Trajectory Analysis of the Mechanism for Westward Propagation of Rossby Waves

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(Manuscript received 25 August 2014, in final form 19 December 2014)

ABSTRACT

An analysis of Rossby wave dynamics in two and three dimensions is carried from the point of view of a reference frame propagating with the zonal phase speed of the wave. Since trajectories and streamlines coincide in this reference frame, the mechanism for (westward) propagation of free waves has a different interpretation than in a reference frame fixed to the ground. In the wave reference frame, propagating free-wave solutions are possible only when parcels approach from the west. When parcels approach from the east, potential vorticity cannot be conserved along trajectories.

1. Introduction

A basic building block for the understanding of atmospheric and oceanic dynamics is the Rossby wave (Rossby 1939). The mathematical solutions for Rossby waves have been motivated through descriptive models that appeal to physical intuition. The arguments developed here have also been developed to supplement physical interpretation of the mathematical solutions and do not appear to have been previously explicitly described in the literature.

Holton and Hakim (2012, their Fig. 5.14) explain the westward propagation of Rossby waves by considering a closed chain of fluid parcels in a resting barotropic atmosphere oriented along a circle of latitude and with zero relative vorticity. A sinusoidal displacement of the chain, with the parcels conserving absolute vorticity, leads to positive relative vorticity for the parcels displaced southward and negative relative vorticity for the parcels displaced northward. The meridional velocity induced by the relative vorticity will then cause the pattern to propagate westward. Gill (1982) and Vallis (2006) present a similar argument, while Pedlosky (1987) includes additionally the effect of vortex stretching based on potential vorticity conservation. Cai and Huang (2013a) provide a physical explanation of the westward barotropic Rossby wave propagation in which the meridional variation of the Coriolis force plays the role of a mechanical barrier and the meridional acceleration is due to the Coriolis force acting on the ageostrophic zonal flow [see also Durran (1988)].

The above arguments combine the Lagrangian vorticity conservation and the Eulerian streamline–vorticity or velocity tendency relationships to describe the mechanism for westward propagation. In general, parcel trajectories do not coincide with the streamlines of the instantaneous velocity. Holton and Hakim (2012, their section 3.3) point out, “only for steady-state motion fields (i.e., fields in which the local rate of change of velocity vanishes) do the streamlines and trajectories coincide.” Some well-known solutions of the potential vorticity equation satisfy this steady-state condition for the coincidence of streamlines and trajectories, including a free Rossby mode viewed from a coordinate system moving with the zonal phase speed of the wave, or the steady Rossby wave response to topographic forcing. As will be shown below, the trajectory–streamline equivalence leads to an intuitively appealing viewpoint for describing the mechanism for westward propagation.

Potential vorticity conservation is the starting point for the discussion below, but with the difference from earlier applications that solutions to the potential vorticity conservation law are considered in a reference frame where the trajectories and streamlines coincide.

2. Barotropic Rossby waves

The motions are taken to satisfy conservation of Rossby potential vorticity in a barotropic incompressible fluid of constant depth. Following the shallow-water
formulation of Holton and Hakim (2012, their sections 4.5 and 5.7.1), the vertical component of the absolute vorticity $\eta$ is conserved following a parcel:

$$\frac{D\eta}{Dt} = 0, \quad (1)$$

where $D\eta/Dt$ denotes the time rate of change of a parcel following the horizontal velocity (Lagrangian time derivative), and $\eta$ is the sum of the vertical component of the relative vorticity $\zeta$ and the planetary vorticity $f$

$$\eta = \zeta + f. \quad (2)$$

A Cartesian midlatitude $\beta$ plane is assumed, so that $f=f_0+\beta(y-y_0)$, where $f_0$ is the planetary vorticity at meridional coordinate $y_0$ and $\beta$ and $f_0$ and $\beta$ can be taken as positive. Using $\delta$ to denote the change along a trajectory, (1) and (2) give $\delta\zeta + \delta f = 0$. Then

$$\delta f = \beta \delta y. \quad (3)$$

It follows that

$$\delta\zeta = -\beta \delta y. \quad (4)$$

Since $\beta > 0$, the relative vorticity change from (4) is anticyclonic, $\delta\zeta < 0$, for a northward displacement, $\delta y > 0$, and cyclonic, $\delta\zeta > 0$, for a southward displacement.

The Eulerian version of (1) (i.e., with respect to the spatial coordinate system) is obtained from $D\eta/Dt = \partial\eta/\partial t + \mathbf{u}\partial\eta/\partial x + \mathbf{v}\partial\eta/\partial y$, where the horizontal velocity $(\mathbf{u}, \mathbf{v})$ is nondivergent and given in terms of the streamfunction $\psi$ as

$$\mathbf{u} = -\frac{\partial\psi}{\partial y}, \quad \mathbf{v} = \frac{\partial\psi}{\partial x}. \quad (5)$$

The parcel velocities are then tangent to the streamlines, given by the surfaces of constant $\psi$. The relative vorticity is $\zeta = \partial\psi/\partial x - \partial\mathbf{u}/\partial y = \nabla^2 \psi$.

While the conservation law, (1), is concisely expressed in a Lagrangian framework in terms of parcel trajectories, which document the history of parcel positions, the relative vorticity is a property of the instantaneous spatial velocity distribution and then has inherently Eulerian properties.

The well-known solution for barotropic Rossby waves considers a small-amplitude zonally propagating wave of zonal wavenumber $k$ (disturbance wavelength $\lambda = 2\pi/k$), meridional wavenumber $l$, and phase velocity $c$ superimposed on a constant basic-state zonal flow $\mathbf{u}$. Denoting the deviation of a quantity from the basic state with a prime, the wave velocity vectors are $(\mathbf{u} + \mathbf{u}'[x, y, t], \mathbf{v}'[x, y, t])$ and the streamfunction is

$$\psi = -\mathbf{u}y + \psi'(x, y, t). \quad (6)$$

The linearized version of (1) is then

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\frac{\partial}{\partial x} + \mathbf{v}\frac{\partial}{\partial y}\right) \nabla^2 \psi' + \beta \frac{\partial}{\partial x} \psi' = 0. \quad (7)$$

The free barotropic Rossby wave solution to (7) is

$$\psi' = \psi_0 \cos[k(x-ct) + ly + \phi]. \quad (8)$$

The dispersion relation is given by

$$c = \frac{\mathbf{u} - \beta L}{K^2}, \quad (9)$$

where $K^2 = k^2 + \ell^2$. From (9), $(\pi - c) > 0$, and the Rossby waves propagate westward with respect to the basic-state flow.

a. Transforming to the wave reference frame

The coordinate system used in (7) is fixed with respect to the ground and will be referred to as the ground reference frame. The wave reference frame is defined to be one that is traveling with the wave—that is, translating with zonal velocity $c$ with respect to the ground. Denoting quantities evaluated in the wave frame with a tilde, the coordinates in the wave frame are related to those in the ground frame by

$$\tilde{x}, \tilde{y} = (x-ct, y). \quad (10)$$

Constant-amplitude waves propagating in the ground reference frame are steady state when viewed from the wave reference frame, with velocities

$$\bar{u}, \bar{v} = [\mathbf{u} + \mathbf{u}'(\tilde{x}, y, t_0) - c, \mathbf{v}'(\tilde{x}, y, t_0)], \quad (11)$$

where $t_0$ is a constant, and streamfunction

$$\tilde{\psi}(\tilde{x}, y) = (c - \mathbf{u})y + \psi'(\tilde{x}, y, t_0). \quad (12)$$

The streamfunction in (12) differs from the streamfunction in the ground reference frame (6) since the wave frame includes an added zonal velocity relative to the ground frame. Then absolute vorticity conservation along trajectories in the wave frame leads to cancellation of relative and planetary vorticity changes,

$$\delta\tilde{\zeta} = -\beta \delta y. \quad (13)$$

b. Mechanism for westward propagation

The discussion is restricted to the consideration of a single wave [i.e. perturbation streamfunction given by (8)] superimposed on the basic-state zonal flow. First, consider the case with no meridional variation of the
perturbation streamfunction, \( l = 0 \). The westward-propagation direction of a Rossby wave with respect to the basic-state flow can then be inferred without explicit knowledge of the dispersion relation (9). Figure 1 shows the two possible configurations of the sinusoidal-shaped trajectories for a zonally propagating wave viewed from the wave reference frame. Since the wave is steady in this reference frame, the streamlines coincide with the trajectories in both configurations. In Fig. 1a, the wave is hypothetically propagating westward with respect to the basic-state flow, so that in the wave reference frame the parcels are approaching from the west and departing to the east. In Fig. 1b, the wave is hypothetically propagating eastward with respect to the basic-state flow, so that the parcels are entering from the east and leaving to the west.

In the \( y \)-independent example, the sign of the relative vorticity can be inferred from inspection of the streamlines. From a natural coordinate point of view, the relative vorticity at the wave crests and troughs is entirely due to their curvature. It is then evident from Fig. 1a that when the wave propagates westward with respect to the mean flow, the curvature of the trajectories is anticyclonic for northward displacements of the trajectories from their mean latitudes and that the relative vorticity is then negative, while for southward displacements, the curvature is cyclonic and the relative vorticity is positive. On the other hand, if the wave propagates eastward with respect to the mean flow (Fig. 1b), northward displacement corresponds to cyclonic curvature and positive relative vorticity, while southward displacement corresponds to anticyclonic curvature and negative relative vorticity. Absolute vorticity conservation (18) implies that the relationship between meridional displacement and relative vorticity must be northward–anticyclonic and southward–cyclonic. Then only the westward-propagating configuration in Fig. 1a is consistent with absolute vorticity conservation and the phase speed of Rossby waves must be less than \( \pi \), or equivalently, \( (\bar{\pi} - c) > 0 \).

This steady-state explanation of the mechanism exploits the equivalence of the Lagrangian and Eulerian descriptions in the wave frame. The mechanism for westward propagation of a Rossby wave viewed from the wave reference frame is then that wavelike motions can only exist when parcels move from west to east along trajectories. When this is the case, the curvature of trajectories induced by an absolute vorticity–conserving latitudinal parcel displacement tends to return the parcels to their equilibrium latitudes.

When \( l \neq 0 \), the relationship between the sign of the relative vorticity and meridional displacement may not be so visually obvious, but the same argument can be made symbolically. Since streamlines and trajectories coincide in the wave reference frame, \( \delta \psi = 0 \) along a trajectory, and the relationship

\[
\delta y = \frac{\delta \psi(\hat{x}, y)}{\bar{u} - c}
\]  

follows from (12). The relative vorticity in the wave frame is

\[
\delta \zeta = -K^2 \delta \psi(\hat{x}, y).
\]

Then (14) can be written

\[
\delta \zeta = -K^2(\bar{\pi} - c)\delta y.
\]

As \( \beta \) and \( K^2 \) are positive, the sign of the relative vorticity change for a positive (negative) meridional displacement from the kinematic relationship (16) is consistent with that from absolute vorticity conservation (13) only when \( \bar{\pi} - c > 0 \). In the wave frame, the parcels must approach from the west, as before.

The quantitative dispersion relation follows from equivalence between the dynamical relationship between meridional parcel displacements and relative vorticity (13) and the kinematic relationship (16) between meridional streamline displacements and relative vorticity.
vorticity changes in the wave frame. A related Lagrangian derivation of the dispersion relation for the y-independent case is given in Holton and Hakim (2012) in the parcel reference frame (parcel zonal velocity is zero).

c. Topographically forced stationary waves

For stationary waves, streamlines and trajectories coincide in the ground reference frame. A brief discussion is presented here to make the connection with the above argument for westward wave propagation. Barotropic topographically forced stationary waves conserve potential vorticity \( q = \eta h \), where \( h \) is the depth of the fluid. For bottom topography \( h_b = h_k \cos(kx) \) with \( h_b \ll h_0 \), conservation of \( q \) implies approximately that

\[
\delta \zeta + \beta \delta y + \frac{f_0}{h_0} h_b = 0 \tag{17}
\]

along trajectories, where the term proportional to \( h_b \) represents vortex compression. Then, using (16) with \( c = 0 \) and \( K^2 = k^2 \), the meridional displacement of a streamline is given by

\[
\delta y = -\frac{f_0}{h_0(\beta - k^2)h_k}. \tag{18}
\]

For westerly \( \overline{u} \), the relative [(16)] and planetary [(3)] vorticity changes have opposite sign for the same displacement (Fig. 1a), and the sum can be either positive or negative. For easterly \( \overline{u} \), the relative and planetary vorticity changes have the same sign (Fig. 1b). The sign of the phase relationship between the meridional trajectory displacements and the topographic height is determined in (18) by potential vorticity conservation. The signs of the meridional parcel displacement and topography are the same (streamlines in phase with topography) when the absolute vorticity increases for positive displacement and are out of phase when the absolute vorticity decreases for positive displacement. In particular, the displacements and topography are out of phase for easterly \( \overline{u} \) (Holton and Hakim 2012, their Fig. 4.12) or for long waves and westerly \( \overline{u} \) and in phase for short waves and westerly \( \overline{u} \).

3. Baroclinic Rossby waves

It is straightforward to generalize the above analysis to vertically propagating baroclinic quasigeostrophic Rossby waves on an isothermal, constant–zonal velocity, basic state. The flow is governed by conservation of quasigeostrophic potential vorticity \( q \) following the geostrophic horizontal flow (Holton and Hakim 2012, their section 12.3.1):

\[
\frac{D_h q}{Dt} = 0. \tag{19}
\]

The horizontal velocities are represented to lowest order by the geostrophic streamfunction \( \psi \) satisfying (5), where \( \psi = \phi/f_0 \) and \( \phi \) is the geopotential. In log-pressure coordinates,

\[
q = \zeta + f + \frac{f^2_0}{\rho_0 N^2} \frac{\partial}{\partial z} \left( \frac{\partial \psi}{\partial z} \right), \tag{20}
\]

where \( z = -H \ln(p/p_s) \), \( H = RT_s/g \) is a scale height, \( \zeta \) is the geostrophic relative vorticity, \( \rho_0 = \rho_s \exp(-z/H) \) is the density for an isothermal atmosphere of temperature \( T_s \), \( R \) is the gas constant, \( \rho_s \) is the density at pressure \( p_s \), and \( N^2 \) is the buoyancy frequency squared. The third term on the rhs of (20) represents vortex compression.

Assuming a constant-amplitude wavelike disturbance on a constant zonal-mean flow, the geostrophic trajectories lie on horizontal surfaces and are steady in the wave frame. Since \( q \) is constant along trajectories and the basic-state motions make no contribution to \( q \),

\[
\delta q' = -\beta \delta y, \tag{21}
\]

where \( q' \) is the sum of the perturbation relative vorticity and compression terms.

The streamfunction for a wave mode takes the form

\[
\psi = -\overline{u} y + \psi'(x, y, z, t), \tag{22}
\]

where

\[
\psi' = \psi_0 e^{z/2H} \cos[k(x - ct) + ly + mz + \varphi] \tag{23}
\]

and \( m \) is the vertical wavenumber. Analogous to (14),

\[
\delta y = \frac{\delta \psi'(\hat{x}, y, z)}{-\overline{u} - c} \tag{24}
\]

along horizontal trajectories in the wave frame. The relative vorticity plus compression in the wave frame is proportional to the streamfunction

\[
\delta q' = -K^2 \delta \psi'(\hat{x}, y, z), \tag{25}
\]

where \( K^2 = (k^2 + l^2 + m^2 + \lambda^{-2}) \), \( \lambda^2 = (f_0^2/N^2)m^2 \), and \( \lambda = 4N^2 H^2/\overline{f_0}^2 \). Then

\[
\delta \overline{u}' = -K^2 (\overline{u} - c) \delta y. \tag{26}
\]

For consistency between (21) and (26), \((\overline{u} - c) > 0\), and parcels must approach from the west in the wave frame.

4. Conclusions

Rossby wave streamlines and trajectories coincide in a reference frame in which the wave is steady state.
Viewed from this reference frame, the mechanism for westward propagation of Rossby waves is that vorticity conservation is consistent with wave behavior only when the parcels approach from the west. If the parcels flow through the wave from east to west, the relative vorticity and planetary vorticity of a displaced parcel have the same sign. Consequently, absolute vorticity cannot be conserved for parcels approaching from the east; however, potential vorticity conservation is possible for stationary waves with easterly basic-state zonal winds, as the absolute vorticity and stretching changes can cancel, leading to anticyclonic troughs over mountain ridges for stationary topographic forcing.

**Acknowledgments.** This research was supported by National Science Foundation Grant AGS-1338427, National Oceanic and Atmospheric Administration Grant NOAA NA14OAR4310160, and National Aeronautics and Space Administration Grant NASA NNX14AM19G. The figure was prepared using GrADS.

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