the neutrality condition, as defined above, applies only to the two-layer model, and the atmosphere is not generally adjusted to baroclinic neutrality (e.g., Barry et al. 2000).

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Another approach is to argue in terms of constraints on the zonal momentum budget. The vertically integrated QG zonal momentum budget can be used to derive a balance condition between the eddy fluxes of potential vorticity (PV) in the interior and the eddy fluxes of potential temperature at the surface. Using a diffusive closure for the eddy fluxes of PV and surface potential temperature, this budget gives a relation between the criticality parameter and the eddy diffusivity. Most importantly, the relation shows that the criticality parameter can be finite only if the eddy diffusivity varies in the vertical (Green 1970). Held (1978, 1982) used this result (and some assumptions for the vertical structure of the PV gradient) to argue that atmospheric mean states with $\xi \approx 1$ are obtained if the eddy diffusivity decays in the vertical over the depth scale of the troposphere.

Schneider (2004, 2005) calls into question the relevance of the QG results of Green (1970) and Held (1978, 1982), arguing that similar results are not recovered in primitive equations. The breakdown of the QG results in primitive equations is attributed to the inadequate representation of isentropic outcrops at the surface.

In primitive equations, a relation between the turbulent eddy fluxes and the atmospheric mean state is most naturally derived from the zonal momentum budget in isentropic coordinates (Schneider 2005; Koh and Plumb 2004). As in QG, the latter can be used to derive a relation between the vertically integrated isentropic eddy flux of PV and the eddy flux of potential temperature at the surface. Questions, however, arise about the appropriate definition of isentropic averages and eddy fluxes on isentropes that intersect with the surface. Schneider (2005) argues that there are two possible ways to separate the eddy PV flux into a mean and eddy flux component on outcropping isentropes. One approach, which was originally proposed by Koh and Plumb (2004), results in a formulation for the isentropic zonal momentum budget, which more closely resembles the zonal momentum budget in QG but is argued to be less suitable for eddy flux closures because the resulting PV gradients often change sign on small scales. Schneider (2005) proposes an alternative approach, under which the PV fluxes and gradients on outcropping isentropes are generally larger and of uniform sign. This alternative approach is used in Schneider (2004) to derive a constraint for the criticality parameter in primitive equations. The results suggest that, unlike in QG, the zonal momentum budget in primitive equations can be closed, assuming that the eddy diffusivity is constant in the vertical throughout the whole depth of the troposphere. Moreover, the choice of a vertically constant eddy diffusivity implies that $\xi = O(1)$.

In this paper we revisit the arguments presented in Schneider (2005, 2004). We find that the definitions of the isentropic mean and eddy PV fluxes proposed by Koh and Plumb (2004) result from a well-defined average over the existing part of the isentrope, while the definitions used in Schneider (2004) include contributions from isentropes below the surface, which are difficult to justify mathematically. Furthermore, a diffusive closure for the eddy PV flux can be justified physically by analysis of the PV variance budget with the approach of Koh and Plumb (2004), while the physical justification for a diffusive closure with the approach used in Schneider (2004) remains unclear to the authors of this paper. In addition, we show that idealized numerical simulations support the use of a diffusive closure following the definitions of Koh and Plumb (2004), while a diffusive closure generates inconsistent results if the approach of Schneider (2004) is applied to analyze our simulations.

We therefore repeat Schneider's (2004) derivation for the relation between the criticality parameter and the vertical structure of the eddy diffusivity but using the averaging approach advocated by Koh and Plumb (2004). Using this approach, we can extend the QG results for the relation between the criticality parameter and the eddy diffusivity to finite isentropic slopes. In particular, we show that the eddy diffusivity decreases in the vertical for any finite criticality, consistent with results obtained using QG scaling and in contrast to the results in Schneider (2004). The revised scaling relation further shows that criticality parameters much larger than one are possible if the eddy diffusivity decays only weakly in the vertical. This provides theoretical support for recent numerical studies that have found supercritical mean states in primitive equation models (e.g., Zurita-Gotor 2008; Zurita-Gotor and Vallis 2010; Jansen and Ferrari 2012). These results are also in qualitative agreement with the vertical decay of passive tracer eddy diffusivities found in atmospheric GCMs (Plumb and Mahlman 1987) and reanalysis data (Haynes and Shuckburgh 2000).

This paper is organized as follows. In section 2 we illustrate the constraint of the momentum budget on the vertical structure of the eddy PV diffusivity using a simple QG model. In section 3, we discuss the two different approaches that have been proposed to close the isentropic zonal momentum budget in primitive equations. In section 4, we test the diffusive closures against idealized numerical simulations. In section 5, we derive a revised condition for the criticality parameter from the vertically integrated isentropic zonal momentum budget. The conclusions are summarized in section 6.

2. Eddy diffusivity and criticality in QG

This study discusses the relation between the criticality parameter and the vertical structure of the eddy diffusivity in the troposphere. The problem is well understood in models using the QG approximation. Schneider (2004), however, pointed out that the relationship may be quite different in primitive equation systems that allow for steep isentropes that intersect with the surface. In the next few sections we will revisit these arguments and show that the result of QG theory is recovered if care is taken in computing averages along isentropic surfaces. It therefore seems appropriate to start with a review of the QG results to set the stage for the rest of the paper.

We will illustrate the QG argument using what is arguably the simplest configuration: a zonally reentrant channel composed of two separate layers with different density. The QG argument goes back to Green (1970), who discussed the constraint on the vertical structure of the eddy diffusivity in the continuously stratified case. A discussion of the two-layer case was, to our knowledge, first given by Marshall (1981). The qualitative result from the two-layer model, discussed here, is similar to the continuous QG case.

Ignoring frictional forces, the steady state zonal-mean zonal momentum budget in the two-layer QG model can be expressed in terms of a balance between the Coriolis acceleration, acting on the mean flow, and the Reynolds stress:

$$f_0 \bar{v}_i = \partial_y \overline{u'_i v'_i}, \quad (1)$$

where $i = 1, 2$ is the model layer, (u_i, v_i) are the zonal and meridional velocities in each layer, f_0 is the Coriolis frequency, the overbar denotes a time and zonal mean, and primed quantities denote deviations thereof. Introducing a residual mean flow as $\bar{v}_i^* = \bar{v}_i + H_i^{-1} \overline{v'_i h'_i}$, with h_i and H_i denoting the in situ and mean layer depths, respectively, the momentum budget can be rewritten as a balance between the Coriolis acceleration, acting on the residual mean flow, and the eddy flux of PV (e.g., Vallis 2006):

$$f_0 \bar{v}_i^* = -\overline{v'_i q'_i}. \quad (2)$$

Here $q_i = \beta y + \zeta_i - f_0 h_i / H_i$ is the QG potential vorticity, where β is the planetary vorticity gradient and $\zeta_i = \partial_x v_i - \partial_y u_i$ is the relative vorticity. Using a diffusive closure for the eddy fluxes of PV and ignoring the contribution of relative vorticity to the mean PV gradient, we can approximate the residual flow in each layer as

$$\bar{v}_i^* = -f_0^{-1} \overline{v'_i q'_i} \equiv f_0^{-1} D_i \partial_y \bar{q}_i \approx D_i \left(\frac{\beta}{f_0} \mp \frac{s}{H_i} \right), \quad (3)$$

where $s \equiv \partial_y \bar{h}_1 = -\partial_y \bar{h}_2$ denotes the slope of the interface. For ease of discussion, we here want to assume that $f_0 s > 0$, as in Earth's extratropical atmosphere. The negative sign in the last term on the rhs of Eq. (3) holds for the lower layer (layer 1) where the planetary vorticity and thickness contribution to the PV gradient have opposite sign, while the positive sign holds for the upper layer (layer 2) where the planetary vorticity and thickness contribution to the PV gradient have the same sign. Mass conservation provides the additional constraint that

$$H_1 \bar{v}_1^* + H_2 \bar{v}_2^* = 0. \quad (4)$$

Combining Eqs. (3) and (4) yields

$$D_1 \left(\frac{H_1 \beta}{f_0} - s \right) \approx -D_2 \left(\frac{H_2 \beta}{f_0} + s \right). \quad (5)$$

This yields a relation between the vertical structure of the eddy diffusivity and the criticality parameter as

$$\xi \equiv \frac{f_0 s}{\beta H} \sim \frac{[D]}{\Delta D}, \quad (6)$$

where $H = H_1 + H_2$ denotes the total depth, f_0/β is the planetary scale, $[D] \equiv (H_1 D_1 + H_2 D_2)/H$ is the vertical mean eddy PV diffusivity, and $\Delta D \equiv D_1 - D_2$ is the difference in the diffusivities. Equation (6) states that the criticality parameter is inversely proportional to the relative vertical variation of the eddy diffusivity. Criticality parameters $O(1)$ are associated with vertical variations in D on the same order of D itself, while strongly supercritical states are associated with weak vertical variations in D .

The criticality parameter ξ is also related to the condition for baroclinic instability. In the two-layer model, the condition for instability is that the PV gradient in the lower layer is negative—that is, $(\beta - f_0 s / H_1) \leq 0$, or, using the definition of the criticality parameter, $\xi \geq H_1 / H$. Hence, weak criticalities are associated with weakly turbulent states and vertically variable diffusivities, while large criticalities characterize strongly turbulent states with vertically uniform eddy diffusivities.

A similar relation between criticality and vertical structure of the eddy diffusivity is obtained with the continuously stratified QG model as shown by Green (1970). In section 5, we show that a result analogous to Eq. (6) can also be derived in primitive equations if the troposphere is divided into two vertical layers. The

primitive equation problem also illustrates what part of the atmosphere ought to be interpreted as the lower layer, which is otherwise arbitrarily defined in the QG problem.

3. The isentropic zonal momentum budget in primitive equations

The seminal papers of Schneider (2004, 2005) and Koh and Plumb (2004) show that the vertically integrated isentropic zonal momentum budget in primitive equations can be used to relate the meridional mass transport in the troposphere to the eddy fluxes of PV and surface potential temperature. Schneider (2004) points out that the integral of the mass transport approximately vanishes if the integral is extended all the way to the top of the tropopause. One is therefore left with a relationship between the eddy fluxes of PV and surface potential temperature, which can be converted into a relationship between the interior PV gradient and the surface potential temperature gradient if one assumes that the fluxes are down their mean gradients. Schneider further shows that this relationship can also be cast as a condition for the criticality of the extratropical atmosphere.

Before discussing the implications for the criticality parameter, we need an appropriate formulation of the isentropic zonal-mean zonal momentum budget on isentropes that intersect with the surface. This is the crucial step that leads Schneider (2004) to obtain a relationship fundamentally different from the QG equation (6).

For simplicity, we will assume that both large-scale mean and eddy flows have a small Rossby number (Ro), which is generally a good approximation in the troposphere. Departing from the QG approximation, however, we do not make an assumption of small isentropic slope (e.g., Vallis 2006). The latter is inadequate for studies of the large-scale atmospheric circulation—and crucial to understand the differences between the QG results and those presented in Schneider (2004).

For small Ro, the isentropic zonal momentum budget reduces to the geostrophic wind relation

$$fv \approx \partial_x M, \tag{7}$$

where f is the planetary vorticity, v is the meridional velocity, and $M = c_p T + gz$ is the Montgomery streamfunction (c_p denotes the specific heat capacity of air at constant pressure, T is temperature, g is the gravitational acceleration, and z is height). The partial zonal derivative ∂_x is taken at constant potential temperature θ . In addition to terms that are higher order in Ro, we neglected the effect of diabatic and frictional

forcing in the momentum equation. The full budget is discussed in Schneider (2005) and Koh and Plumb (2004), but the neglected terms are not relevant for the key results presented here.

To derive an equation for the isentropic mass transport, we take zonal and temporal averages of the momentum budget along isentropes. This raises the issue of how to define these averages when isentropes intersect the surface. We can introduce a normalized zonal and temporal integral along any isentrope (that may or may not outcrop at some longitude or time) as in Koh and Plumb (2004):

$$\overline{f(\theta, x, y, t)}^\diamond \equiv \frac{1}{TL} \iint_{\theta > \theta_s(x, y, t)} f(\theta, x, y, t) dx dt, \tag{8}$$

where L is the width of the (zonally reentrant) domain and the temporal integral is taken over some time period T . For simplicity we use Cartesian coordinates, but the generalization to spherical coordinates is straightforward. If $f(\theta, x, y, t)$ is continually extended to isentropes below the surface, the normalized integral may for convenience be written as

$$\begin{aligned} \overline{f(\theta, x, y, t)}^\diamond &= \frac{1}{TL} \iint \mathcal{H}(\theta - \theta_s) f(\theta, x, y, t) dx dt \\ &\equiv \overline{\mathcal{H}(\theta - \theta_s) f(\theta, x, y, t)}, \end{aligned} \tag{9}$$

where the overbar denotes the isentropic zonal and temporal mean, and $\mathcal{H}(\theta - \theta_s)$ is the Heaviside function, which guarantees that there are no contributions to the integral from the subsurface part of an isentrope.

Using the definition in Eq. (9), the normalized integral of the momentum budget in Eq. (7) can be written as

$$\overline{\mathcal{H}(\theta - \theta_s) fv} \approx \overline{\mathcal{H}(\theta - \theta_s) \partial_x M}. \tag{10}$$

Equation (10) is the starting point to illustrate the two different approaches to rewrite the zonal-mean zonal momentum budget in terms of mean and eddy fluxes of PV, as discussed in Schneider (2005).

a. Two formulations for mean and eddy fluxes of PV

To derive an equation for the isentropic mass transport from the momentum budget in Eq. (10), we express the Coriolis term on the lhs of Eq. (10) in terms of thickness-weighted mean and eddy fluxes of PV. Upon division by the mean PV, this yields an equation for the thickness-weighted velocity, which can be integrated in the vertical to yield the total isentropic mass transport.

Two different approaches have been recommended to separate the term $\mathcal{H}(\theta - \theta_s)vf$ into mean and eddy flux

terms. The approach proposed by Koh and Plumb (2004) can be written as

$$\begin{aligned} \overline{\mathcal{H}(\theta - \theta_s)vf} &= \overline{\frac{\mathcal{H}(\theta - \theta_s)\sigma vf}{\sigma}} \\ &= \overline{\mathcal{H}(\theta - \theta_s)vP} \\ &= \overline{\rho_\theta v P^*} \\ &= \overline{\rho_\theta}(\overline{v^* P^*} + \hat{v} P^*), \end{aligned} \quad (11)$$

where we introduced the isentropic thickness, $\sigma \equiv -g^{-1}\partial_\theta p$, and the planetary PV, $P \equiv f/\sigma$. We further introduced a generalized thickness as $\rho_\theta = \mathcal{H}(\theta - \theta_s)\sigma$ and a generalized thickness-weighted zonal average, $(\overline{\quad})^* = \overline{\sigma(\quad)}/\overline{\sigma} = \overline{\rho_\theta(\quad)}/\overline{\rho_\theta}$ (see also Andrews 1983; Juckes et al. 1994). Note that $(\overline{\quad})^*$ is simply the thickness-weighted zonal average, taken over the above-surface part of the isentrope. Deviations from this average are denoted by $(\hat{\quad}) \equiv (\overline{\quad}) - (\overline{\quad})^*$.

Schneider (2004, 2005) instead recommends to define

$$\begin{aligned} \overline{\mathcal{H}(\theta - \theta_s)vf} &= \overline{\frac{\rho_\theta \mathcal{H}(\theta - \theta_s)vf}{\rho_\theta}} \\ &= \overline{\rho_\theta \mathcal{H}(\theta - \theta_s)vP_{\text{ext}}} \\ &= \overline{\rho_\theta \mathcal{H}(\theta - \theta_s)vP_{\text{ext}}^*} \\ &= \overline{\rho_\theta}(\overline{v_{\text{ext}}^* P_{\text{ext}}^*} + \hat{v}_{\text{ext}} \hat{P}_{\text{ext}}^*), \end{aligned} \quad (12)$$

and introduces an extended planetary PV, $P_{\text{ext}} \equiv f/\rho_\theta$ (notice the generalized thickness, ρ_θ , in the denominator), and an extended velocity, $v_{\text{ext}} \equiv \mathcal{H}(\theta - \theta_s)v$, defined to exist on isentropes both above and below the surface. The first step in Eq. (12) is mathematically ill defined since it implies multiplying and dividing by zero whenever the isentrope is below the surface and ρ_θ vanishes. Furthermore, the extended PV, $P_{\text{ext}} \equiv f/\rho_\theta$, is infinite on isentropes below the surface, and thus both the generalized thickness-weighted average of P_{ext} and $\hat{v}_{\text{ext}} \hat{P}_{\text{ext}}^*$ are ill defined. Schneider (2004) evaluates $\overline{P_{\text{ext}}^*}$ as

$$\overline{\rho_\theta P_{\text{ext}}^*} = \overline{\left(\frac{\rho_\theta f}{\rho_\theta}\right)} \equiv f; \quad (13)$$

that is, the ratio of ρ_θ in the numerator and ρ_θ in the denominator of Eq. (13) is assumed equal to one everywhere, including the part of isentropes below the surface where $\rho_\theta = 0$.

Schneider (2005) discusses the relative merits of the two formulations. He argues that the two approaches can be understood as arising from different evaluations of the ill-defined ratio in Eq. (13). The formulation of

Koh and Plumb (2004) is recovered if the term $(\rho_\theta f/\rho_\theta)$ is assumed to vanish when isentropes are below the surface, while Schneider's (2004) approach assumes that it is equal to f . Notice, however, that the approach of Koh and Plumb can instead be derived without making any assumptions for quantities on the subsurface part of isentropes, as per Eq. (11). The problem is that, by introducing the generalized quantities P_{ext} and v_{ext} , the extended PV approach requires arbitrary assumptions down the line to obtain well-defined thickness-weighted averages.

Taking a different point of view, which we will adopt in the following, the separation into mean and eddy terms in Eq. (12), together with the definition for the thickness-weighted average of the extended PV in Eq. (13), may be regarded as a definition for the extended eddy PV flux $\hat{v}_{\text{ext}} \hat{P}_{\text{ext}}^*$. One could then regard Eq. (12) as a different choice for separating mean and eddy fluxes. The question is whether one can rationalize what the mean and eddy flux terms physically represent in this case, as they are no longer defined as the mean and eddy contributions from a Reynolds decomposition.

The physical meaning of the mean and eddy flux terms becomes particularly important if we want to relate the eddy PV flux to the mean PV gradient through a diffusive closure so as to close the momentum budget in terms of mean state variables. Tung (1986) offers support for a diffusive closure for the thickness-weighted isentropic eddy PV flux using small amplitude theory. He does, however, not consider isentropic outcrops, which are crucial for the discussion here. In appendix A, we present a general justification for a diffusive closure for Koh and Plumb's (2004) interior eddy PV flux, $\hat{v} P^*$, based on the PV variance budget, that accounts for isentropic outcrops. We were not able to derive a similar relation for the extended PV variance since \hat{P}_{ext}^2 is not well defined on outcropping isentropes. Schneider (2004, 2005) justifies his closure by showing that the extended eddy flux $\hat{v}_{\text{ext}} \hat{P}_{\text{ext}}^*$ is down the gradient of $\overline{P_{\text{ext}}^*}$ in his numerical simulations and proceeds to use a diffusive closure $\hat{v}_{\text{ext}} \hat{P}_{\text{ext}}^* = -D\partial_y \overline{P_{\text{ext}}^*}$. However, we will argue below that such a relationship is not justified in general.

It may be argued that the approach of Koh and Plumb (2004) becomes problematic when averages are taken along the coldest isentropes, which exist only in a few places. In particular, one could question whether a downgradient closure can be defended when averages are taken on small regions. These isentropes, however, contribute little to the net PV flux and associated mass transport. In section 4 we will show that the closure for Koh and Plumb's interior PV is well supported by idealized numerical experiments.

b. An equation for the meridional mass transport

In this section we will discuss how the isentropic zonal momentum budget can be transformed into an equation for the meridional isentropic mass transport, following the same approach outlined in Schneider (2005) and Koh and Plumb (2004). Focusing on the role of isentropic outcrops, we will here analyze the mass transport integrated over the surface layer (SL), defined as the layer including all isentropes that intersect the surface at some longitude or time.

Schneider (2005) shows the mass transport over the SL can be estimated by dividing the zonal momentum equation (10) by \overline{P}^* and integrating vertically over the SL. Substituting Eq. (11) into Eq. (10), dividing by \overline{P}^* , and integrating from the surface to the top of the SL, one obtains an estimate for the transport in the SL:

$$\int_{\theta_b}^{\theta_i} \overline{\rho_\theta v^*} d\theta \approx - \underbrace{\int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta \hat{v} \overline{P}^*}}{\overline{P}^*} d\theta}_{(1)} - \underbrace{\frac{f}{\overline{P}^*(\theta_s)} \overline{v'_s \theta'_s}}_{(2)}. \quad (14)$$

Here θ_b is the minimum potential temperature over the domain, and θ_i is the potential temperature at the top of the SL, the subscript s denotes surface quantities, and $\overline{(\)^s}$ denotes an average along the surface. The surface potential temperature flux term on the rhs of Eq. (14) represents the bottom form stress, which arises as the boundary contribution from the Montgomery streamfunction term in Eq. (10) (see Koh and Plumb 2004; Schneider 2005). The derivation of this term is unambiguous and therefore not repeated in detail here. Equation (14) states that the total meridional mass transport between the surface and the isentropic surface θ_i , is driven by isentropic eddy PV fluxes and the eddy flux of surface potential temperature.

The same equation can be derived using the extended PV approach advocated by Schneider (2005), who finds that

$$\int_{\theta_b}^{\theta_i} \overline{\rho_\theta v^*} d\theta \approx - \underbrace{\int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta \hat{v}_{\text{ext}} \overline{P}_{\text{ext}}^*}}{\overline{P}_{\text{ext}}^*} d\theta}_{(1)} - \underbrace{\frac{f}{\overline{P}_{\text{ext}}^*(\theta_s)} \overline{v'_s \theta'_s}}_{(2)}. \quad (15)$$

The total mass transport—the term on the lhs of Eqs. (14) and (15)—is independent of the approach, but the separation between the contributions associated with the eddy flux of PV, term 1, and the contribution

associated with the eddy flux of surface potential temperature, term 2, differs in the two approaches. This can be seen by noting that $\overline{P_{\text{ext}}^*}(\theta_s) \neq \overline{P}^*(\theta_s)$. The prefactor multiplying the eddy flux of surface potential temperature, term 2, thus differs in the two approaches. This difference is absorbed by a corresponding difference in the PV flux contribution, term 1.

Using diffusive closures for the eddy fluxes of interior PV and surface potential temperature, Eq. (14) becomes

$$\int_{\theta_b}^{\theta_i} \overline{\rho_\theta v^*} d\theta \approx \int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta} D \partial_y \overline{P}^*}{\overline{P}^*} d\theta + \frac{f}{\overline{P}^*(\theta_s)} D_s \partial_y \overline{\theta}_s^s; \quad (16)$$

that is, the SL mass transport is given by a contribution associated with the interior PV gradient and a contribution associated with the surface potential temperature gradient (describing the form drag on outcropping isentropes). Using instead a diffusive closure for the eddy flux of extended PV, along with a closure for the surface potential temperature flux, Eq. (15) becomes

$$\int_{\theta_b}^{\theta_i} \overline{\rho_\theta v^*} d\theta \approx \int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta} D_{\text{ext}} \partial_y \overline{P}_{\text{ext}}^*}{\overline{P}_{\text{ext}}^*} d\theta + \frac{f}{\overline{P}_{\text{ext}}^*(\theta_s)} D_s \partial_y \overline{\theta}_s^s. \quad (17)$$

One would now like to assume that the eddy diffusivity is a fundamental property of the flow, describing the rate of mixing by the turbulent eddies independently of the tracer being stirred. Since the SL is centered around the median surface temperature, it seems appropriate to assume that $D_{\text{ext}} \approx D \approx D_s$ near the surface. Numerical simulations discussed in section 4 show that the budget in Eq. (16) can be closed, assuming $D \approx D_s$, while Eq. (17) strongly overestimates the near-surface mass transport if we assume that $D_{\text{ext}} \approx D_s$.

4. Numerical simulations

We proceed to test the arguments presented in the previous section, and in particular the applicability of diffusive closures for the eddy PV fluxes in the two approaches, in an idealized numerical simulation. The model is set up to limit boundary effects that arise from explicit parameterization of boundary layer physics. While the model setup might be a less realistic description of the real atmosphere, it helps to focus exclusively on the dynamics that are the topic of the present paper and keep any additional physics as simple as possible.

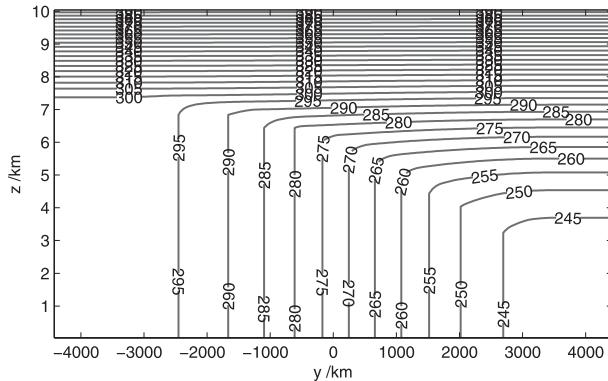


FIG. 1. Equilibrium potential temperature (K) for thermal restoring, contour interval (CI): 5 K.

a. Model setup

We use a hydrostatic, Boussinesq,¹ Cartesian coordinate configuration of the Massachusetts Institute of Technology general circulation model (Marshall et al. 1997). The setup is similar to the simulations discussed in Jansen and Ferrari (2012): a zonally reentrant β -plane channel 15 000 km long, bounded meridionally by sidewalls at $y = \pm 4500$ km and vertically by rigid lids at $z = H = 10.2$ km and $z = 0$. Differing from the simulations in Jansen and Ferrari (2012), we use free-slip boundary conditions on all boundaries to limit frictional effects. Kinetic energy is removed by a linear Rayleigh drag, with a constant drag coefficient $r = (40 \text{ days})^{-1}$, acting over the whole domain. Following Jansen and Ferrari (2012), we use a linear equation of state with a thermal expansion coefficient of $\alpha = 2.25 \times 10^{-4} \text{ K}^{-1}$; that is, $b = g\alpha(\theta - \theta_0)$ in which b is buoyancy, θ_0 a reference potential temperature, and g the acceleration of gravity. The thermal expansion coefficient is smaller than that of air to obtain a better separation between the eddy and domain scale and to allow the model to set its criticality (see Jansen and Ferrari 2012).

The simulations are forced through relaxation to the equilibrium temperature profile shown in Fig. 1, which is the radiative convective equilibrium solution of the profile used in Jansen and Ferrari (2012). It is characterized by a baroclinic zone 7000 km wide, across which the temperature drops by about 55 K. The equilibrium stratification is statically neutral over the lower part of the domain, while a statically stable equilibrium

¹ Notice that the theoretical discussions above are for the more general case of a compressible fluid, in direct analogy to the derivations in Schneider (2004) and Koh and Plumb (2004). However, as discussed in Jansen and Ferrari (2012), the same results are obtained in the Boussinesq limit.

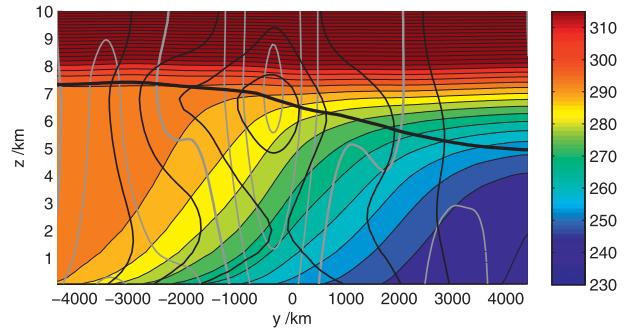


FIG. 2. Time- and zonal-mean fields of potential temperature (K) (color shading, CI: 5 K), eddy kinetic energy (thin black lines, CI: $2 \text{ m}^2 \text{ s}^{-2}$), zonal wind (gray lines, CI: 1 m s^{-1}), and the tropopause height, defined as the height at which $d\bar{\theta}/dz = 10^{-2} \text{ K m}^{-1}$ (thick black line).

stratification is prescribed at higher altitudes to mimic the stable radiative-equilibrium profile of the stratosphere. The relaxation time scale is constant over the domain at $\tau = 50$ days.

The simulation is spun up until a quasi-steady state is reached. Isentropic diagnostics are calculated from 200 snapshots saved every 10 days after the solution is equilibrated.

b. Results

The equilibrated time- and zonal-mean state of the simulation is shown in Fig. 2. Significant baroclinicity is found over most of the domain, with a jet centered around $y = -400$ km. The jet is associated with a maximum in baroclinicity, as well as maxima in the barotropic westerly wind and eddy kinetic energy.

Figure 3a shows the three terms that appear in the zonal momentum budget as per Eq. (14). The integral is taken from the surface to the top of the surface layer, defined to include all isentropes up to the 95% quantile of potential temperature values found at the surface, along a latitude circle. The budget is shown only for the SL, because that is where we expect differences in the averaging approaches (all approaches are equivalent aloft where there are no isentropes that intersect the surface). The continuous gray line shows the total SL isentropic mass flux:

$$F_{\text{tot}} = \int_{\theta_b}^{\theta_i} \bar{\rho}_\theta \bar{v}^* d\theta. \quad (18)$$

The dotted black lines show the sum of the mass fluxes associated with the eddy PV flux and the surface potential temperature flux. Using the averaging approach of Koh and Plumb (2004), these are

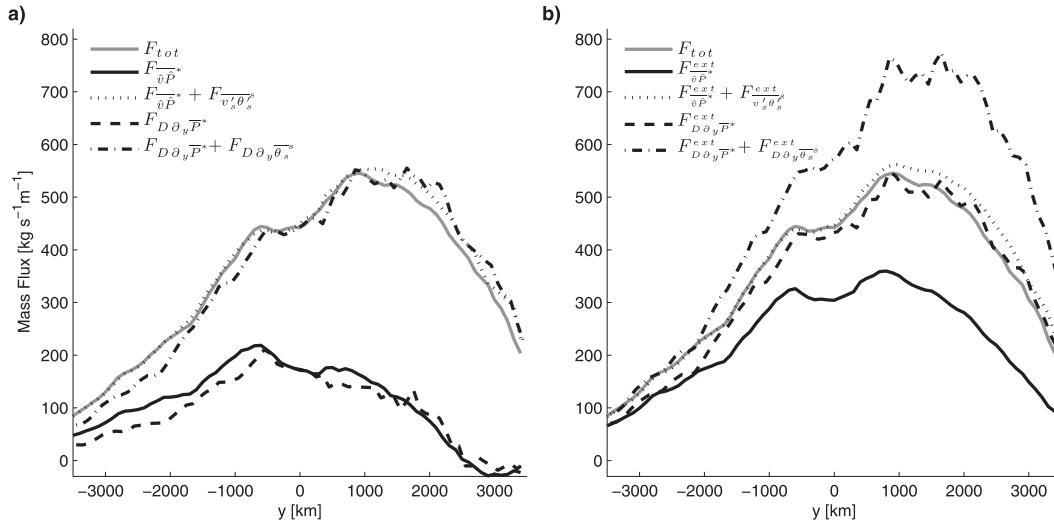


FIG. 3. Surface layer mass fluxes over the baroclinically forced zone. Shown is the total mass transport integrated vertically over the SL, its contribution due to eddy PV and surface potential temperature flux, as well as estimates of the respective fluxes resulting from a diffusive closure using the surface eddy diffusivity (see legend). Results (a) using the interior PV approach for the separation of SL mass fluxes into a contribution from the eddy PV flux and surface potential temperature fluxes and (b) using the extended PV approach. The SL is here defined as including all isentropes up to the 95% quantile of the surface potential temperature.

$$F_{\hat{v}P^*} \equiv - \int_{\theta_b}^{\theta_i} \frac{\overline{\bar{\rho}_\theta \hat{v}P^*}}{\overline{P^*}} d\theta \quad \text{and} \quad F_{v'_s \theta'_s} \equiv - \frac{f}{\overline{P^*}(\theta_s)} \overline{v'_s \theta'_s}. \quad (19)$$

The budget derived in Eq. (14) predicts that

$$F_{\text{tot}} \approx F_{\hat{v}P^*} + F_{v'_s \theta'_s}. \quad (20)$$

Figure 3a shows that this relation is satisfied by the numerical simulation. We find that most of the SL mass transport is associated with the surface contribution $F_{v'_s \theta'_s}$, with a smaller (but significant) contribution from the interior eddy PV flux component $F_{\hat{v}P^*}$.

Alternatively we can separate the eddy mass flux components according to the extended PV approach of Schneider (2004), given in Eq. (15):

$$F_{\text{tot}} \approx F_{\hat{v}P^*}^{\text{ext}} + F_{v'_s \theta'_s}^{\text{ext}}, \quad (21)$$

where

$$F_{\hat{v}P^*}^{\text{ext}} \equiv - \int_{\theta_b}^{\theta_i} \frac{\overline{\bar{\rho}_\theta \hat{v}_{\text{ext}} P_{\text{ext}}^*}}{\overline{P_{\text{ext}}^*}} d\theta \quad \text{and} \quad F_{v'_s \theta'_s}^{\text{ext}} \equiv - \frac{f}{\overline{P_{\text{ext}}^*}(\theta_s)} \overline{v'_s \theta'_s}. \quad (22)$$

The extended eddy PV flux $\hat{v}_{\text{ext}} P_{\text{ext}}^*$ is defined as the residual between the total PV flux $\hat{v}P^*$ and the mean advection of the generalized PV $\bar{v}_{\text{ext}} P_{\text{ext}}^*$, as discussed in section 3a. The mass flux budget in Eq. (21) is similarly

confirmed by the model simulations (Fig. 3b). Compared to the budget in Fig. 3b, the total mass transport has a larger contribution from the extended eddy PV flux term and a correspondingly smaller contribution from the eddy flux of surface potential temperature. In fact, it can be shown easily that $F_{v'_s \theta'_s}^{\text{ext}} \approx (1/2)F_{v'_s \theta'_s}$ by comparing Eqs. (19) and (22), but realizing that

$$\overline{P^*}(\theta_s) = \frac{\overline{f\mathcal{H}(\bar{\theta}_s - \theta_s)}}{\overline{\bar{\rho}_\theta(\theta_s)}} \approx \frac{1}{2} \frac{f}{\overline{\bar{\rho}_\theta(\theta_s)}} = \frac{1}{2} \overline{P_{\text{ext}}^*}(\theta_s). \quad (23)$$

Here we used that the mean surface potential temperature is well approximated by its median value and, hence, $\overline{\mathcal{H}(\bar{\theta}_s - \theta_s)} \approx 1/2$. Since the total mass transport is independent of the averaging approach, a similar but opposite difference exists between the eddy PV flux contributions $F_{\hat{v}P^*}$ and $F_{\hat{v}P^*}^{\text{ext}}$.

We now test the diffusive closures for the eddy fluxes of PV using the two approaches. We calculate the eddy diffusivity from the surface potential temperature flux/gradient relationship as²

² Notice that the surface eddy diffusivity is here calculated using the actual surface potential temperature flux. Following the full derivation shown in Schneider (2005), or Koh and Plumb (2004), the surface term in Eqs. (14) and (15) should generally be described by the geostrophic eddy flux of surface potential temperature. In the presented simulation, where the Rossby number is small and no enhanced surface drag is used, the difference between the full eddy flux and the geostrophic eddy flux is negligible.

$$D_s \equiv -\overline{v'_s \theta'_s} / \partial_y \overline{\theta_s^s}. \quad (24)$$

Using this definition, the diffusive closures for the mass transport associated with the surface eddy flux of potential temperature,

$$\begin{aligned} F_{D\partial_y \overline{\theta_s^s}} &\equiv \frac{f}{\overline{P^*}(\overline{\theta_s^s})} D_s \partial_y \overline{\theta_s^s} \quad \text{and} \\ F_{D\partial_y \overline{\theta_s^s}}^{\text{ext}} &\equiv \frac{f}{\overline{P^*}_{\text{ext}}(\overline{\theta_s^s})} D_s \partial_y \overline{\theta_s^s}, \end{aligned} \quad (25)$$

reproduce, by construction, the corresponding flux terms $F_{\overline{v'_s \theta'_s}}$ and $F_{\overline{v'_s \theta'_s}}^{\text{ext}}$.

Schneider (2004) argues that a diffusive closure holds also for the eddy PV flux with the same eddy diffusivity D_s , as one ought to expect if eddies mix all tracers at the same rate and if this mixing rate varies little over the depth of the SL. This is tested by computing

$$\begin{aligned} F_{D\partial_y \overline{P^*}} &\equiv \int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta} D_s \partial_y \overline{P^*}}{\overline{P^*}} d\theta \quad \text{and} \\ F_{D\partial_y \overline{P^*}}^{\text{ext}} &\equiv \int_{\theta_b}^{\theta_i} \frac{\overline{\rho_\theta} D_s \partial_y \overline{P^*}_{\text{ext}}}{\overline{P^*}_{\text{ext}}} d\theta. \end{aligned} \quad (26)$$

For the interior PV approach, we find that the PV flux and therefore the full SL mass transport is described reasonably well by the diffusive closure (Fig. 3a), suggesting that the PV flux in the SL can be described as a diffusive flux with an eddy diffusivity similar to that for surface potential temperature. However, with the extended PV approach, we find that the eddy PV flux, and therefore the total SL mass flux, is considerably overestimated by the diffusive closure. Interestingly, the diffusive approximation for the extended PV flux contribution, $F_{D\partial_y \overline{P^*}}^{\text{ext}}$, alone is a better approximation of the total SL mass transport F_{tot} (Fig. 3b). This result can be explained in terms of the different approaches taken to treat outcropping isentropes, as discussed in the following.

c. Explaining the results

The difference between the SL mass balance in the two approaches can be understood by rewriting the extended PV flux and gradient in terms of the contributions associated with fluxes and gradients of interior PV and surface potential temperature. Using again that $F_{\overline{v'_s \theta'_s}}^{\text{ext}} \approx (1/2)F_{\overline{v'_s \theta'_s}}$, as shown above in Eq. (23), we find that (see appendix B for a more detailed derivation)

$$F_{\overline{v'_s \theta'_s}}^{\text{ext}} \approx F_{\overline{v'_s \theta'_s}} + \frac{1}{2}F_{\overline{v'_s \theta'_s}}. \quad (27)$$

In the extended PV formulation of the mass budget [Eq. (21)], about half of the surface potential temperature flux contribution to the isentropic mass transport is thus absorbed into the extended PV flux term.

However, the diffusive closure for the flux associated with the extended PV gradient contains the contribution associated with the interior PV gradient, plus the full contribution associated with the surface potential temperature gradient (see appendix B for a derivation):

$$F_{D\partial_y \overline{P^*}}^{\text{ext}} \approx F_{D\partial_y \overline{P^*}} + F_{D\partial_y \overline{\theta_s^s}}. \quad (28)$$

If the interior eddy PV flux near the surface can be closed with a diffusivity similar to that for surface potential temperature (as confirmed in our simulation), a similar closure for the extended eddy PV flux thus automatically overestimates the flux. Consistent with the numerical results shown in Fig. 3, a comparison of Eqs. (28) and (20) suggests that $F_{D\partial_y \overline{P^*}}^{\text{ext}}$ is by itself an approximation for the full SL mass transport, F_{tot} .

Before returning to the implications of our findings for the criticality parameter, we remark that the numerical results discussed here have been confirmed in a whole suite of simulations using a similar idealized setup but with differing parameters and restoring profiles—the results hold as long as the system does not become convective, in which case the transformation into isentropic coordinates fails. Results from these simulations will be reported in an upcoming paper, where we discuss their implications for our understanding of the equilibration of the extratropical troposphere.

Schneider (2005) argues that in his simulations a diffusive closure for the extended eddy PV flux is better justified than for the eddy flux of interior PV since the gradients of the latter are generally weak and vary on small scales in the surface layer. While this makes it impractical to estimate an eddy diffusivity from a flux-gradient relationship for the interior PV, it does not mean that the integrated SL momentum budget cannot be closed with a diffusive closure for the eddy flux of interior PV. If the eddy flux is similarly weak, the integrated momentum budget can still be closed (and in fact becomes rather independent of the exact choice for the interior PV diffusivity). We ran some example simulations that are in this limit and were able to close all budgets.

In cases where both the eddy flux and gradient of interior PV are very weak, the extended eddy PV flux may appear to be well defined and downgradient, as both the flux and the gradient are dominated by the surface potential temperature contribution [see Eqs. (27) and (28)]. Such a scenario was considered in Schneider (2005)

and used to conclude that the extended PV approach is more suitable for a closure. However, as discussed above, the mass transport associated with the extended eddy PV flux in this case is overestimated by a factor of 2 if it is closed with an eddy diffusivity similar to that of surface potential temperature. This becomes crucial in the derivation of a relation between the criticality parameter and the vertical structure of the eddy diffusivity, as will be discussed in section 5.

The extended eddy PV flux closure can correctly represent the SL mass budget only if a very weak interior PV gradient were associated with a strong eddy flux of interior PV. This may be true in some of the simulations analyzed by Schneider (2005). However, we surmise that such situations may as well be explained by nonconservative boundary layer effects, which can cause both interior PV and surface potential temperature to behave differently from conservative tracers. The effect of nonconservative boundary layer terms, however, cannot be salvaged by redefining the zonal average. The problem in such situations is that a simple diffusive closure is no longer appropriate. We plan to carefully analyze more realistic atmospheric simulations and observational data to test whether the closure advocated here holds up in a more realistic setting.

5. The integrated momentum budget and the criticality

Schneider (2004) derives a condition for the state of the extratropical atmosphere by integrating the isentropic mass flux balance (15) to the top of the troposphere θ_t . He further makes a diffusive closure for both the extended eddy PV flux $\hat{v}_{\text{ext}}\hat{P}_{\text{ext}}$ and the surface potential temperature flux $v'_s\theta'_s$, with the same vertically constant eddy diffusivity. We showed above that this is generally not expected to hold. Instead, we argued that a diffusive closure for the eddy flux of interior PV, as defined in Koh and Plumb (2004), is better supported by our numerical simulations and can be justified by physical arguments. We will therefore repeat the derivation of Schneider (2004), but using diffusive closures for the interior eddy PV fluxes $\hat{v}\hat{P}$ and surface potential temperature flux $v'_s\theta'_s$. We find that, except in the limit case of infinite supercriticality, the momentum balance cannot be satisfied with an eddy diffusivity that is vertically constant throughout the entire troposphere, consistent with theories based on the QG approximation (Green 1970). We then discuss how the revised momentum balance can be used to derive a “criticality” condition, or more generally a relation between the criticality parameter and the vertical structure of the eddy diffusivity.

a. The vertically integrated isentropic momentum budget

If the integration in the mass balance equation (14) is extended all the way up to the tropopause and it is assumed that the mass transport above the troposphere is negligible, one gets

$$\int_{\theta_b}^{\theta_t} \frac{\bar{\rho}_\theta \hat{v}\hat{P}}{\bar{P}^*} d\theta \approx -\frac{f}{\bar{P}^*(\theta_s)} \bar{v}'_s \bar{\theta}'_s. \tag{29}$$

Assuming a diffusive closure for the eddy flux of PV and surface potential temperature then yields

$$-\int_{\theta_b}^{\theta_t} \frac{\bar{\rho}_\theta D \partial_y \bar{P}^*}{\bar{P}^*} d\theta \approx \frac{f}{\bar{P}^*(\theta_s)} D_s \partial_y \bar{\theta}'_s. \tag{30}$$

While we found in the previous section that D may be assumed similar to D_s within the SL, we now use Eq. (30) to show that D cannot be constant throughout the depth of the entire troposphere, but generally decreases toward the tropopause.

The thickness-weighted mean PV can be approximated as $\bar{P}^* \approx \Pi f / \bar{\rho}_\theta$; thus

$$\partial_y \bar{P}^* \approx \frac{\bar{P}^*}{f} \beta + \frac{\bar{P}^*}{\Pi} \partial_y \Pi - \frac{\bar{P}^*}{\bar{\rho}_\theta} \partial_y \bar{\rho}_\theta. \tag{31}$$

Equation (30) then becomes

$$-\int_{\theta_b}^{\theta_t} D \left(\frac{\bar{\rho}_\theta}{f} \beta - \partial_y \bar{\rho}_\theta \right) d\theta - \int_{\theta_b}^{\theta_t} D \frac{\bar{\rho}_\theta}{\Pi} \partial_y \Pi d\theta \approx D_s \frac{\bar{\rho}_\theta(\theta_s)}{\Pi(\theta_s)} \partial_y \bar{\theta}'_s. \tag{32}$$

The second term on the lhs of Eq. (32) can be approximated as

$$\begin{aligned} -\int_{\theta_b}^{\theta_t} D \frac{\bar{\rho}_\theta}{\Pi} \partial_y \Pi d\theta &= -\int_{\theta_b}^{\theta_t} D \frac{\bar{\rho}_\theta}{\Pi} \overline{\mathcal{H}(\theta - \theta_s)} d\theta \\ &= D_s \overline{\left[\frac{\bar{\rho}_\theta(\theta_s)}{\Pi(\theta_s)} \partial_y \theta_s \right]} \approx D_s \frac{\bar{\rho}_\theta(\theta_s)}{\Pi(\theta_s)} \partial_y \bar{\theta}'_s, \end{aligned} \tag{33}$$

which leaves us with

$$-\int_{\theta_b}^{\theta_t} D \left(\frac{\rho_\theta}{f} \beta - \partial_y \rho_\theta \right) d\theta \approx 0. \tag{34}$$

Equation (34) states that the diffusive flux associated with the planetary vorticity gradient and the generalized thickness gradient has to integrate to zero over the depth of the troposphere.

Comparison of Eq. (32) to Eq. (11) in Schneider (2004) shows that the first integral on the lhs is similar in the two approaches, but Schneider does not include the second integral on the lhs, which appears because of the factor Π in the definition of the thickness-weighted mean PV. This term balances the surface contribution on the rhs of Eq. (32). Schneider (2004) therefore remains with an additional surface potential temperature gradient contribution on the rhs of Eq. (34). This additional term allows him to balance the momentum budget with a vertically constant eddy diffusivity.

Equation (34) shows that D cannot be constant in general. This can be seen by noting that the second term in the integral in Eq. (34) approximately integrates to zero if D is assumed vertically constant between θ_b and $\bar{\theta}_i$:

$$\int_{\theta_b}^{\bar{\theta}_i} D \partial_y \rho_\theta d\theta = -Dg^{-1} [\partial_y|_{\theta} \bar{p}(\bar{\theta}_i) - \partial_y \bar{p}_s^s] \approx 0. \quad (35)$$

As justified in Schneider (2004), we here assumed that both the isentropic ‘‘slope’’ at the tropopause, $\partial_y|_{\theta} \bar{p}(\bar{\theta}_i)$, and the surface pressure gradient, $\partial_y \bar{p}_s^s$, are negligible. The first term in the integral in Eq. (34), however, is positive definite. As in the QG model, we thus find that the eddy diffusivity D in general cannot be vertically constant throughout the entire depth of the troposphere.

b. A scaling relation for the criticality parameter

Equation (34) can be used to obtain a scaling relation between the criticality parameter and the vertical structure of the eddy diffusivity. As argued above, the momentum budget cannot generally be satisfied if the eddy diffusivity is assumed vertically constant. Instead, we will show that the eddy diffusivity is generally expected to be larger near the surface and decay toward the tropopause, as in the QG case discussed in section 2.

To derive a scaling argument for the vertical structure of the eddy diffusivity, we separate the integral of the mass transport in Eq. (34) into two parts above and below some level $\theta_1(y)$, which yields

$$D_1 \int_{\theta_b}^{\theta_1} \left(\frac{\rho_\theta}{f} \beta - \partial_y \rho_\theta \right) d\theta = -D_2 \int_{\theta_1}^{\bar{\theta}_i} \left(\frac{\rho_\theta}{f} \beta - \partial_y \rho_\theta \right) d\theta, \quad (36)$$

where we defined bulk diffusivities D_1 and D_2 for the two layers, which formally can be described as weighted vertical averages of the eddy diffusivity over the respective layer. We further assume that the lower layer is chosen such that it includes the entire surface layer, which, at any given latitude, comprises all isentropes that intersect with the surface at some time or longitude. We

thus require that $\mathcal{H}(\theta_1 - \theta_s) = 1$ at all times and longitudes. Ignoring again contributions due to the isentropic slope at the tropopause, $\partial_y|_{\theta} \bar{p}(\bar{\theta}_i)$, and the surface pressure gradient, $\partial_y \bar{p}_s^s$, we then find that

$$D_1 \left\{ \frac{[\bar{p}(\theta_1) - \bar{p}_s^s] \beta}{f} - \partial_y|_{\theta} \bar{p}(\theta_1) \right\} \approx -D_2 \left\{ \frac{[\bar{p}(\bar{\theta}_i) - \bar{p}(\theta_1)] \beta}{f} + \partial_y|_{\theta} \bar{p}(\theta_1) \right\}. \quad (37)$$

Equation (37) is the generalization of the two-layer QG relation in Eq. (5), where layer thicknesses have been replaced by the corresponding pressure ranges, and the isentropic slope becomes $\partial_y|_{\theta} \bar{p} \approx \partial_z \bar{p} \partial_y|_{\theta} \bar{z}$. As in the QG problem, the mass fluxes in the two layers can be written in terms of the eddy diffusivity multiplied by the sum of a contribution associated with the layer-integrated planetary vorticity gradient β and a contribution associated with the thickness gradient in each layer, which in turn is given by the isentropic slope at the interface.

Analog to the QG case, Eq. (37) can be used to derive a scaling relation for the criticality parameter as

$$\xi \sim \frac{fs}{\beta H} \sim \frac{[D]}{\Delta D}, \quad (38)$$

where $[D] = \{[\bar{p}_s^s - \bar{p}(\theta_1)]D_1 + [\bar{p}(\theta_1) - \bar{p}(\bar{\theta}_i)]D_2\} / [\bar{p}_s^s - \bar{p}(\bar{\theta}_i)]$ denotes the vertical mean of the eddy diffusivity, $\Delta D \equiv D_1 - D_2$, $s \equiv \partial_y|_{\theta} \bar{z}(\theta_1)$, and $H \equiv -\partial_p \bar{z}(\theta_1) [\bar{p}_s^s - \bar{p}(\bar{\theta}_i)]$.

Notice that we required that the lower layer include the entire surface layer. Apart from that requirement, Eq. (38) can technically be derived for any choice of the height of the layer interface. However, the bulk diffusivities D_1 and D_2 represent poorly defined layer averages if the weighting factor, $[(\rho_\theta/f)\beta - \partial_y \rho_\theta]$, takes on large positive and negative values within a single layer. The sign of this factor is given by the sign of the extended PV gradient and is typically negative throughout the SL, owing to the dominant contribution associated with the isentropic outcrops. An adequate choice of the layer interface is thus given by the first level above the SL where the PV gradient becomes positive. Schneider (2005) finds that this is typically just above the SL; in the simulation discussed here, the sign change occurs somewhat higher than the top of the SL.

Relation (38) shows that vertical variations in the eddy diffusivity over the troposphere may be small (compared to its mean value) only if $\xi \gg 1$, which is in agreement with the fact that eddies tend to be more barotropic in the limit of large supercriticalities. If, on the other hand, the criticality is $O(1)$, vertical variations

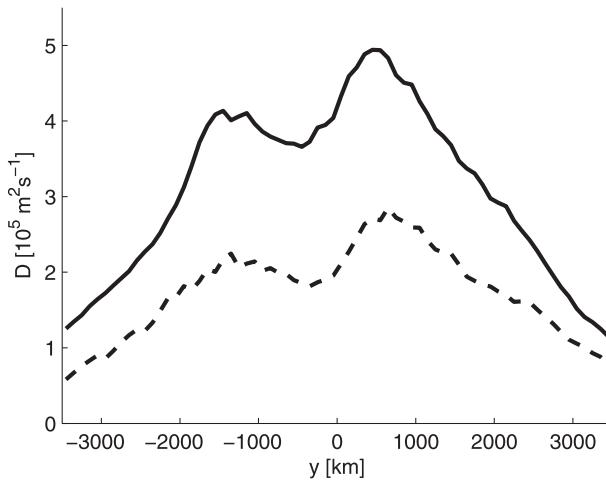


FIG. 4. Surface (solid) and upper-tropospheric (dashed) eddy diffusivity. The surface eddy diffusivity is calculated from a flux-gradient relationship of surface potential temperature as in Eq. (24). The upper-tropospheric eddy diffusivity is calculated as an effective bulk eddy diffusivity integrated from the first level above the SL where the PV gradients becomes positive, θ_1 , to the top of the troposphere as $D_{UT} = \int_{\theta_1}^{\theta_t} (\bar{\rho}_\theta \bar{v} \bar{P}^* / \bar{P}^*) / \int_{\theta_1}^{\theta_t} (\bar{\rho}_\theta \partial_y \bar{P}^* / \bar{P}^*) d\theta$. Notice that both quantities are independent of the definition of PV fluxes and gradients on near-surface isentropes.

in the eddy diffusivity are expected to be on the same order as the eddy diffusivity itself. This is in agreement with the numerical simulation discussed in section 4. A rough estimate for the criticality in the simulation can be obtained from Fig. 2. With $a = f_0/\beta = 6250$ km, we have $\xi = (a/H)s \approx 2$, which is in qualitative agreement with the observed upper-tropospheric eddy diffusivity being approximately 50% smaller than the eddy diffusivity near the surface (see Fig. 4). An extensive series of numerical simulations supporting the scaling relation (38) will be discussed in a follow-up paper.

The argument for the relationship between the criticality and the vertical structure of D is closely related to the analysis by Held (1978, 1982), who argues that the criticality parameter needs to be close to one, if the depth of the tropopause is set by the depth scale of the baroclinic eddies, and therefore equals the vertical scale over which the eddy diffusivity decays. This is likely to be the case in Earth's present atmosphere and for a considerable range of parameters around the present state, which would explain why the criticality of the atmosphere is $O(1)$, and is reported to be rather insensitive to changes in the external forcing in numerical simulations (Schneider 2004; Schneider and Walker 2006). However, Zurita-Gotor (2008) and Jansen and Ferrari (2012) find strongly supercritical mean states when they force the system to be very different from today's atmosphere. According to relation (38), such states are possible if the eddy diffusivity decays only weakly over the depth of the troposphere.

6. Conclusions

We used the isentropic zonal momentum budget of the troposphere to derive a scaling relation, which links the criticality parameter to the vertical structure of the eddy diffusivity [Eq. (38)]. We found that the criticality parameter is inversely proportional to the relative change of eddy diffusivity between the surface and the tropopause. Marginally critical states are expected if the scale over which the eddy diffusivity decays is similar to the depth of the troposphere—that is, if the relative vertical variation of the eddy diffusivity is $O(1)$. Supercritical states are predicted if the eddy diffusivity varies only weakly in the vertical. Subcritical states would require the eddy diffusivity to decrease rapidly over a scale much smaller than the depth of the troposphere.

This result is in contrast to that of Schneider (2004), who argued that, in primitive equations, marginally critical states are obtained with an eddy diffusivity that is vertically constant throughout the whole depth of the troposphere, while strongly supercritical states are impossible (Schneider and Walker 2006). The difference stems from the different approaches taken to compute averages along isentropes in the surface layer. We use a closure for the interior eddy PV flux as defined in Koh and Plumb (2004), which can be justified by analysis of the PV variance budget and is supported by idealized numerical simulations. Schneider (2004) instead uses a closure for an extended eddy PV flux, which includes contributions from isentropes below the surface. We find that this closure overrepresents the surface layer mass transport associated with the form drag on outcropping isentropes.

With our approach, the relation between the criticality and the vertical structure of the eddy diffusivity is a direct extension of Green's (1970) result based on QG theory. Differences from the QG results are only quantitative in nature, and arise because of the finite isentropic slope and intersections of isentropes with the ground. While these results are illustrated through one simulation in this paper, we ran many more examples that confirm our results (Jansen 2012). It would also be instructive to test our scaling relation in less idealized simulations.

The revised scaling relationship in Eq. (38) has important implications for the macroturbulent equilibration of an atmosphere. In particular, the criticality parameter can become much larger than one. Both marginally critical and strongly supercritical states are possible if the magnitude and vertical structure of the eddy diffusivity can change. Notice, however, that the scaling relation is not a predictive theory for the criticality. Such a theory would require an independent prediction for the magnitude and vertical structure of the eddy diffusivity.

Theories for the eddy diffusivity are therefore needed to predict the response of the criticality parameter to changes in external parameters.

A number of additional effects important for the real atmosphere have been ignored in this study. For example, complex boundary layer dynamics and moisture both introduce sources and sinks of PV, raising questions about whether diffusive closures for the eddy PV fluxes hold, especially near the surface. Our work should be seen as a step in a hierarchy of studies. We extended results from QG theory to an idealized primitive equation system. Next we will need to connect our scaling for the extratropical adjustment to more realistic settings. This is left for future work.

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APPENDIX A

The PV Variance Budget and the Closure Problem

We want to relate the eddy PV flux to mean quantities in terms of a diffusive closure, which assumes that eddy PV fluxes are directed down the mean PV gradient. In the following we present justification for such a closure using the interior PV approach, based on the PV variance budget.

a. An equation for the thickness-weighted zonal-mean PV

We start with the adiabatic potential vorticity equation in isentropic coordinates (e.g., Vallis 2006):

$$\partial_t P + \mathbf{u} \cdot \nabla_\theta P = -rP, \quad (\text{A1})$$

where ∇_θ denotes the horizontal nabla operator, with derivatives taken along isentropes, and \mathbf{u} denotes the horizontal velocity vector. Dissipation is here represented by a simple linear Rayleigh sink term, but other dissipation mechanisms would lead to qualitatively similar results as long as they tend to reduce the PV variance (see also Plumb 1979). Taking the generalized thickness-weighted isentropic average of Eq. (A1), and rearranging terms, yields

$$\begin{aligned} \partial_t \overline{P^*} - \frac{\overline{P}}{\overline{\rho_\theta}} \partial_t \overline{\rho_\theta} + \frac{\overline{\rho_\theta P}}{\overline{\rho_\theta^2}} \partial_t \overline{\rho_\theta} + \frac{1}{\overline{\rho_\theta}} \nabla_\theta \cdot (\overline{\rho_\theta \mathbf{u}^* P^*}) \\ + \frac{1}{\overline{\rho_\theta}} \nabla_\theta \cdot \overline{\rho_\theta \hat{\mathbf{u}} \hat{P}} - \frac{\overline{P}}{\overline{\rho_\theta}} \nabla_\theta \cdot (\mathbf{u} \rho_\theta) = -r \overline{P^*}. \end{aligned} \quad (\text{A2})$$

We can now use the (adiabatic) thickness equation

$$\partial_t \rho_\theta + \nabla_\theta \cdot (\mathbf{u} \rho_\theta) = 0, \quad (\text{A3})$$

which can be shown to hold similarly for the generalized thickness^{A1} ρ_θ to cancel the second and the last terms on the lhs of Eq. (A2). If we further use the isentropic average of Eq. (A3) to rewrite the third term on the lhs of Eq. (A2) as

$$\frac{\overline{\rho_\theta P}}{\overline{\rho_\theta^2}} \partial_t \overline{\rho_\theta} = \frac{\overline{P^*}}{\overline{\rho_\theta}} \partial_t \overline{\rho_\theta} = -\frac{\overline{P^*}}{\overline{\rho_\theta}} \nabla_\theta \cdot (\overline{\rho_\theta \mathbf{u}^*}),$$

we arrive at

$$\partial_t \overline{P^*} + \overline{\mathbf{u}^*} \cdot \nabla_\theta \overline{P^*} + \frac{1}{\overline{\rho_\theta}} \nabla_\theta \cdot (\overline{\rho_\theta \hat{\mathbf{u}} \hat{P}}) = -r \overline{P^*}. \quad (\text{A4})$$

b. An equation for PV variance

Subtracting Eq. (A4) from Eq. (A1), we obtain an equation for the PV perturbations $\hat{P} = P - \overline{P^*}$:

$$\partial_t \hat{P} = -\mathbf{u} \cdot \nabla_\theta \hat{P} - \hat{\mathbf{u}} \cdot \nabla_\theta \overline{P^*} + \frac{1}{\overline{\rho_\theta}} \nabla_\theta \cdot (\overline{\rho_\theta \hat{\mathbf{u}} \hat{P}}) - r \hat{P}. \quad (\text{A5})$$

To derive a variance equation, we multiply both sides by $\rho_\theta \hat{P}$ and average, which after some algebra yields

$$\begin{aligned} \frac{1}{2} \partial_t \overline{\rho_\theta \hat{P}^2} - \frac{1}{2} \overline{\hat{P}^2} \partial_t \overline{\rho_\theta} = -\frac{1}{2} \nabla_\theta \cdot (\overline{\rho_\theta \mathbf{u} \hat{P}^2}) + \frac{1}{2} \overline{\hat{P}^2} \nabla_\theta \cdot (\overline{\rho_\theta \mathbf{u}}) \\ - \overline{\rho_\theta \hat{\mathbf{u}} \hat{P}} \cdot \nabla_\theta \overline{P^*} - r \overline{\rho_\theta \hat{P}^2}. \end{aligned} \quad (\text{A6})$$

The thickness equation (A3) implies that the second term on the lhs and the second term on the rhs cancel. For a zonally reentrant channel, the budget simplifies to

$$\frac{1}{2} \partial_t (\overline{\rho_\theta \hat{P}^2}) = -\frac{1}{2} \partial_y (\overline{\rho_\theta v \hat{P}^2}) - \overline{\rho_\theta \hat{v} \hat{P}} \partial_y \overline{P^*} - r \overline{\rho_\theta \hat{P}^2}. \quad (\text{A7})$$

Note that the triple correlation is included in the flux term (i.e., the first term on the rhs), which describes the flux of PV variance associated with the total meridional velocity $v = \bar{v}^* + \hat{v}$. If the turbulence is sufficiently homogeneous such that the meridional advection of variance can be ignored, we find that in a statistically steady state

$$\overline{\hat{v} \hat{P}} \partial_y \overline{P^*} = -r \overline{\hat{P}^2}. \quad (\text{A8})$$

Since the rhs of Eq. (A8) is negative definite, the thickness-weighted eddy PV flux has to be down the thickness-weighted mean gradient. If the turbulence

^{A1} The conservation equation for the generalized thickness [Eq. (A3)] can be derived from the conservation equations for thickness $\partial_t \sigma + \nabla_\theta \cdot (\mathbf{u} \sigma) = 0$ (e.g., Andrews et al. 1987) and the thermodynamic equation for surface potential temperature $\partial_t \theta_s + \mathbf{u} \cdot \nabla_\theta \theta_s = 0$.

is not homogeneous and the meridional advection of PV variance is a dominant contribution, a local downgradient closure is not defensible, though the eddy PV flux still needs to be downgradient in a domain-averaged sense.

APPENDIX B

Relations between the Interior and Extended PV Fluxes and Gradients in the Surface Layer

Here we want to show how the mass transport associated with the extended eddy PV flux can be written in terms of the mass transport associated with the eddy flux of interior PV plus roughly half of the mass transport associated with the surface eddy potential temperature flux. The extended PV gradient, instead, can be written in terms of the interior PV gradient plus a contribution associated with the full surface potential temperature gradient.

The extended eddy PV flux contribution to the mass transport can be rewritten by subtracting Eq. (20) from Eq. (21), which yields

$$F_{\bar{v}\bar{P}}^{\text{ext}} = F_{\bar{v}\bar{P}}^* + F_{\bar{v}_s'\bar{\theta}_s'} - F_{\bar{v}_s'\bar{\theta}_s'}^{\text{ext}} \approx F_{\bar{v}\bar{P}}^* + \frac{1}{2}F_{\bar{v}_s'\bar{\theta}_s'}, \quad (\text{B1})$$

where for the last step we used that $F_{\bar{v}_s'\bar{\theta}_s'}^{\text{ext}} \approx (1/2)F_{\bar{v}_s'\bar{\theta}_s'}$ because $\overline{P_{\text{ext}}^*}(\bar{\theta}_s) \approx 2\overline{P^*}(\bar{\theta}_s)$, as shown in Eq. (23).

Using that $\overline{P_{\text{ext}}^*} = \Pi^{-1}\overline{P^*}$, where we defined $\Pi(\theta) \equiv \overline{\mathcal{H}(\theta - \theta_s)}$, we find that the estimated mass transport associated with the extended PV gradient, on the other hand, becomes

$$\begin{aligned} F_{D\partial_y\bar{P}}^{\text{ext}} &= \int_{\theta_b}^{\theta_i} \frac{\bar{\rho}_\theta D_s \partial_y \overline{P_{\text{ext}}^*}}{\overline{P_{\text{ext}}^*}} d\theta \\ &= \int_{\theta_b}^{\theta_i} \frac{\bar{\rho}_\theta D_s \partial_y \overline{P^*}}{\overline{P^*}} d\theta - \int_{\theta_b}^{\theta_i} \bar{\rho}_\theta \Pi^{-1} D_s \partial_y \Pi d\theta \\ &= \int_{\theta_b}^{\theta_i} \frac{\bar{\rho}_\theta D_s \partial_y \overline{P^*}}{\overline{P^*}} d\theta + \overline{\frac{f}{\overline{P^*}} D_s \delta(\theta - \theta_s) \partial_y \theta_s} \\ &\approx \int_{\theta_b}^{\theta_i} \frac{\bar{\rho}_\theta D_s \partial_y \overline{P^*}}{\overline{P^*}} d\theta + \frac{f}{\overline{P^*}(\bar{\theta}_s)} D_s \partial_y \bar{\theta}_s \\ &= F_{D\partial_y\bar{P}^*} + F_{D\partial_y\bar{\theta}_s}. \end{aligned} \quad (\text{B2})$$

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^{B1} Note that $\overline{P_{\text{ext}}^*} \equiv (f/\bar{\rho}_\theta)$, while $\overline{P^*} \equiv [\overline{\mathcal{H}(\theta - \theta_s)} f/\bar{\rho}_\theta] = \Pi \overline{P_{\text{ext}}^*}$.