An Analysis of a Barotropically Unstable, High–Rossby Number Vortex in Shear

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ABSTRACT

The interactions of the barotropic instability found at low levels in tropical cyclones and a shear forcing are presented. Previous works have indicated that at low levels of tropical cyclones, the inner edge of the core may be barotropically unstable and thereby able to support counterpropagating vortex Rossby wave interactions. It has also been demonstrated that hurricanes and other barotropic vortices possess innate, dry abilities to maintain themselves when under the duress of vertical wind shear. This work will address how these two separate processes interact with each other.

In this study, the barotropic ring is given additional vorticity in the outer regions to mimic observations more closely. This allows for the outward propagation of energy and simultaneous reduction of the radius of maximum wind. When this vortex is sheared, it is found that the shear forcing, which acts as a de facto wavenumber-1 forcing, does not noticeably alter the growth of the most unstable mode, wavenumber 3. The tilt precession of the vortex is altered greatly, as the tilt becomes both larger and slower. Palinstrophy and deformation analysis indicates that overall peak mixing is also reduced, owing to changes in the axi-symmetrization process. Energetics analyses show that the radial component of the shear forcing acts to generate eddies while the tangential component of the shear tends to destroy eddies. The calculations are carried out a second time with another center-finding method, which shows the tilt to be much smaller and more variable while imparting a large wavenumber-1 signal in Fourier analyses.

1. Introduction

Over the past few decades, the skill in the prediction of tropical cyclone (TC) track has steadily increased as model physics and our general understanding of hurricanes and their interactions with environments have improved. Hurricane intensity forecasting, however, has remained relatively unchanged in the same period with very little improvement (Krishnamurti et al. 2005). With recent improvements in computing power, higher-resolution hurricane models can better resolve inner-core dynamical processes (Braun 2002). While this complicates the analysis of model output, coarser models only prescribe TCs as monopolar vortices. Monopolar vortices are those vortices where the vorticity peaks in the center and decreases radially outwards. Observations indicate that TCs can sometimes possess a vorticity structure at low levels that is ringlike in nature: moderate vorticity in the eye, a peak of vorticity inside the eyewall (Kossin and Eastin 2001; Hendricks et al. 2012), and lower values of vorticity in the outer regions (skirt) of the storm (Mallen et al. 2005). Barotropic models indicate that this ringlike structure of vorticity could be barotropically unstable, thus allowing for the generation of mesovortices and polygonal eyewalls (Schubert et al. 1999, hereafter S99). In an idealized barotropic model, similar to the one used in S99, Rozoff et al. (2009) showed that an unstable ring of vorticity will break down because of barotropic instability; however, the constant convergence at low levels, parameterized in their 2D simulations, prevents total vortex collapse, as seen in previous barotropic simulations (S99). Additionally, Hendricks et al. (2009) demonstrated that for certain ranges of inner-core vorticity values and vorticity peak values, the end result of dynamic rearrangement could be persistent mesovortices that do not resolve into a monopolar profile.

There are differing opinions on how the presence of low-level eddies impacts the evolution of a vortex. While internal eddies, perhaps caused either by counterpropagating vortex Rossby waves’ interactions with each other (S99) or

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through a Rossby–inertia–buoyancy instability with the skirt (Ford 1994; Hodys and Nolan 2008), were initially thought possibly to result in intensification of a vortex through eddy momentum flux mechanisms (Montgomery and Kallenbach 1997), other studies indicate that these eddies generally cause a weakening of the vortex (Nolan and Grasso 2003) if they affect the vortex’s evolution at all (Kwon and Frank 2008). Nolan et al. (2007) confirmed the results of Nolan and Grasso (2003) by demonstrating that in more realistic 3D simulations, when the perturbations are thermal in nature (as opposed to vorticity perturbations), they act to decay the storm slowly after an initial increase in intensity. Memelaou and Yau (2014), however, readdressed those simulations and showed that thermal perturbations can, in fact, intensify a vortex through eddy flux mechanisms and spiral windup from damped, sheared vortex Rossby waves. While spontaneous inertia–gravity wave emission could act to deform a vortex if the critical-layer vorticity rearrangement is not sufficient to balance the wave emission (Schecter and Montgomery 2006; Schecter 2008), more recent work has shown that the effects of these waves on the mean flow are negligible when compared with the “slow mode” vorticity and PV evolution (Moon and Nolan 2010; Hendricks et al. 2010).

Previous works have shown that, when comparing symmetric simulations versus 3D simulations, there is a 15% reduction in intensity of the 3D vortex (Yang et al. 2007). A subsequent experiment performed by Bryan and Rotunno (2009) found that, in an axisymmetric model, when inner-core mixing is turned off, there is no appreciable change in hurricane evolution. Conversely, Persing et al. (2013) demonstrated fundamental differences between 3D and axisymmetric dynamics and showed how small-scale eddies could possibly act to intensify a storm. Full-physics 3D simulations by Nguyen et al. (2011) show that TCs go through periods where the low-level vorticity profile possesses a ringlike structure followed by an episodic ring breakdown and inner-core mixing that redistributes the low-level vorticity, thus stabilizing the system. This also acts as a brake on intensification. Aggregated observations by Rogers et al. (2013) also indicate that inner-core vorticity profiles of TCs fall into these two regimes, and each regime is associated with either intensification (ringlike structure) or nonintensification (monopole).

A unique case where mesovortices may have actually acted to cause a rapid deepening of a TC has been identified by Reasor et al. (2009): eastern Pacific Hurricane Guillermo (1997). While it is difficult to isolate mesovortices as the cause of the rapid deepening, circumstantial evidence points to the fact that the environment was not altogether favorable for rapid intensification (Sikowski and Barnes 2009). What is more remarkable is that Guillermo’s rapid intensification occurred while it was under duress from environmental wind shear. The effects of environmental shear on TCs are generally thought to be negative (Bender 1997). Modeling studies indicate the storm can weaken from ventilation of the warm core aloft (Frank and Ritchie 2001); from the interruption of moist inflow at low levels from dry, cool downdrafts (Riemer et al. 2010); or from midlevel entrainment of dry air (Tang and Emanuel 2010). Despite all these negative effects of shear, Guillermo still managed to intensify rapidly. This could be due in part to the interaction of these mesovortices with a TC’s response to shear: the enhanced convection and the generation of convective available potential energy (CAPE) in the downshear-left quadrant of the storm (Frank and Ritchie 2001; Corbosiero and Molinari 2003; Molinari and Vollaro 2010). Full-physics simulations have shown that passage of mesovortices through these CAPE zones produces intense convection trailing the mesovortex because of enhanced convergence (Wu and Braun 2004; Braun and Wu 2007).

It is unclear why, exactly, Guillermo rapidly intensified. It might perhaps be that the mesovortices played a nontrivial role in modification of the vortex system as a whole. As Reasor et al. (2009) pointed out, Guillermo was undergoing moderate (8 m s$^{-1}$) vertical wind shear while maintaining these discrete, low-level eddies traversing the eyewall. There are several interesting facets of this phenomenon, both dynamically and thermodynamically, including but not limited to, eye–eyewall mixing (Cram et al. 2007), mean vortex profile modification through mixing (S99), modification of the environment through convection (C. Velden 2014, personal communication), and perhaps a unique path to rapid intensification. This work will focus on the dynamical interactions between the mesovortices and the shear forcing. Previous work has demonstrated that TCs possess an innate dry dynamical ability to maintain physical coherence when under the duress of an external environmental forcing, either through differential vortex advection and precession in the vertical (Jones 1995; Smith et al. 2000; Jones 2000) or through conservation of angular momentum via critical-layer damping (Schecter et al. 2002; Reasor et al. 2004; Schecter and Montgomery 2006; Schecter 2008). No previous study has addressed how the shear forcings and the low-level eddies interact with each other from a pure dynamical framework. Does shear affect unstable mode growth? Do mesovortices affect tilt rotation? This work will focus on these questions. To simplify the first steps of this very complicated problem, moisture will be removed to isolate the pure dynamics of the situation. This work is
meant to be a purely dry dynamical analysis of this interaction, providing a baseline of the dynamics against which more complicated, full-physics simulations can be compared.

To address this particular problem, the research performed in this study includes the analysis of the evolution of a barotropically unstable, high–Rossby number vortex in shear simulated in a fully nonlinear, compressible, adiabatic model. By “barotropically unstable,” it is meant that the vortex satisfies the Rayleigh necessary condition for instability: the radial gradient of PV changes sign (S99). By “high Rossby number,” it is meant that the vortex’s Rossby number is assumed to be greater than 10, thus placing it in the cyclostrophic balance regime (Holton 2004, 66–67).

Section 2 will present the description of the model used in the simulations. Section 3 will provide a discussion of the barotropic instability and the tilt. Section 4 will present an energetics analysis. Section 5 will discuss the sensitivity of results to the definition of the center. Section 6 will discuss conclusions drawn from the analysis.

2. Experiment description

The model used for these simulations is the compressible, nonlinear cloud model CM1 (Bryan and Fritsch 2002), version 12 (CM1r12). The primary goals of this research are to investigate the adiabatic dynamics of unstable, high–Rossby number vortices in shear. As a result, the special considerations given to the treatment of moist physics in the CM1 will unfortunately not be used at present in this research. Three simulations will be run: a perturbed ring that mimics the initial conditions of the S99 ring, except modified with a significant skirt of vorticity; a sheared monopole that is based on the modified S99 ring profile, preserving the radius of maximum winds (RMW) and the skirt profile; and a sheared, perturbed ring that is the same profile as the modified, nonsheared ring with skirt. All storms are initially barotropic such that they do not vary vertically and are symmetric. The perturbations, much like S99, are small, finite-vorticity perturbations composed of wavenumbers 1–8. The profiles of each storm are defined using the Hermite polynomial to shape the vorticity field. When a profile is modified, it is given an additional skirt of vorticity. The original formulation of the S99 profile is almost Rankine-like in nature, as the maximum vorticity decreases to negative vorticity values radially very quickly. The modified profiles yield a much more gradual radial decrease of vorticity. The vorticity profile of each model run, compared with those profiles without skirts of vorticity, is given in appendix A and can be seen in Fig. 1, as can the corresponding initial tangential wind profiles. In the CM1, all vortices are initialized—moist or dry—through an iterative process. For this case, the vorticity profile is integrated radially using the trapezoid rule to yield the wind field. The gradient wind equation is used to solve for the perturbation pressure field. This pressure field is then used to solve the hydrostatic equation. Were moisture a factor in this simulation, a density potential temperature correction would then be made at this point. In the latest version of the CM1, release 17 as of this writing, this process is repeated approximately 20 times to reach a thermodynamic equilibrium. In release 12, used here, this process was only performed once.

The maximum wind is approximately 55 m s\(^{-1}\) at approximately 61 km from the center of the domain. While the vorticity profile in the core is somewhat unnatural compared with observational records in Reasor et al. (2009), the skirt profiles of vorticity are not significantly different from the observed profile of skirt vorticity (Mallen et al. 2005). There is a slight increase in maximum tangential winds, but the structure is not deemed to be significantly different. It should be noted, however, that a linear stability analysis performed on this profile (not shown) indicates that it does not meet the
requirements of critical-layer damping theory (Schecter and Montgomery 2006; i.e., the radial gradient of PV is zero at the critical radius), so the expected evolution of the sheared vortices will be more akin to Jones (1995) and Smith et al. (2000) than to Schecter et al. (2002) and Reasor et al. (2004).

Each simulation used 1-km horizontal resolution and 500-m vertical resolution with 20 levels. The lateral breadths of the domains were different: 1708 km × 1708 km for the nonsheared run and 2002 km × 2002 km for the sheared run. The reason for the larger domain for the sheared simulations is that during the sheared runs, the vortices move. This larger domain allowed the sheared vortices to move without coming into contact with the 100-km Rayleigh damping layers on all lateral sides, thus maintaining zero circulation in the domain. Each model run was 48 h with 15-min output times. A fifth-order advection was recommended (G. Bryan 2008, personal communication) because the diffusion is implicit. CM1r12 does not allow for a variation in the Coriolis parameter, so the simulations are limited to an f-plane set to about 15°N. A simplified vertical sounding that holds the Brunt–Väisälä frequency constant at 1.44 × 10^{-2} Hz was also used. All analyses will be performed on the half-gridpoint scalar grid, as the CM1 utilizes a C-grid configuration (Arakawa and Lamb 1977).

In most prior works that use a shear forcing, such as Jones (1995) and Frank and Ritchie (2001), the shear is initialized as a perturbation. Typically, as shear is introduced as a perturbation, its effects are maintained via periodic domain boundaries. Since we have implemented sponge layers to absorb the inevitable emission of acoustic, gravity, and inertia–gravity waves, it is necessary that the shear be relegated to the background flow. The shear is entirely zonal and has the following profile:

\[ U_0 = u_0 \cos \left( \frac{\pi z}{z_i} \right), \tag{1} \]

where \( u_0 \) is the scaling factor, \( z \) is the model level, and \( z_i \) is the top of the model. In all simulations with shear, the scaling factor is \(-2 \text{ m s}^{-1}\), for a total wind shear value of \( 4 \text{ m s}^{-1}\).

To maintain geostrophic and thermal wind balance of the background, a thermal perturbation and a pressure perturbation must be added. The full derivation is in appendix B. The thermal perturbation is given by

\[ \theta_{bg} = \theta_0 + \left( f_0 \frac{d u_0}{c_p} \right) \left[ e^{-\left( \frac{N_0^2}{g}\right)(z-z_d)} \right] \times \left[ \frac{N_0^2}{g} \cos \left( \frac{\pi z}{z_T} \right) + \frac{\pi}{z_T} \sin \left( \frac{\pi z}{z_T} \right) \right]. \tag{2} \]

The pressure field must also be brought into balance. This can be accomplished with the following:

\[ \Pi_{bg} = \Pi_0 + \left( \frac{f_0 u_0 y}{c_p \theta_{sfc}} \right) \left[ e^{-\left( \frac{N_0^2}{g}\right)(z-z_d)} \right] \times \cos \left( \frac{\pi z}{z_T} \right). \tag{3} \]

All symbols for Eqs. (2) and (3), which allow for the background to be in linear geostrophic and hydrostatic balance, are defined in appendix B.

3. Vortex evolution

a. Determining the center of the vortex

A significant hurdle in analyzing a storm undergoing inner-core PV rearrangement is determining where, exactly, the center of the storm is located. Ryglicki and Hart (2015) described the three main classes of center calculation: local extreme (LE), weighted grid point (WGP), and minimization of azimuthal variance (MAV). For this research, the pressure-ring centroid (WGP class) was developed when the other methods, such as the PV centroid (also WGP), proved problematic. The theory behind using the pressure-ring centroid is that it completely ignores the dynamically evolving center of the storm and finds the center of mass based on the broader structure of the storm.

The mathematical formulation is not dissimilar from the calculations of PV centroid as presented in Reasor et al. (2004), Jones (2004), or Riemer et al. (2010). A first guess is required to start the process: here, pressure minimum at a given level is used. From there, a search box is created. Next, only the grid points whose pressure values fall within a certain range are used. For this specific vortex, the pressure contour around which the ring is built at all levels was determined by trial and error. Ultimately, the pressure contour at each level that was at a distance of 1.5 × RMW at the initial time (approximately 90 km) was chosen, and all grid points that fell within 50 Pa in either direction of that contour were used for the weighting. An exploration of traditional PV centroid techniques for use in finding the center will be presented in section 5.

b. Unstable mode evolution

The evolutions of a variety of unstable, barotropic rings in two dimensions were investigated by Hendricks et al. (2009). They investigated the difference of ring breakdown as they varied the innermost region of vorticity. Here, the innermost section of vorticity is similar to that of S99, but the outer regions are changed. This alters the most unstable mode, and the resultant
evolution is a hybrid wavenumber-3–4 structure, as seen in Fig. 2 for the control ring. Figure 3 shows the magnitudes of PV wavenumbers 0–5 normalized by maximum value for the control ring at the lowest model level. Wavenumber 3 appears to be the dominant mode, as it grows the largest and the fastest initially, which is what gives the breakdown a more triangular shape, as opposed to the square shape seen in S99. After the primary mixing event, the axisymmetrization process returns energy to the mean.

The initial evolution of the inner core of the sheared ring, as seen in Fig. 4, is nearly identical to the control ring. The differences seen here are due to the distortion of the PV field by the shear as the vortex begins to rotate. As with the vortices in Jones (1995), the skirts of the vortices begin to distend as the rotation commences. The maximum mean tangential winds, the minimum surface pressures, and the RMW evolutions of the two runs can be seen in Fig. 5. Aside from more noise in the RMW calculation and a slightly larger pressure minimum (owing to the perturbation fields), the evolutions are similar. Much like in S99 and in Kossin and Schubert (2001), spinning up the eye also results in a pressure fall via gradient wind-balance considerations: as the

![Fig. 2. Evolution of the PV field of the control ring at z = 1 (0.25 km) and at t = 0, 2, 4, and 6 h.](image-url)
centrifugal term grows, so must the pressure gradient term to maintain balance. The Fourier decomposition, with the shear forcing removed, can be seen in Fig. 6. The clear difference is the existence of a prominent wavenumber-1 signature at early times, a result that can be explained entirely by the presence of the shear forcing, but there are more phenomena of interest in this evolution.

Figure 7 shows the normalized differences of wave-numbers 0, 1, and 3, respectively, between the two vortices. There are three important features. As seen for wavenumber 1, the shear forcing induces growth at early times. It becomes larger again at later times at inner radii (less than 40 km). Conversely, in the control ring, the PV redistribution leads to a stable monopole. There is a secondary rotation of a new PV max (not

FIG. 3. Thumbnails of Hovmoller diagrams of PV azimuthal wavenumbers (wv#) 0–5 for the control ring at z = 1 (0.25 km). Values are PV normalized by maximum value of that specific wavenumber in time. Maximum values are 13.70, 6.54, 5.15, 6.21, 3.42, and 3.60 PV units (1 PVU = 10^{-6} K kg^{-1} m^{2} s^{-1}) for wavenumbers 0–5, respectively.
shown) after the collapse of the sheared ring. The shear forcing is most likely causing this additional rotation in the core, most likely from a combination of cross-vortex flow and induced differential advection from other layers. A second feature—albeit very subtle—of note in Fig. 7 is the difference in wavenumber 3. At the earliest times, when the modes are growing and can be thought of in a linear sense, the instability grows faster and larger in the nonsheared case, as evidenced by the warmer colors at 40 and 55 km, respectively. These results indicate that the instabilities are shifted inward slightly more quickly. Conversely, for the third feature, between 8 and 12 h in the annulus between 30 and 40 km, wavenumber 3 has a few notable peaks in the sheared-ring time series. This would indicate that, perhaps, the mixing is not as complete in the sheared run as it is in the control run.

A succinct way to visualize differences in mixing is by calculating palinstrophy, as a measure of the sharpness of vorticity gradients and, by proxy, a quantification of mixing. Since this is a three-dimensional model with forcing and damping layers, established conservation laws that directly relate energy, enstrophy, and palinstrophy will not hold (S99). Regardless, palinstrophy—in
In this case, the palinstrophy of the PV field—can still reveal details about the evolution of the mixing. PV palinstrophy (PVP) is defined as

$$PVP = \frac{1}{2}(VPV \cdot VPV). \quad (4)$$

Figure 8 shows the comparison of normalized volume-integrated PVP of the two vortices. The normalized maximum PVP value of the sheared ring is less than that of the control ring, indicating that shear is limiting the most vigorous mixing of the vortex. One way to analyze this evolution physically is to contemplate each mesovortex as a transient individual element of the surrounding fluid: the vortex itself. By maintaining this perspective, a field that can provide insight into what is happening to these individual elements is the deformation field. The total deformation field can be defined as

$$def = \sqrt{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)^2}, \quad (5)$$

where \(u\) and \(v\) are the Cartesian wind components. The first term in Eq. (5) is the stretching deformation, while the second term is the shearing deformation. While the specific components of deformation are quite sensitive to the coordinate system, the magnitude of the total deformation field as presented here mathematically is coordinate insensitive (Bluestein 1993, 99–106). Deformation has been used most often in midlatitude analyses for frontogenesis and frontolysis studies, but by maintaining the proper perspective, it can prove useful in this context.

Figure 9 shows the PV field and the deformation field of both rings at the lowest level at 8 h into their respective simulations (sheared simulation again with shear removed). In this image, it appears that there is still some
semblance of a coherent wavenumber-3 asymmetry in the sheared simulation. What is most striking about this image is the mesovortex on the southern side of the core. In the control run, the element is being intensely deformed on both sides by the flow. In the sheared run, however, the deformation appears to be less severe. Part of the reason for this appears to be the location of the element in the center of the vortex. Looking again at the Fourier decomposition differences in Fig. 7, in the 7–8-h time frame, wavenumber 0 is considerably larger in the control ring at the innermost radii. Contrast that with wavenumber 1: there exists a relative maximum in the sheared-ring time series at the same time at 0–10 km. This, plus inspection of PV fields in Fig. 9, indicates that in the evolution of the sheared ring, one of the specific mesovortex fluid elements is not able to maintain its position at the center of the vortex, which is thus reducing its effects, and the effect of the vortex core as a whole as this mesovortex...
becomes the new core, on deforming the other two mesovortices. To express this more succinctly, the non-linear aspects of the axisymmetrization process are being altered by the external forcing. The question that can be posed here is if this evolution is due to the cross-vortex flow or some other aspect of induced flow by the other levels. Figure 10 is a plot halfway up the domain at a height of 4.75 km (z = 10) of PV and PVP at 10 h into the respective simulations, the time at which the PVP is at its largest value in the control run. The sheared ring is showing two coherent mesovortices, while the control ring evidently has a dominant singular mesovortex developing into a new center. It is important to keep in mind that at this level, due to the chosen cosine shear profile as expressed in Eq. (1), the background wind field is near zero, so the PV evolution seen here might be more a result of flow induced from other levels of the vortex. Deducing specific contributions to this complicated and highly nonlinear problem is reserved for future work, but from this analysis, it appears that both cross-vortex flow and induced flow from other layers play a role in altering the axisymmetrization process.
c. Tilt evolution

To examine the evolution of the tilt of the vortex, a companion vortex simulation was run to act as a benchmark against which the barotropically unstable vortex could be compared. This second vortex in shear possesses the same RMW and the same skirt of vorticity, as seen in Fig. 1, but the inner core is completely stable: a monopole of vorticity. Figure 11 shows the evolution of the top of both vortices at the highest model level ($z = 20$, 9.75 km) normalized to their respective surface centers that have been repositioned to the origin. As mentioned, since the skirt of vorticity does not satisfy the requirements for critical-layer damping (Schecet and Montgomery 2006), the vortices evolve as those seen in Jones (1995). The monopole eventually settles into a fairly distinct regime while the ring can clearly be seen to expand its tilt and to have its precession speed slowed down. A comparison of both can be seen in Fig. 12. The weakening caused by the mixing nearly doubles the second radial tilt peak and slows down the second rotation of the ring vortex by nearly 4 h. The divergence of the rotational speed, however, does not happen until 24 h into the simulation, the point at which the top vortex and the bottom vortex come closest to realignment. This also has a noticeable effect on track. Figure 13 shows the track of the vortex at $z = 10$. This level was chosen since it displays the smallest amount of orbital characteristics. Assuming that mass loss due to inertia–gravity wave radiation is negligible, this demonstrates that a rearranging core can alter the motion of a vortex given the same amount of total mass in the system.

To show why this particular vortex’s tilt evolves as described, it is necessary to follow the work done by Jones (1995) and Smith et al. (2000) and investigate the penetration depth. Following Hoskins et al. (1985), penetration depth can be defined as

$$ D = \left[ \xi (f_0 + \zeta) \right]^{1/2} \left( \frac{L}{N} \right), $$

where $\xi$ is the modified Coriolis parameter ($f_0 + 2\Omega$), $\Omega$ is the angular velocity, $\zeta$ is the relative vorticity, $L$ is the length scale, and $N$ is the Brunt–Väisälä frequency. Figure 14 shows the Hovmöller diagram of penetration depth at the lowest model level. The penetration depth drops considerably as the ring collapses, yielding a storm that is less resilient in shear. When inspecting the initial profile, the penetration depth of the RMW is 8.7 km. When using the final profile, the penetration depth of the RMW is around 4.5 km. Even at the same initial RMW, 61 km, the penetration depth decreases to around 5.5 km. It should be noted that at the initial time, the penetration depth is slightly larger than the scale height (calculated to be approximately 8.2 km in the core, virtually time invariant) and then proceeds to drop considerably below that. Jones (1995) admitted not knowing how to properly implement the metric, so a quantitative estimation is somewhat difficult. Acknowledging that fact, the analysis does give some physical insight into what is happening, as explored in Smith et al. (2000), since the vertical depth to which the inertial flow of a given layer of the vortex is felt is reduced after mixing. Had these vortices been resonantly damped even after the mixing event, as in Reasor et al. (2004), the tilt evolution probably would have shown nutation around an axis in the downshear-left vicinity. Despite that fact, the qualitative difference is expected to be the same, as mixing increases the Rossby deformation radius as well (not shown). After mixing, the RMW moves outward, and the mean tangential wind decreases, thus decreasing the depth to which the inertial flow of each individual level is felt, so the tilt will increase.

4. Energetics

Having explored the changes in internal mixing and tilt in the sheared ring, it would be worthwhile to investigate how eddies and the shear interact with the mean vortex from an energetics perspective in order to investigate more deeply how the elements interact in time. The process for deriving a wave-mean energy equation is described in Kwon and Frank (2005). In their work, the derived wave-mean energy equation for a dry, barotropic vortex is problematic for the current situation with the presence of the shear since the shear itself adds a constant forcing on the mean vortex. The derivation of a new energetic equation is found in appendix C. The new energy equation is
\[
\frac{d}{dt} \left( \text{EKE} \right) dV = \left[ -ru'v' \frac{\partial \Omega}{\partial r} dV \right]_{\text{BT}} - \left[ \left( \frac{g}{\theta N_0} \right)^2 u' \theta' \frac{\partial \bar{\theta}}{\partial r} dV \right]_{\text{BC}} - \left[ v'' \frac{\partial \bar{\theta}}{\partial z} dV \right]_{\text{DVDZ}} + \left[ \bar{\Omega} u' v' - u' v' \frac{\partial \bar{\theta}}{\partial r} - \left( \frac{g}{\theta N_0} \right)^2 \left( \frac{u' \theta' \frac{\partial \bar{\theta}}{\partial r}}{\bar{\Theta} \bar{\Theta}_0} + \bar{\Omega} \bar{\theta}' \frac{\partial \bar{\theta}}{\partial \lambda} \right) \right] dV.
\]

(7)

All terms are defined and explained in appendix C. The energetics calculation is slightly different from previous work. In previous work (e.g., Kwon and Frank 2005; Wang 2002), it is implied that all calculations are performed using one surface center. When a storm is as highly tilted as this vortex, using a surface center would result in wavenumber-1 signatures that would dominate the signal. To compensate for this tilt, the center is

Fig. 9. (left) PV (PVU) and (right) total deformation (10^{-3} \text{s}^{-1}, see text for definition) fields of the (top) control ring and (bottom) sheared ring at height z = 1 (0.25 km), 8 h into the simulation.
determined at each level from the pressure-ring centroid method. Afterward, a subdomain of the vortex is extracted at that level. For instance, for model level 10, the domain is relocated to the pressure-ring centroid. Then levels 9, 10, and 11 are extracted using the pressure-ring centroid of level 10. This way, the horizontal dynamics are adequately accounted for, as are the vertical derivative terms. This is repeated at each level except for at the boundaries, where second-order upward- and downward-in-space (relative to the vertical level) calculations are used, respectively. It should be noted that the way Eq. (7) is calculated is not exactly conservative, as opposed to one calculated from a single center, but the results should be more illuminating for the evolutions of interest: the internal eddies.

The shorthand used for each term in Eq. (7) is BT for the first term on the right, BC for the second term, DVDZ for the third term, UVs for the fourth term, UsV for the fifth term, UsTh for the sixth term, and ThThs for the seventh term. An a posteriori analysis indicates that only four of these terms are significant: BT, UVs, UsV, and ThThs. These terms are on the order of $10^{12}$ kg m$^{-3}$ s$^{-5}$, while the other three (BC, DVDZ, and UsTh) are of an order less. The time series of all terms are shown in Fig. 15. Only the nonnegligible terms will be discussed.

For the nonnegligible terms, as seen in Fig. 15, the BT term, as expected, is the dominant feature, matching Kwon and Frank (2005). The initial peak at near 5 h is consistent with the control ring (not shown) and is a primary result of the internal mixing. The oscillations...
following that are caused by an altering of the axisymmetrization process (as explored in section 3b) and could possibly be a result of the center-location uncertainty (to be explored in section 5). After the primary mixing event, the changes in BT are dominated by the tilt, as the only times when BT actually lowers eddy kinetic energy (EKE) are when the storm is coming out of its tilt, which would reduce the distortion of the horizontal PV field. For the shear barotropic terms, UVs and UsV, they act in opposite manners. For UsV, much like BT, this term appears to follow the distortion of the PV field as a result of the tilt evolution. UVs almost always acts to generate EKE. Finally, ThThs is interestingly the smoothest field of all seven rhs terms. As the storm tilts, it produces a wavenumber-1 potential temperature anomaly (Jones 1995). ThThs is greatest when the storm is tilting into the warm side of the perturbation theta field from the geostrophic balancing derived in section 2. EKE grows when the storm tilts up the gradient, since the cold, tilt-induced potential anomalies of the vortex are in the direction of the tilt. The main points presented by this analysis are that when the storm “views” the shear as a radial wind, it only acts to generate eddies when the storm is highly tilted, whereas the interaction of the tangential component of the shear and radial wind anomalies almost always act to generate eddies.

5. Sensitivity to center location

It is natural to question how valid these results are in light of the method found to determine the centroid. Ryglicki and Hart (2015) demonstrated how different center-finding methods can affect tilt calculations and tangential and radial wind calculations. To test how sensitive the results are to the center location, the analyses performed in sections 3 and 4 will be performed again with a new center location.

Before moving on to the more complicated scenario of a dynamically evolving core, the monopole will be tested first. We tested 11 centroids: 10 PV centroids and the pressure-ring centroid. The PV centroids are a collection of various weights, ranging from PV to PV cubed (PV³), and summation boxes, ranging in size from 120 km × 120 km to 500 km × 500 km. Figure 16a shows the evolution of the top of the sheared monopole using a select number of methods, again normalized by a low-level
center repositioned to the origin. The PV centroid with the largest box with no additional weighting possesses the largest tilt, while the PV$^3$ weighting appears to asymptote along with the smallest box (120 km × 120 km). An increase in the value of PV by any weight would place more weight on the higher PV values in the core. This exercise would indicate that using a PV centroid for determining the center of a vortex is sensitive to the vorticity distribution outside the core, echoing results of Ryglicki and Hart (2015). The pressure-ring centroid method stands apart from two of the PV centroids as its evolution indicates a less elliptical tilt path and a more rounded one.

Figure 16b shows the evolution of the top of the sheared ring using the various centroid metrics. From looking at the PV centroids, it is apparent that some begin to falter as the ring breaks down. The larger boxes, like the monopole, are skewed with the skirt. With smaller boxes, the evolution becomes more chaotic, becoming practically unusable when applying the smallest box. Combining a larger box with a weighting applied to the PV does smooth out the results slightly, but there is still a higher-order rotation postmixing within the broader tilting envelope of motion. The pressure-ring centroid, while still possessing a certain amount of noise in the signal, is able to display a better

![Tilt Magnitude](image1)

![Tilt Angle](image2)

**FIG. 12.** Evolution of the two vortices: (top) the distance from the top level ($z = 20$) to the bottom level ($z = 1$) and (bottom) the degree of rotation of the separation from the top level to the bottom level, where $0^\circ$ is east in the domain.
element of stability in attempting to analyze the tilt of the vortex, as it is ignorant of internal mixing processes.

To demonstrate how a different center-finding metric would have altered results, a PV$^3$ centroid with a weighting box of 300 km × 300 km was selected for this part of the analysis. First, Fig. 17 is a comparison of the tilt evolution of the two centers. This further exemplifies the idea of a larger-scale tilt, as captured by the pressure-ring centroid, versus a smaller-scale tilt, as captured by the PV$^3$ centroid. The center that is based on the PV centroid oscillates in the broader envelope of the tilt, as captured by the pressure-ring centroid. Figure 18 shows differences of Fourier amplitudes of the two cylindrical wind components between the two centroids at the lowest model level. Wavenumber 0 for both components is the azimuthal mean, while the plots of wavenumbers 1–10 can be thought of as the azimuthal perturbations. In the tangential wind mean, for the PV$^3$ centroid, much of the inner-radius wind is interpreted as tangential mean, while the pressure-ring centroid maintains larger eddies at inner radii for both radial and tangential winds.

The differences in how perturbation and mean winds are established by the two different center-finding algorithms will strongly affect the components of Eq. (7). Since this is essentially a closed system, the total EKE differences should be relatively small. What will change,
however, are the components that contribute to the generation and destruction of EKE. Figure 19 shows the normalized differences between the values of that particular component at that moment in time. What is being plotted is

\[ \frac{x_{\text{PRC}} - x_{\text{PV}}}{\sigma_{\text{PRC}}}. \] (8)

This is the difference of the two components normalized by the standard deviation of the component from the original pressure-ring centroid calculation. Essentially, this quantifies significance relative to the original metric. Four terms show the largest variations between the center choices: BT, BC, UVs, and UsV. Each of these terms involves a component of radial wind, which is unsurprising given previous work (Ryglicki and Hart 2015). UsV and UVs act in exactly opposite ways, which is expected, as reorientation of the center will change the conversion of the shear flow into cylindrical components as a matter of perspective. BT changes follow the UVs changes, again owing to how the wind is perceived using different centers.

6. Summary and conclusions

In this paper, the evolution of a barotropically unstable, high–Rossby number vortex in shear has been investigated. A vortex based on the techniques outlined in Schubert et al. (1999) is expanded to three dimensions and inserted into CM1 (Bryan and Fritsch 2002). The vortex is then sheared using a simple cosine profile, which is geostrophically balanced with the addition of thermal and pressure perturbations to the background. To compare the results of the sheared vortex, two additional vortices were run to isolate the distinct components of the sheared, unstable vortex: a barotropically unstable vortex but not in shear, and a sheared vortex that possesses a stable radial profile of PV. The first component, the breakdown of the ring, is shown through PV palinstrophy, deformation, and Fourier analysis to mix less in the sheared vortex than in the control vortex. The second component, the tilt, is shown to both increase in displacement and decrease in rotational velocity regardless of center-finding metric.

The goals of this research were to ascertain what changes in evolution the shear forcing had on the growth of the unstable mode and, conversely, what changes in evolution the unstable mode and mixing had on the tilt. The initial unstable ring was run to serve as a control. For the modification of the mixing, it was shown that the cross-vortex flow of the shear causes an increase in wavenumber 1 at the edges of the ring. This should quicken the mixing processes, and looking at the differences in wavenumber 3 (the most unstable mode), there is a slight radial displacement inwards of the wavenumber-3 anomaly in the sheared ring. Overall, it appears that the effects of shear on the initial unstable-mode growth are negligible. After the growth of the unstable mode, however, the nonlinear axisymmetrization...
process is greatly affected, as the mesovortex fluid elements are able to maintain coherency as the runs progress, owing to a reduced deformation field. The reason for this appears to be that, because of a combination of vortex precession and cross-vortex flow, a new center caused by advection of a mesovortex into the core is not allowed to complete in the same manner as if the vortex were not externally forced. Determining which component—precession or cross-vortex flow—is more significant is an interesting problem for future work.

The change in tilt evolution is quite drastic. As shown, the radius of maximum winds moves out, and its magnitude decreases from 56 to 48 m s\(^{-1}\). As a result, the penetration depth of the flow decreases. At initial times, the penetration depth at the RMW is 8.7 km. At later times, it decreases to 4.5 km. The induced flow caused by

---

**FIG. 16.** Evolutions of the top levels of the (top) sheared monopole and (bottom) sheared ring normalized to the surface center repositioned to the origin using the pressure-ring centroid, PV centroid (120 km \(\times\) 120 km), PV centroid (300 km \(\times\) 300 km), and PV\(^3\) centroid (300 km \(\times\) 300 km).
each layer on the others is dramatically decreased. The tilt of the postmixed sheared ring increases radially and decreases rotationally. The rotational timeframe does not significantly alter until the near realignment occurs at 22 h into the simulation. The vortex never realigns nor approaches a steady-state tilt since it is not critically damped. As a result, the vortex evolves following non-linear paths, as shown in Jones (1995) and Smith et al. (2000). Were the vortex resonantly damped, it would most likely approach a downshear-left tilt with nutations in the tilt as a result of nonlinear advective characteristics of the vortex itself. Regardless, the tilt should still increase because the storm weakens and the core broadens considerably postmixing.

An energetics analysis was performed on the sheared ring using two different centers. First, it was shown that in the derived energetics equation, both shear terms contribute nonnegligibly to eddy kinetic energy generation. Despite attempting to account for tilt-relative asymmetries—as opposed to using the “quasimode” view, where the tilt is a perturbation on the vortex itself (Reasor et al. 2004)—the signal of eddy generation at later times was dominated by the tilt, since the skirt of vorticity distorts with the tilt. Second, it was shown that differences in energetics conversion terms dealing with radial wind, regardless of origin, are very sensitive to center. When using a center that follows higher values of PV, the decomposition of the winds favors the mean.
tangential wind, while a broader-scale center, the pressure-ring centroid, reduces mean values in favor of inner-core and far-field perturbations. The two terms that involve the effects of the background flow on the vortex act in completely opposite ways when altering the center.

Unfortunately, without a wave–wave analysis using scale interactions (Krishnamurti et al. 2005), it is not possible to discover the real wave–wave interaction during nonlinear mixing. This would give a clearer indication of wave–wave nonlinear interactions after the storm has mixed. At initial times, only certain wavenumbers are exponentially unstable and can be thought of in a linear sense, and something substantive can be said about their differences. After wavenumbers 3 and 4 break, wave–wave evaluation would give a better

![Fig. 18. Normalized azimuthal Fourier amplitude differences between the pressure-ring centroid and the PV$^3$ centroid for mean and perturbation (left) radial winds and (right) tangential winds at the lowest model level. Cold (warm) colors indicate that the PV$^3$ centroid (pressure-ring centroid) is greater.](image-url)
explanation of the differences between the sheared and nonsheared rings. Since the mesovortices remained coherent, a Lagrangian analysis would also be a worthwhile investigation to augment the deformation field (Haller and Yuan 2000). Additionally, Kwon and Frank (2008) showed that when adding moisture, the barotropic results of Kwon and Frank (2005) were muted. While adding moisture will no doubt damp development of unstable modes (Schecter and Montgomery 2007), the redistribution of PV and weakening of the maximum wind will be detrimental to storms’ dynamical resilience to shear, and this phenomenon most likely reduced the resilience of storms like Guillermo.

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APPENDIX A

Vorticity Profiles for Model Runs

These are the vorticity profiles used to prescribe the initial vortex for the adiabatic runs. The Hermite polynomial \( S(s) = 1 - 3s^2 + 2s^3 \), satisfying \( S(0) = 1 \), \( S(1) = 0 \), and \( S'(0) = S'(1) = 0 \) is used as a transition between two constants of vorticity. The general form of the vorticity profile follows Schubert et al. (1999). In cylindrical coordinates, the vorticity profile is as follows:

\[
\zeta(r, \lambda, z) = \begin{cases} 
\zeta_i, \\
\zeta_i S[(r - r_i + d_i)/2d_i] + \zeta_{i+1} S[(r_i + d_i - r)/2d_i], \\
\zeta_{i+1},
\end{cases} \]

where \( i \) subscripts indicate a particular segment of the profile, and \( r_n \) and \( d_n \) are constants that allow for smooth transitions between vorticity segments. Table A1 summarizes the values used for each
vorticity profile. For the two runs that use a ring-like profile, a small vorticity perturbation is initialized to encourage the barotropic instability mechanism:

\[
0, \quad S[(r_1 + d_1 - r)/2d_1], \quad 0 \leq r \leq r_1 - d_1,
\]
\[
r_1 - d_1 \leq r \leq r_1 + d_1,
\]
\[
r_1 + d_1 \leq r \leq r_2 - d_2,
\]
\[
r_2 - d_2 \leq r \leq r_2 + d_2,
\]
\[
r_2 + d_2 \leq r \leq r_{\text{ext}},
\]
where \( \xi_{\text{amp}} \) is the amplitude of the perturbation, \( 1.0 \times 10^{-3} \) s\(^{-1} \), \( r_{\text{ext}} \) is the termination radius at which the winds are set to zero, \( m \) is a wavenumber ranging from 1 to 8, and \( \lambda \) is the azimuthal angle. For the small amplitude perturbation, the values of \( r, d, \) and \( r_{\text{ext}} \) are from the S99 ring without a skirt of vorticity.

**APPENDIX B**

Derivation of Background Thermal and Pressure Fields

Geostrophic balance on an \( f \) plane in the CM1 with a purely zonal wind and a dry atmosphere can be defined as follows:

\[
c_p \theta \frac{\partial \Pi}{\partial y} = -f_0 u, \quad (B1)
\]
where \( c_p \) is the specific heat of dry air, \( \theta \) is the potential temperature, \( f_0 \) is the Coriolis term, \( u \) is the zonal wind, and \( \Pi \) is the Exner function:

\[
\Pi = \left( \frac{p}{p_0} \right)^{Rc_p} \theta_0, \quad (B2)
\]
where \( p \) is pressure, \( p_0 \) is a reference pressure (1000 hPa), \( R \) is the dry gas constant, and \( c_p \) is the specific heat capacity of dry air. Assuming that products of perturbations are negligible and that the base-state horizontal pressure gradient is zero, Eq. (B1) can be integrated and rearranged to yield an expression for the perturbation pressure:

\[
\Pi' = -\frac{f_0 u y}{c_p \theta_0}, \quad (B3)
\]
where prime indicates perturbations and \( u \) is the wind profile:

\[
u(z) = u_0 \cos \left( \frac{\pi z}{z_T} \right), \quad (B4)
\]
where \( u_0 \) is the scaling factor (\( -2 \) m s\(^{-1} \) at the surface), \( z \) is the height, and \( z_T \) is the top of the domain. Since the vertical profile is dry with constant Brunt–Väisälä frequency, the background temperature profile is as follows:

\[
\theta_0 = \theta_{\text{slc}} \exp \left[ -N_0 \frac{(z - z_{\text{slc}})}{g} \right], \quad (B5)
\]
where \( \theta_{\text{slc}} \) is the surface potential temperature (300 K), \( g \) is gravity, and \( N_0 \) is the Brunt–Väisälä frequency. The total background (subscript “bg”) pressure field is

---

**Table A1. Values of each segment in each profile: ring with no skirt (Ring, NS), monopole with no skirt (MP, NS), ring with skirt (Ring, S), and monopole with skirt (MP, S). Vorticity is in units of per second and \( r \) and \( d \) are in units of kilometers.**

<table>
<thead>
<tr>
<th>Values</th>
<th>Ring, NS</th>
<th>MP, NS</th>
<th>Ring, S</th>
<th>MP, S</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_1 )</td>
<td>( 3.5759 \times 10^{-4} )</td>
<td>( 2.0730 \times 10^{-3} )</td>
<td>( 3.5759 \times 10^{-4} )</td>
<td>( 2.050 \times 10^{-3} )</td>
</tr>
<tr>
<td>( r_1 )</td>
<td>37.5</td>
<td>57.5</td>
<td>37.5</td>
<td>57.5</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>7.5</td>
<td>7.5</td>
<td>7.5</td>
<td>7.5</td>
</tr>
<tr>
<td>( \xi_2 )</td>
<td>( 3.3999 \times 10^{-3} )</td>
<td>( -4.0653 \times 10^{-4} )</td>
<td>( 3.3999 \times 10^{-3} )</td>
<td>( 3.5759 \times 10^{-4} )</td>
</tr>
<tr>
<td>( r_2 )</td>
<td>57.5</td>
<td>414.5</td>
<td>57.5</td>
<td>120.5</td>
</tr>
<tr>
<td>( d_2 )</td>
<td>7.5</td>
<td>12.5</td>
<td>7.5</td>
<td>52.5</td>
</tr>
<tr>
<td>( \xi_3 )</td>
<td>( -4.0653 \times 10^{-5} )</td>
<td>0.0</td>
<td>( 3.5759 \times 10^{-4} )</td>
<td>( -4.0653 \times 10^{-5} )</td>
</tr>
<tr>
<td>( r_3 )</td>
<td>414.5</td>
<td>-</td>
<td>120.5</td>
<td>535.5</td>
</tr>
<tr>
<td>( d_3 )</td>
<td>12.5</td>
<td>-</td>
<td>52.5</td>
<td>12.5</td>
</tr>
<tr>
<td>( \xi_4 )</td>
<td>0.0</td>
<td>-</td>
<td>-4.0653 \times 10^{-5}</td>
<td>0.0</td>
</tr>
<tr>
<td>( r_4 )</td>
<td>-</td>
<td>-</td>
<td>535.5</td>
<td>-</td>
</tr>
<tr>
<td>( d_4 )</td>
<td>-</td>
<td>-</td>
<td>12.5</td>
<td>-</td>
</tr>
<tr>
<td>( \xi_5 )</td>
<td>-</td>
<td>-</td>
<td>0.0</td>
<td>-</td>
</tr>
<tr>
<td>( r_{\text{ext}} )</td>
<td>450.0</td>
<td>450.0</td>
<td>550.0</td>
<td>550.0</td>
</tr>
</tbody>
</table>
\[ \Pi_{bg} = \Pi_0 - \frac{f_0 u y}{c_p \theta_{sfc}} \exp \left[ -N_0^2 \frac{z - z_{sfc}}{g} \right]. \]  

(B6)

We next assume that the shear perturbation is in hydrostatic balance. In the CM1, this can be expressed as

\[ c_p \frac{\partial \Pi'}{\partial z} = g \frac{\partial \theta'}{\partial z}. \]  

(B7)

Substituting Eqs. (B3) and (B4) into a linearized Eq. (B7) and performing the necessary calculus and algebra while remembering that \( N_0 \) is constant yields an expression for the background thermal field:

\[ \theta_{bg} = \theta_0 + \frac{f_0 u_0 y}{g} \left\{ \exp \left[ \frac{N_0^2 (z - z_{sfc})}{g} \right] \right\} \times \left[ \frac{N_0^2}{g} \cos \left( \frac{\pi z}{z_T} \right) + \frac{\pi}{z_T} \sin \left( \frac{\pi z}{z_T} \right) \right]. \]  

(B8)

APPENDIX C

Derivation of the Eddy Kinetic Energy Equation

For deriving an equation for eddy kinetic energy (EKE), the relevant equations of motion, including a background shear flow in cylindrical coordinates, are as follows:

\[
\begin{align*}
\frac{\partial}{\partial t} (u + u_s) &= -(u + u_s) \frac{\partial}{\partial r} (u + u_s) - (v + v_s) \frac{\partial}{\partial \lambda} (u + u_s) \\
&\quad - w \frac{\partial}{\partial z} (u + u_s) + \frac{(v + v_s)^2}{r} \\
&\quad + f_0 (v + v_s) - \frac{\partial}{\partial r} (\theta + \theta_s); \\
\frac{\partial}{\partial t} (v + v_s) &= -(u + u_s) \frac{\partial}{\partial r} (v + v_s) - (v + v_s) \frac{\partial}{\partial \lambda} (v + v_s) \\
&\quad - w \frac{\partial}{\partial z} (v + v_s) - \frac{(u + u_s)(v + v_s)}{r} \\
&\quad - f_0 (u + u_s) - \frac{1}{r \lambda} (\theta + \theta_s); \\
\frac{\partial}{\partial t} (\theta + \theta_s) &= -(u + u_s) \frac{\partial}{\partial r} (\theta + \theta_s) - (v + v_s) \frac{\partial}{\partial \lambda} (\theta + \theta_s) \\
&\quad - w \frac{\partial}{\partial z} (\theta + \theta_s),
\end{align*}
\]

(C1a)

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= \frac{\partial \theta}{\partial r} - \frac{\partial \phi}{\partial \lambda} - \frac{\partial \theta}{\partial z};
\end{align*}
\]

(C1b)

where \( u \) is the radial wind, \( v \) is the tangential wind, \( w \) is the vertical wind, \( \theta \) is the potential temperature, and \( \phi \) is the geopotential, the use of which is for the ease of notation and will be addressed further in the derivation. The subscript \( s \) indicates shear forcings. Each variable can be separated into its azimuthal mean and its eddy term. Since these vortices are initially barotropic, further assumptions can be made. The symmetric component of both radial velocity and vertical velocity can be assumed to be zero. The symmetric components of tangential velocity, of potential temperature, and of pressure vary in height, radius, and time only. The fluctuations in density are assumed to be small. The relevant assumptions of a barotropic vortex are as follows:

\[
\begin{align*}
u &= \bar{v}(r, z, t) + v'(r, \lambda, z, t), \\
u &= u'(r, \lambda, z, t), \\
w &= w'(r, \lambda, z, t), \\
\theta &= \bar{\theta}(r, z, t) + \theta'(r, \lambda, z, t), \\
\rho &= \bar{\rho}(r, z, t), \text{ and} \\
p &= \bar{p}(r, z, t) + p'(r, \lambda, z, t).
\end{align*}
\]

(C2)

With the absence of an initial secondary circulation, the symmetric components of both radial velocity \( u \) and vertical velocity \( w \) of the vortex can be neglected. The components of the shear forcing can be defined as follows:

\[
\begin{align*}
u_s &= -u_0 \sin \lambda \cos \left( \frac{\pi z}{z_T} \right) \quad \text{and} \\
v_s &= -u_0 \cos \lambda \sin \left( \frac{\pi z}{z_T} \right).
\end{align*}
\]

(C3a) and (C3b) into Eqs. (C1a) and (C1b) yields the following:

\[
\begin{align*}
\frac{\partial \bar{u}}{\partial t} &= -u \frac{\partial \bar{u}}{\partial r} - \frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{u}}{\partial \lambda} + \frac{v^2}{r} + f_0 v - \frac{\partial \phi}{\partial r} \\
&\quad - u_s \frac{\partial \bar{u}}{\partial r} + \frac{v v_s}{r} + \frac{\partial \bar{u}}{\partial \lambda} - \frac{w \partial \bar{u}}{\partial z}, \quad \text{and} \\
\frac{\partial \bar{v}}{\partial t} &= -u \frac{\partial \bar{v}}{\partial r} - \frac{\partial \bar{v}}{\partial z} - \frac{\partial \bar{v}}{\partial \lambda} - \frac{u v_s}{r} \frac{\partial \bar{v}}{\partial \lambda} - \frac{\partial \bar{v}}{\partial \lambda} - \frac{\partial \bar{v}}{\partial z}.
\end{align*}
\]

(C4a)

Substituting Eq. (C2) in Eqs. (C4a) and (C4b) while under the properties of strict linearization, the equations become
\[ \frac{\partial u'}{\partial t} = -\frac{\Omega}{\partial r} \frac{\partial u'}{\partial \lambda} + 2\Omega u' + f_0u' - \frac{\partial \phi'}{\partial r} + \Omega \nu_s \quad \text{(C5a)} \]

\[ \frac{\partial v'}{\partial t} = -u' \frac{\partial \Omega}{\partial r} - \frac{\partial u'}{\partial \lambda} - \frac{w' \partial \phi'}{\partial z} - \frac{\partial \nu_s}{\partial r} - \frac{1}{r} \frac{\partial \phi'}{\partial \lambda} - \frac{\partial \nu_s}{\partial \lambda}, \quad \text{(C5b)} \]

where \( \Omega \) is the mean angular rotation \( \Omega/\lambda \). It is assumed that the vortex is in gradient wind balance. The product of shear terms and eddy terms has also been neglected. Equation (C5a) is multiplied by \( u' \), and Eq. (C5b) is multiplied by \( v' \), then the two equations are added together. Taking the azimuthal mean and using the product rule yields

\[ \frac{\partial}{\partial t} \left( \frac{u'^2 + v'^2}{2} \right) = -ru' v' \frac{\partial \Omega}{\partial r} - u' w' \frac{\partial \phi'}{\partial z} - \left( \frac{\partial \phi'}{\partial r} + \frac{1}{r} \frac{\partial \phi'}{\partial \lambda} \right) + \Omega \nu_s v_s - u_s \nu_s \frac{\partial \phi'}{\partial r}. \quad \text{(C6)} \]

A similar process can be undertaken for the thermodynamic equation. Implementing Eq. (C2) in Eq. (C1c) while using a strict linearization on the resulting equation yields

\[ \frac{\partial \theta'}{\partial t} = -u' \frac{\partial \theta'}{\partial r} - \frac{\partial \theta'}{\partial \lambda} - w' \frac{\partial \phi'}{\partial z} - \frac{\partial \nu_s}{\partial r} - \Omega \frac{\partial \phi'}{\partial \lambda}. \quad \text{(C7)} \]

To get this equation into the proper form, it must be multiplied by the factor \( [g/(\partial N_0)]^2 \theta' \). Upon azimuthal averaging, the thermodynamic equation becomes

\[ \frac{d}{dt} \left[ \text{(EKE)} \right] dV = - \int ru' v' \frac{\partial \Omega}{\partial r} dV - \int \left( \frac{g}{\partial N_0} \right)^2 u' \theta' \frac{\partial \theta'}{\partial r} dV - \int v' w' \frac{\partial \phi'}{\partial z} dV + \int \left[ \Omega \nu_s v_s - u_s \nu_s \frac{\partial \phi'}{\partial r} - \left( \frac{g}{\partial N_0} \right)^2 u_s \theta' \frac{\partial \theta'}{\partial r} + \Omega \theta' \frac{\partial \phi'}{\partial \lambda} \right] dV, \quad \text{(C10)} \]

where

\[ dV = (\rho g) (d\lambda)(d\xi), \quad \text{(C11)} \]

and

\[ \text{EKE} = \left\{ \frac{1}{2} \left[ u'^2 + v'^2 + \left( \frac{g}{\partial N_0} \right)^2 \theta'^2 \right] \right\}, \quad \text{(C12)} \]

which is the azimuthally averaged EKE.

The first three terms on the right-hand side of Eq. (C10) have been explained in Kwon and Frank (2005). They are the barotropic conversion term, the baroclinic conversion term, and the vertical shear conversion term. The final integral includes the shear terms separate from the vortex-exclusive terms. The first two are barotropic terms, while the last two are baroclinic terms. The first two terms deal with the modification of perturbation flows by the shear flow. In the first term, when the sign of the shear terms is positive, eddies grow when the radial perturbation velocity is positive, scaled by the mean angular velocity. The second barotropic term will contribute to eddy growth in areas when, given that the winds decrease (increase) radially outward, the radial winds caused by the shear and the tangential velocity perturbations are the same (opposite) sign.
The first baroclinic term involving shear behaves in the same way as the first baroclinic term: its sign is determined by the direction of eddy fluxes with respect to the gradient of mean temperature field, with its magnitude determined by a collocation of mechanical eddies with thermal ones. The second baroclinic term is the advection of thermal eddies along the background potential temperature profile. Cold (warm) eddies advected along a trajectory where the background temperature gradient increases (decreases) will cause eddy growth.

REFERENCES


