Broad-Spectrum Mountain Waves

RONALD B. SMITH AND CHRISTOPHER G. KRUSE

Department of Geology and Geophysics, Yale University, New Haven, Connecticut

(Manuscript received 21 October 2016, in final form 16 January 2017)

ABSTRACT

Recent airborne mountain-wave measurements over New Zealand in the lower stratosphere during the Deep Propagating Gravity Wave Experiment (DEEPWAVE) campaign allow for improved spectral analysis of velocities $u$, $v$, and $w$, pressure $p$, and temperature $T$ fluctuations. Striking characteristics of these data are the spectral breadth and the different spectral shapes of the different physical quantities. Using idealized complex terrain as a guide, the spectra are divided into the long-wave “volume mode” arising from airflow over the whole massif and the short-wave “roughness mode” arising from flow into and out of valleys. The roughness mode is evident in the aircraft data as an intense band of $w$ power from horizontal wavelength $l = 8–40$ km. The shorter part of this band (i.e., $l = 8–15$ km) falls near the nonhydrostatic buoyancy cutoff ($l = 2\pi U/N$). It penetrates easily into the lower stratosphere but carries little $u$ power or momentum flux. The longer part of this roughness mode (i.e., $l = 15–40$ km) carries most of the wave momentum flux. The volume mode for New Zealand, in the range $l = 200–400$ km, is detected using the $u$-power, $p$-power, and $T$-power spectra. Typically, the volume mode carries a third or less of the total wave momentum flux, but it dominates the $u$ power and thus may control the wave breakdown aloft. Spectra from numerical simulations agree with theory and aircraft data. Problems with the monochromatic assumption for wave observation and momentum flux parameterization are discussed.

1. Introduction

Gravity waves transport momentum and energy vertically, causing secondary circulations and heating in the stratosphere and above (Eliassen and Palm 1960; Bretherton 1969; Lilly and Kennedy 1973; Lindzen 1981, 1988; Holton 1982; Palmer et al. 1986; McFarlane 1987; Fritts and VanZandt 1993; Lu and Fritts 1993; Fritts and Alexander 2003; Smith et al. 2008; Alexander et al. 2010; McLandress et al. 2012; Placke et al. 2013; Geller et al. 2013). While many clever methods have been devised to observe gravity waves [e.g., balloon soundings, vertically pointing lidar and frequency-modulated continuous-wave (FMCW) radar, and limb and nadir infrared detection from satellites], they usually observe only one or two physical variables. For example, recent advances in superpressure balloon technology (Vincent and Hertzog 2014) provide good horizontal structure of pressure and wind, but vertical air motion must be inferred and temperature is not available.

Compared to other methods, aircraft observations, obtained by flying horizontally though a field of waves, provide the best available multivariable observations with good horizontal scale information. Wind components $u$, $v$, and $w$, pressure $p$, and air temperature $T$ can all be determined at high resolution along a flight leg. Aircraft data analyzed by Jasperson et al. (1990) showed that the zonal wind spectrum with respect to horizontal wavelength $\lambda$ is “red” (i.e., more long-wave than short-wave power) with a log–log exponent near $-5/2$ and a notably larger variance over mountainous terrain than over the sea. Cho et al. (1999) report red wind and temperature spectra with exponents in the range from $-5/3$ to $-2$. Bacmeister et al. (1996) agree that the spectra of temperature and wind are red with exponents between $-3/2$ and $-2$, while the spectrum for vertical velocity is flatter or perhaps “white”.

Other authors have examined the spectra of Earth’s terrain. Bretherton (1969) found a power law relating terrain variance and wavenumber with a log–log slope of $-5/2$. Young and Pielke (1983) and Steyn and Ayotte (1985) found exponents between $-2$ and $-5/2$, with some dependence on region and transect orientation. These exponents roughly match the wind and temperature exponents mentioned above.

In comparison to these earlier projects, the recent Deep Propagating Gravity Wave Experiment (DEEPWAVE)
campaign over New Zealand in 2014 (Fritts et al. 2016; Smith et al. 2016, hereinafter S16) provides an improved dataset for gravity wave spectral studies over mountains. All the standard physical variables (e.g., $u$, $v$, $w$, $p$, and $T$) were measured independently and redundantly. The Southern Alps of New Zealand are surrounded by ocean and are therefore compact. Spectral and physical analyses are easier if the disturbance is compact. The Southern Alps have rapid tectonic uplift and erosion rates (Williams 1991) and one of the most rugged terrains in the world. Small-scale relief exceeds 1 km in the high mountain areas (Korup et al. 2005). This roughness broadens the terrain spectrum and the associated wave spectra found in the atmosphere. The DEEPWAVE project design included longer flight legs flown repetitively over a small set of predefined flight tracks. This repetition allows multiple spectral analyses within each wave event.

Typically, regionwide or project-total gravity wave spectra are broad and smooth, motivating a broad spectral approach to momentum flux description (Bretherton 1969; Bannon and Yuhas 1990; Fritts and VanZandt 1993; Lu and Fritts 1993; Wright and Gille 2013). Alternatively, individual remote sensing–or aircraft-derived spectra may show uneven, peaked, or even nearly monochromatic spectra. When this occurs, a common type of gravity wave analysis uses the monochromatic assumption. If one or two physical variables are measured along a horizontal leg, other variables may be inferred using the linearized monochromatic “dispersion relation,” “impedance relation,” or “polarization relation” (Gossard and Hooke 1975; Gill 1982; Nappo 2002; Ern et al. 2004). These relations are physically based formulae that assume that all the physical variables (e.g., $u$, $v$, $w$, $p$, and $T$) take the form of a single plane wave. The monochromatic assumption is also used in writing the “saturation condition” for marginally breaking waves (Lindzen 1981, 1988). With broad spectra, however, these relations may be invalid, as different physical variables may be dominated by different part of the spectrum. In this study, we examine mountain-wave events that mostly violate the monochromatic assumption. We consider whether the monochromatic approach has validity in mountain-wave analysis.

2. Broad mountain-wave spectra

An important characteristic of a broad gravity wave spectrum is the different spectral shape of the $u$, $v$, $w$, $p$, and $T$ variance and covariance power spectra. To illustrate this property, we imagine a two-dimensional steady flow of a stratified hydrostatic atmosphere with streamlines and air parcels oscillating up and down about a horizontal reference level. The vertical displacement of air parcels as a function of horizontal distance is given by $\eta(x)$. These oscillations are assumed to be part of a steady vertically propagating mountain wave. The Fourier transform of $\eta(x)$ is

$$\tilde{\eta}(k) = \int_{-\infty}^{\infty} \eta(x) \exp(-ikx) \, dx. \tag{1}$$

Using Rayleigh’s energy or Parseval’s theorem, the variance of $\eta(x)$ is

$$\text{Var}(\eta) = \int_{-\infty}^{\infty} \eta^2(x) \, dx = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} \tilde{\eta}(k) \tilde{\eta}(k^*) \, dk, \tag{2}$$

where the integrand $\tilde{\eta}(k) \tilde{\eta}(k^*)$ is the power spectrum of $\eta(x)$. The circumflex indicates the Fourier transform. This displacement variance in (2) is not normalized by the domain of integration, so the units are cubic meters.

In a similar way, we can compute variances of other physical quantities using the linear steady hydrostatic 2D nonrotating Boussinesq equations of motion in an unsheared basic state (e.g., Queney 1948; Gossard and Hooke 1975; Gill 1982; Durran 1986; Smith 1979b, 1989; Nappo 2002). Three important quantities are the variances and covariance of vertical ($w$) and horizontal ($u$) velocity:

$$\text{Var}(w) = \int_{-\infty}^{\infty} w^2(x) \, dx = \left(\frac{U^2}{2\pi}\right) \int_{-\infty}^{\infty} k^2 \tilde{\eta}(k) \tilde{\eta}(k^*) \, dk, \tag{3}$$

$$\text{Cov}(u, w) = \int_{-\infty}^{\infty} u(x)w(x) \, dx = -\left(\frac{NU}{2\pi}\right) \int_{-\infty}^{\infty} |k| \tilde{\eta}(k) \tilde{\eta}(k^*) \, dk, \quad \text{and} \tag{4}$$

$$\text{Var}(u) = \int_{-\infty}^{\infty} u^2(x) \, dx = \left(\frac{N^2}{2\pi}\right) \int_{-\infty}^{\infty} \tilde{\eta}(k) \tilde{\eta}(k^*) \, dk. \tag{5}$$

These quantities in (3), (4), and (5) will also be called $w$ power, momentum flux (MF) power, and $u$ power respectively. Expressions (3)–(5) differ primarily in the wavenumber weighting in the integrand (i.e., $k^2$, $k^1$, and $k^0$, respectively). This difference arises from the fact that the ratio of vertical to horizontal wind perturbations (i.e., $\bar{w}/\bar{u}$) in a steady mountain wave depends on the horizontal wavenumber $k$. In two dimensions, the continuity equation is $\bar{k}u + \bar{m}w = 0$, and, in the hydrostatic case, the vertical wavenumber is $m = N/U$ for $k > 0$. Thus, with the vertical wavenumber $m$ independent of $k$, the ratio $\bar{w}/\bar{u} = -kU/N$ is proportional to $k$. It follows that, for a single wavenumber, the variances satisfy
\[ \dot{w} = (-kU/N)\dot{w} \quad \text{and} \quad \dot{u} = (-N/kU)\dot{w}. \]

With wavenumber \( k \) in the numerator and denominator, respectively, these variance spectra are blue and red shifted, respectively, relative to the covariance spectrum. These shifts are described in the classical “polarization relations” for gravity waves (e.g., Gossard and Hooke 1975; Gill 1982; Nappo 2002).

According to linear theory, with constant background wind \( U \) and stability \( N \), the vertical displacement \( h(x) \) at any level \( z \) is related to the underlying terrain \( h(x) \) in Fourier space by

\[ \tilde{h}(k, z) = \hat{h}(k) \exp(imz). \]  

(6)

When \( m \) is real, the phase shift factor \( \exp(imz) \) cancels out from (2)–(5) when \( \tilde{h}(k) \) is multiplied by its complex conjugate \( \hat{h}(k)^* \). In that case, all the variances and covariances in (2)–(5) are independent of altitude. Whether or not (6) quantitatively describes wave launching from terrain, it may represent the tilted vertical structure of the vertical displacement \( h(x, z) \) aloft.

The wave drag on a mountain ridge per unit length is the cross-ridge integrated momentum flux \( MF = \bar{p}\dot{u}\dot{w}' \), which from (4) is \( \text{Drag} = \bar{p}\text{Cov}(u, w) \), with units of newtons per meter. Using (2), the variances of horizontal velocity in (5), temperature, and pressure can be written in terms of \( \text{Var}(\eta) \) as

\[ \text{Var}(\eta) = \frac{1}{2} \text{Var}(\eta), \quad \text{Var}(T) = \frac{1}{C_1} \text{Var}(\eta), \quad \text{and} \quad \text{Var}(p) = (\rho UN)^2 \text{Var}(\eta). \]  

(7)  

(8)  

(9)

The horizontal energy flux is also proportional to \( \text{Var}(\eta) \); that is,

\[ \text{EFx} = \int_{-\infty}^{\infty} p'\dot{u}' \, dx = (-\rho UN^2) \text{Var}(\eta). \]  

(10)

Another useful proportionality is between the vertical fluxes of energy (EFz) and horizontal MF (Eliassen and Palm 1960). Using (4),

\[ \text{EFz} = \int_{-\infty}^{0} p'\dot{w}' \, dx = (-U)\text{MF}. \]  

(11)

We can categorize all the above quantities into three types. Type K2 has a \( k \)-squared weighting with respect to...
the displacement field ($w$ power only). Type K1 has a single $k$ weighting (e.g., MF and EFz). Type K0 has no $k$ weighting (e.g., $u$ power, $T$ power, $p$ power, and EFx).

If the displacement power spectrum $\eta(k)$ is narrow, these weightings will not significantly alter the shape of the various power spectra. In the extreme case, imagine that $\eta(k) = A\delta(k - k_0)$, where $\delta$ is a delta function. Each spectral type will be a delta function too. Only in such a monochromatic case is there a simple relationship between quantities with different $k$ weightings. For example, using (4) and (8), drag (type K1) and temperature variance (type K0) are related by

$$\text{Drag} = \left[ \frac{\rho N U k_0}{(d\theta/dz)^4} \right] \text{Var}(T), \quad (12)$$

where the dominant wavenumber $k_0$ appears in the coefficient. Formulas of this type are frequently used to estimate momentum flux from satellite-derived temperature variance (Preusse et al. 2002; Ern et al. 2004; Alexander 2015; Wright et al. 2016), but the wavenumber $k_0$ must be known.

Examples of narrow and broad terrain spectra are given in Figs. 1–3. Each figure shows the terrain (or parcel displacement) shape and the three different $k$-weighted spectra discussed above: types K0, K1, and K2. The spectra are plotted with linear scales, so the area under each curve is the total variance. Figures 1 and 2 are the cosine–parabola terrain discussed in the next section with well-defined analytical properties. The first spectrum is narrow, and the second is broad. Figure 3 is an actual transect across the Mt. Cook section of the Southern Alps with 1.4-km sampling and 4-point smoothing. Similar to Fig. 2, it has a broad spectrum. The terrain-height spectrum (K0) is dominated by the small-wavenumber “volume mode.” As the $k$ weighting is increased (K1 and K2), the higher-wavenumber “roughness mode” becomes evident. With K2 weighting, the volume mode is barely seen. According to (2)–(11), these curves are predictions for the mountain-wave spectra of $w$ power (type K2); MF and EFz (both type K1); and displacement, $u$ power, $p$ power, $T$ power, and EFx (all type K0).

### 3. Idealized complex terrain

To aid our interpretation of complicated aircraft and model data, we study an idealized terrain family

![Fig. 2. Idealized terrain with a broad spectrum: (a) terrain shape, (b) terrain power spectrum (type K0), (c) $k$-weighted power spectrum (type K1), and (d) $k$-squared weighted power spectrum (type K2). Terrain parameters are $h_m = 1800$ m, $d = 100$ km, $\beta = 0.3$, and $n = 11$. The roughness wavelength is about $\lambda = 20$ km. The blue reference line is the buoyancy cutoff $k = N/U = 0.0005$ m$^{-1}$ for $N = 0.01$ s$^{-1}$ and $U = 20$ m s$^{-1}$. The plots in (b), (c), and (d) should approximate the $u$-power, momentum flux, and $w$-power spectra, respectively.](image-url)
with broad spectra. Most of the theoretical literature on mountain-wave generation used either a periodic sinusoidal terrain or a single isolated bump (e.g., Queney 1948; Durran 1986; Smith 1979a, 1989). An exception is Grubišić and Stiperski (2009), who studied a double bump. We consider terrain with a much broader spectrum than these. One simple broad-spectrum terrain has a large elevated volume above the reference plane (i.e., \( z = 0 \)) and an embedded roughness. A convenient choice is a parabolic hill with an embedded cosine:

\[
   h(x) = h_m \left[ 1 - \left( \frac{x}{d} \right)^2 \right] \left[ 1 - \beta + \beta \cos(k_m x) \right]
   \text{ for } -d \leq x \leq d.
\]  

(13)

The terrain height is defined as zero for \( |x| > d \). Expression (13) has four free parameters: \( h_m \) is the maximum height; \( d \) is the half-width; \( k_m \) is the wavenumber of the embedded cosine; and \( \beta \) is the relative amplitude of the cosine. This formula drops the mean level below the envelope parabola by an amount \( \beta \) and then puts an amplitude \( \beta \) oscillation on it. The peak height remains \( h(x = 0) = h_m \). The number of bumps is \( n = dk_m/\pi \). The mathematical properties of (13) are given in the appendix.

The parabola–cosine shape given by (13) allows one to vary both the width and the depth of the valley structures. It does not capture the randomness of real terrain, but it allows us to do prototype calculations with a broad-spectrum hill. The four terrain parameters could be chosen to match various New Zealand cross sections by a rough visual comparison or by matching quantitative measures, such as volume, variance, slope variance, or wave drag coefficient. A visual match for the Mt. Cook section in Fig. 3 could give \( h_m = 1800 \) m, \( d = 100 \) km, \( \beta = 0.3 \), and \( n = 11 \). Their spectra can be compared between Figs. 2 and 3.

The drag spectra in (4) for terrain given by (13) is shown in Fig. 4 for \( h_m = 1800 \) m, \( d = 100 \) km, \( \beta = 0.3 \), and \( n = 3–17 \). The area under each curve is the total wave drag. For any \( n \geq 3 \), the drag spectrum has two independent peaks. The volume peak is comprised of long waves describing the flow over the whole mountain range. It is related to the volume of the hill, which is reflected in the value of the Fourier transform at zero wavenumber (see the appendix). Its drag peak is located approximately at \( k = 1/d \) and is nearly unchanged as \( n \) increases.

The roughness peak is related to airflow into and out of interior valleys of the terrain. The roughness-drag peak shifts to higher wavenumber and increases in magnitude linearly with \( n \) because shorter waves carry more flux per unit distance. With the modes well separated, the total drag in (4) can be accurately approximated as the sum of the two modes:
and 17. Airflow parameters are $N = 0.01 \text{ s}^{-1}$ and $U = 20 \text{ m s}^{-1}$.

Hydrostatic results are blue, and nonhydrostatic results are black. The volume mode at $k = 0.02 \text{ km}^{-1}$ (marked “all”) is nearly unchanged with $n$. The hydrostatic roughness-mode peak shifts right and increases its amplitude with $n$. The non-hydrostatic drag decreases as the dominant roughness wave-number $k = \pi n/d$ approaches $k_N = N/U$. For $n > 15$, the roughness drag is zero.

$\text{Drag} = \text{Volume Drag} + \text{Roughness Drag}$ and

$$C_D \approx \frac{4}{\pi} (1 - \beta)^2 + \frac{8\pi}{15} n \beta^2,$$

where $C_D = \text{Drag}/(\rho N U h_m^2)$. See the appendix for details. For $\beta = 0.3$ and $n = 11$, the ratio $R_{MF}$ of the second to the first term in (14b) is $R_{MF} \approx 3.6/0.62 \approx 5.8$ (see Fig. 2c). The volume drag has strong $u'$ and weak $w'$, while the roughness drag has weak $u'$ and strong $w'$, as shown in Figs. 2b and 2d.

The variance of horizontal velocity (i.e., $u$ power) is also the sum of the two modes if $n \geq 3$. Using (7) and (A3), we obtain

$$\text{Var}(u) \approx h_m^2 N^2 \left(\frac{16}{15}\right) d \left[ (1 - \beta)^2 + \frac{\beta^2}{2} \right].$$

(15)

independent of $n$. For the value of $\beta = 0.3$, the ratio $R_u$ of the second to the first term in the square brackets of (15) is $R_u = 0.045/0.49 \approx 0.1$. Thus, the volume mode dominates the $u$ power (Fig. 2b). This result is important as the perturbation $u$ field usually dominates the breakdown of gravity waves by stagnating the flow, by overturning the flow, or by shear instability. The volume mode will increase its $u$-power dominance even further when we consider nonhydrostatic effects in the next section.

Similar to (15), the variance of vertical velocity from (3) and (A4) has volume and roughness components:

$$\text{Var}(w) \approx \frac{8U^2 h_m^2}{3d} \left\{ (1 - \beta)^2 + \frac{\beta^2}{2} + \frac{1}{5} n^2 \beta^2 \right\}.$$

(16)

With $\beta = 0.3$ and $n = 11$, the ratio $R_w$ of the second to the first term in the curly braces in (16) is $R_w = 7.2/0.18 \approx 40$, so the roughness mode dominates the $w$ power (Fig. 2d).

To illustrate the relationship between different variances, we define the ratio $R_{wu}$ of $w$ power to $u$ power. For a monochromatic hydrostatic wave, this ratio,

$$R_{wu} = \frac{\text{Var}(w)}{\text{Var}(u)} \approx \left(\frac{k_0 U}{N}\right)^2,$$

(17a)

depends on the wavenumber $k_0$. For terrain (13), using the dominant volume mode from (15) and roughness mode from (16), the ratio is

$$R_{wu} = \frac{\text{Var}(w)}{\text{Var}(u)} \approx \left(\frac{\pi^2 U^2}{2N^2 d^2}\right) \left[ \frac{n^2 \beta^2}{1 - \beta^2} \right].$$

(17b)
If desired, (17a) and (17b) can be equated to derive an effective wavenumber for $R_{wu}$:

$$k_0 \approx \left(\frac{\pi}{\sqrt{2d}}\right) \left[\frac{\eta \beta}{(1 - \beta)}\right],$$

but this value applies to this ratio in (17) only. There is no effective wavenumber that completely characterizes a broadband spectrum.

4. Nonhydrostatic effects

It is well known that the intrinsic frequency ($\sigma_I = Uk$) of steady mountain waves must lie between the Coriolis parameter $f = 2\Omega \sin(\phi)$ and the Brunt–Väisälä frequency $N = [(g/\theta)(d\theta/dz)]^{1/2}$ (e.g., Blumen 1965). Typical values are $f = 0.0001 \text{s}^{-1}$ and $N = 0.01 \text{s}^{-1}$. Writing these frequency limits in terms of wavenumbers, we obtain the following, with a sample wind speed of $U = 40 \text{m s}^{-1}$:

$$k_C = \frac{f}{U} = 2.5 \times 10^{-4} \text{m}^{-1} \quad \text{and} \quad k_N = \frac{N}{U} = 2.5 \times 10^{-4} \text{m}^{-1},$$

The corresponding wavelengths are $\lambda_C = 2\pi/k_C = 2512 \text{km}$ and $\lambda_N = 2\pi/k_N = 25.1 \text{km}$. Clearly, for New Zealand, the nonhydrostatic effects associated with $k_N$ will be the more important constraint (Fig. 3). According to classical linear mountain-wave theory (e.g., Scorer 1949; Smith 1979a,b; Durran 1986), nonhydrostatic effects are captured in the generic formula for the vertical wavenumber for steady flow:

$$m = \left(\frac{N}{U}\right) \left[1 - \left(\frac{kU}{N}\right)^2\right]^{1/2}.$$  \hspace{1cm} (20)

For small $k$, (20) reduces to its hydrostatic form [$m = (N/U) \text{sign}(k)$], satisfying the radiation condition. As $k$ approaches $k_N = N/U$, $m$ vanishes. When $k > k_N$, vertical wavenumber becomes imaginary. The vertical wavenumber determines the structure of the wave. If it is imaginary, the wave pattern decays with height, the phases of $u'$ and $w'$ are in quadrature, and the momentum flux is zero. For the present purposes, however, we are interested in the cases when $k = k_N$ and $m$ is small. In this case, the disturbance penetrates vertically very well, but the amplitude of the horizontal velocity perturbations decreases with $m$ according to $u = (m/k)\hat{w}$. Near
the cutoff wavenumber \( k = k_N \), MF vanishes because there are no \( u' \) perturbations.

We illustrate this important nonhydrostatic effect in Figs. 4 and 5. In Fig. 4, we hold \( N, U, \) and width \( d \) constant and increase the number of bumps \( n \) in the terrain. According to hydrostatic theory, the roughness drag increases linearly with \( n \) [(14b)]. Using (20), however, as the roughness wavenumber increases and approaches the buoyancy cutoff wavenumber \( k_N = N/U \), nonhydrostatic effects reduce the drag. In Fig. 4, the nonhydrostatic roughness drag decreases with \( n \) and vanishes for \( n > 15 \). The nonhydrostatic effect can be easily incorporated into the drag equation [see (14b)]. It becomes

\[
C_D = \left( 4 \pi \right) \left( 1 - \beta \right)^2 + \left( \frac{8 \pi}{15} \right) n \beta^2 \left[ 1 - \left( \frac{\pi n U}{d N} \right) \right]^{1/2}
\]

for \( d N > \pi n U \).

Note that the nonhydrostatic correction is applied only to the roughness term, as only these shorter waves are affected, and only this peak is narrow enough to be characterized by a single wavenumber \( k_m = \pi n / d \). For a given overall mountain width \( d \), as \( n \) increases, the \( C_D \) eventually falls as the roughness wavenumber \( k_m \) approaches \( k_N \). For example, with \( U = 20 \text{ m s}^{-1}, N = 0.01 \text{ s}^{-1}, \) and \( d = 100 \text{ km}, \) the cutoff value of \( n = d N / \pi U \) lies between \( n = 15 \) and 16. Implicit in (21) is that, when the roughness wavenumber exceeds \( k_N \), it makes no drag contribution.

The influence of wind speed on drag is shown in Fig. 5 for \( n = 11 \). Two reference curves are given: the full hydrostatic drag in (14b) and the volume drag alone [first term in (14b) or (21)]. The full nonhydrostatic drag, computed from the analytical spectrum in (A5), is compared with the modal equation [see (21)]. The agreement is excellent. The total drag rises rapidly with \( U \) and is dominated by the roughness drag. Eventually, however, the high wind speed makes the short waves hit the cutoff. The total drag plummets, and only the smaller volume drag remains.
This drop in the roughness drag due to nonhydrostatic effects is further illustrated in Fig. 6 for a case with strong winds. These curves are computed using a fast Fourier transform incorporating (1), (4), (6), and (20). Shown are $w'(x)$, $u'(x)$, and accumulated MF$(x)$ at 12 km, the altitude of the aircraft data in section 5. When the nonhydrostatic effect in (20) is included, the $w$ field is shifted slightly downwind as a result of dispersion, but its amplitude remains large. The disturbance easily reaches the 12-km level in both cases. The roughness part of the $u$ field is nearly zero as $m \approx 0$. The flow streamlines move up and down together with constant vertical spacing. With only weak $u'$ oscillations, the drag is much reduced. The roughness contribution to $u$ power vanishes when $k_m = k_N$. Note also that the $u$ field has long-wave power while the $w$ field does not.

Sections 2–4 contain many predictions for wave characteristics over mountains, like those in New Zealand, that have both volume and roughness. First, the wave spectrum is expected to be broad, perhaps spanning a factor of 10 or more in scale, as it comprises both the volume and roughness modes. Second, the power spectra for $K_0$, $K_1$, and $K_2$ quantities will be very different from each other. Third, different power spectra will be dominated by different parts of the wavelength spectrum. The volume mode will dominate the $u$-power, $p$-power, and $T$-power spectra (i.e., type $K_0$), while these three spectra may not show the roughness mode. Conversely, the roughness mode will dominate the $w$-power spectra (i.e., type $K_2$). The MF and EF$z$ spectra may have contributions from both modes. Fourth, near the buoyancy cutoff wavelength, we expect to find waves with strong penetrating $w$ power but little $u$ power or MF because of nonhydrostatic effects. For still shorter waves, evanescence (i.e., exponential decay aloft) may prevent detection at flight level.

5. Power spectra from aircraft transects over New Zealand

During the DEEPWAVE project in 2014, the NSF/NCAR Gulfstream V research aircraft flew 14 missions over New Zealand comprising 97 cross-terrain legs (Fig. 7). The legs were mostly flown at an altitude of $z = 12.1$ km in the core of the subtropical jet. These flight-level data included winds $u$, $v$, and $w$, pressure $p$, and temperature $T$, so several different variance and covariance spectra can be computed. To illustrate this type of data, we show nine legs from research flight 05 (RF05) in Fig. 8. Each trace shows the vertical parcel displacement computed from $\eta(x) = \int_0^x w'(x)u(x) \, dx$. The vertical energy fluxes (EF$z$ = $(p'w')/u'$) were all positive for these legs, indicating upward wave propagation. These EF$z$ values were 5.0, 3.5, 3.6, 5.4, 5.3, 5.1, and 6.7 W m$^{-2}$ for the seven 12.1-km legs and 3.1 and 1.7 W m$^{-2}$ for the two 13.8-km legs. The reduction of EF$z$ aloft is probably caused by nondissipative wave energy absorption in the negative shear above the jet stream core.

As shown by S16, these observed waves over New Zealand have many characteristics of steady linear mountain waves, such as positive EF$z$, negative MFx, and small nonlinearity ratio (i.e., $|u'|/U < 1$). They showed that the horizontal energy flux vector (i.e., EF$x$ = $\langle p'u' \rangle$ and EF$x$ = $\langle p'v' \rangle$) was oriented into the mean wind. They also verified that the relationship EF$z$ = $-U \cdot MF$ between energy and momentum flux for steady mountain waves was well satisfied. Several types of scale analysis were also attempted by S16, including wavelet and bandpass filtering. They noted that the $w$ power was dominated by the shorter waves, while the $u$ power was dominated by the longer waves. The momentum and vertical energy fluxes had contributions from both wave scales, with the short waves dominating in the stronger events.

For a more detailed discussion, we select four representative legs (Table 1) from four different research flights (i.e., RF05, RF08, RF09, and RF16). These examples are chosen to be representative of characteristics seen in the full dataset. For each chosen leg, several other similar legs can be found in the full dataset. A cutoff wavelength is computed for each leg from $\lambda_N = 2\pi U/N$, where the leg-average wind speed and a standard stratospheric value of $N = 0.02$ s$^{-1}$ are used.

In Figs. 9–12, we compare the power spectra from these flight legs over New Zealand. The abscissa has a log scale covering almost three orders of magnitude in wavelength. The total variance is the area under the curves. All signals have had their means and trends

<table>
<thead>
<tr>
<th>RF</th>
<th>Leg</th>
<th>Date</th>
<th>Mountain</th>
<th>Spectrum</th>
<th>$U$ (m s$^{-1}$)</th>
<th>Std dev (u': m s$^{-1}$)</th>
<th>$\lambda_c$ (km)</th>
<th>Std dev (w': m s$^{-1}$)</th>
<th>EF$z$ (W m$^{-2}$)</th>
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TABLE 1. Four analyzed DEEPWAVE aircraft legs, all during 2014.
removed prior to spectral analysis. Of the five spectral quantities plotted in Figs. 9–12, the two most similar are $u$ power and $p$ power. These two fields approximately satisfy the steady unsheared linearized integrated horizontal momentum equation $p'(x) = -pUu'(x)$ so that their power spectra are proportional with a factor $(pU)^2 \approx 100 \text{ kg}^2 \text{ m}^{-4} \text{ s}^{-2}$ [see (7) and (9) and S16].

One common characteristic is seen in all four examples: a strong $w$-power peak in the range $\lambda = 8$–40 km. In each case, the shorter part of this range, $\lambda = 8$–15 km, nearly coincides with the buoyancy cutoff and has little MFx, $u$ power, or $p$ power. Following sections 3 and 4, we interpret this feature as a highly nonhydrostatic part of the roughness peak. The longer part of the $w$-power roughness peak, $\lambda = 15$–40 km, has significant MFx, $u$-power, and $p$-power spectra. All four legs (Figs. 9–12) have significant $u$ power, $p$ power, and $T$ power from 40 to 300 km.

In Fig. 9, the MFx, $u$ power, and $p$ power are distributed widely from $\lambda = 15$–200 km, thus including both volume and roughness modes. In Fig. 10, the MFx, $u$-power, and $p$-power peaks at $\lambda = 26$ km give a more monochromatic appearance. In Fig. 11, a strong MFx peak at $\lambda = 23$ km is supplemented by significant peaks at $\lambda = 50$–200 km. In Fig. 12, all the variances have strong nearly monochromatic roughness peaks at $\lambda = 26$ km. This last case was the largest MFx leg in the entire DEEPWAVE project (S16).

There is certainly no single “dominant wavelength” for these fields of waves. It depends on which diagnostic quantity one uses. This property verifies the predictions from sections 2 and 3 concerning $K_0$, $K_1$, and $K_2$ spectral types. The lack of $u$ power and MFx power near the buoyancy cutoff, in spite of the $w$-power peak, verifies the nonhydrostatic dynamics there. The lack of any signal with wavelength shorter than the cutoff verifies the idea of evanescence.

A key question concerns the existence of a volume mode indicating airflow lifting over the whole New Zealand massif. We should look for this signature in the $K_0$ spectral types, such as $u$ power, $p$ power, and $T$ power. This feature is clearly seen in Figs. 9 and 11.
with wavelengths from 200 to 400 km but is mostly absent in Figs. 10 and 12. Its identification is compromised slightly by the limited length of the aircraft transects (e.g., 350 km) and the presence of large-scale nonorographic disturbances. Note that these long waves contribute to the momentum flux but have small $w$ power.

These data allow us to speculate about the relationship between wave breaking, $u$ power, and momentum flux. In RF09, (Fig. 11 and Table 1) there is considerable $u$ power, suggesting that wave breaking aloft may be possible. Indeed, as shown in S16, this case had stagnation-induced wave breaking at $z = 13$ km. In contrast, RF16 has a much larger MFx but smaller $u$ power. The aircraft found no evidence of wave breaking in this case. Thus, the validity of the saturation momentum flux concept is called into question (see section 8).

We end this section by summarizing the 92 best aircraft transects across New Zealand in Table 2. We use a double boxcar filter as described by S16 to divide the flight-level data into the volume and roughness mode with a $\lambda = 60$ km dividing wavelength. The filtered records are used to compute normalized $w$ power, MFx, and $u$ power. All units are square meters per square seconds. The long-wave volume mode dominates the $u$ power. The two modes contribute equally to MFx. The short-wave roughness mode dominates the $w$ power. These results qualitatively agree with the linear theory of broad-spectrum mountain waves in section 3 (e.g., Figs. 2, 3) and the spectral analysis of aircraft data (Figs. 9–12). Note that the modal variances in Table 2 do not exactly sum to the total variance because of filter damping in the region of spectral overlap.

6. Flow fields and spectra from numerical simulation

For another perspective on broad-spectrum mountain waves, we use numerical simulation. Numerically simulated waves have imperfect accuracy (Reinecke and Durran 2009) but allow full 3D wave analysis and area averaging superior to the narrow leg averages from...
Numerical simulation allows us to include several fluid dynamical aspects that are missing from the theory in section 3, especially unsteadiness, nonlinearity, variable wind and stability with height, boundary layer dynamics, and terrain three-dimensionality.

Five 2-km-resolution full-physics Weather Research and Forecasting (WRF) Model simulations of 3D time-dependent airflow over New Zealand (Table 3) were recently completed by Kruse et al. (2016) to understand wave generation and attenuation in the lower stratosphere. The WRF domain was nested into the global ECMWF analysis. Model terrain is shown in Fig. 7. The 2-km grid will capture most of the roughness mode and the entire volume mode. The model predictions were compared with balloon and aircraft data with remarkable success. For the present purpose, we focus on two events: 24 June (Figs. 13, 14) and 4 July 2014 (Figs. 15, 16).

In Fig. 13, we show the WRF wind and temperature fields on 24 June 2014 along an east–west cross section through Mt. Cook (43.6°S) and the spectra at \( z = 12 \) km. In Fig. 13a, we see that the low-tropospheric wind speed slows upstream and then increases over the mountain. There is a hint of plunging flow over the massif, with the axis of fast winds descending. Aloft at \( z = 13 \) km, there is a large-scale stagnation and streamline steepening that will cause wave dissipation (Kruse et al. 2016). The vertical velocity in Fig. 13b shows a few convective updrafts upwind and weak roughness oscillations and general descent over the mountains.

A spectral analysis of the flow in Fig. 13 is shown in Fig. 14. Here we extract data from the WRF run along a 400-km cross-mountain transect at \( z = 12 \) km to simulate an east–west aircraft leg. In Fig. 14, two \( w \)-power peaks are seen at \( \lambda = 12 \) and 20 km. Because of the lack of \( u \) power there, neither contributes to MFx. Smaller but significant \( w \)-power is seen at long wavelengths \( \lambda = 50, 90, \) and 300 km. All the other spectra (i.e., \( u \) power, MFx, \( p \) power, and \( T \) power) are dominated by these long waves. Their spectra are red, and the volume mode dominates, associated with the plunging flow in Fig. 13.
The WRF run for 4 July is shown in Figs. 15 and 16. In contrast to the 24 June case, Fig. 15a shows weaker speed variation on any scale but strong roughness oscillations in the theta lines. In Fig. 15b, we see dramatic columns of positive and negative vertical velocity over the terrain. Approximately 10 updrafts and 10 downdrafts lie within a 150-km distance, giving a \( \lambda = 10-30 \text{ km} \) wavelength. Without strong \( u \) power aloft, wave breaking is delayed but finally occurs at \( z = 15 \text{ km} \) in small regions.

The spectra in Fig. 16 for 4 July has strong \( w \)-power peaks at \( \lambda = 10 \) and 30 km. A small peak is seen at 80 km. The \( \lambda = 10 \text{ km} \) peak does not appear in MFx, \( u \) power, or \( p \) power, probably because it coincides with the buoyancy cutoff. The \( \lambda = 30 \text{ km} \) peak dominates all of the other spectra and gives all except \( T \) power a monochromatic appearance with roughness mode dominance. This case was unique as the strongest MF case in the project. The model spectra in Fig. 16 agree well with the aircraft spectra (Fig. 12) for this event.

These spectra suggest that 2-km WRF simulations capture the broad spectrum of New Zealand disturbance, including the volume and roughness modes. The shorter part of the roughness mode makes no contribution to the drag because of the effect of the nonhydrostatic cutoff. The longer part of the roughness mode and the volume mode contribute to the drag. Another view of this pattern is seen from model statistics in the next section.

7. Bandpass analysis of waves from numerical simulation

The previous section indicated that the numerical simulation of airflow over New Zealand captures the
breadth of the mountain-wave spectra. Therefore, we can use it to paint a fuller picture of the role of the volume and roughness modes in mountain-wave dynamics. We separate the two modes by applying a spectral bandpass filter described by Kruse and Smith (2015). Our volume-mode filter retains wavelengths from 60 to 600 km. Our roughness-mode filter retains wavelengths shorter than 60 km. These spectral cutoffs are not sharp, but, with broad spectra, they should give meaningful estimates of the two modes. We average over the large box shown in Fig. 7 with an area of about 711 000 km².

The WRF runs for five events are summarized in Table 3, and two of these events are shown in Figs. 17 and 18. In Table 3, we give the maximum momentum flux at \( z = 12 \) km and corresponding 4-km winds for each event. Also given is the dominant mode for each variance type. The two figures show the time development of low-tropospheric wind speed and direction at \( z = 4 \) km, as well as \( w \) power, MF, and \( u \) power in the lower stratosphere at flight level \( z = 12 \) km. The component of MF in the direction of the regionally averaged 4-km wind is shown. The two variances are normalized to units of square meters per square seconds, while MF is multiplied by air density to obtain units of millipascals. Generally speaking, the variances and the wind speed reach their peaks at about the same time, but there are exceptions.

In all five events, the volume mode dominates the \( u \) power, and the roughness mode dominates the

<table>
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<td>Vol</td>
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**Fig. 13.** WRF 2-km simulation of flow over Mt. Cook at 1200 UTC 24 Jun 2014 (RF09 case). (a) The wind and theta fields; (b) the vertical velocity and theta fields. Note convective updrafts upwind of New Zealand and dominant descent over the terrain. Spectral analysis is in Fig. 14.
In four of the five events, the roughness mode dominates the MF. For 24–25 June (Fig. 17), however, the volumemode drag slightly exceeds the roughness drag. The 3–5 July simulation (Fig. 18) is more typical, with the roughness mode dominating MF. This event has the largest winds and fluxes of the DEEEPWAVE project, as seen on both the numerical simulation and the aircraft observations (RF16).

We can understand the variation in partitioning between volume and roughness modes by examining the triads of figures: Figs. 13, 14, and 17 for the 24 June event and Figs. 15, 16, and 18 for the 4 July event. In Fig. 13, the dominance of the volume mode can be seen by the plunging flow and the weakness of the small-scale up- and downdrafts over terrain. In contrast, Fig. 15 has strong roughness-driven vertical motion. This contrast is also seen in the spectral diagrams (Figs. 14, 16) and in the bandpass diagrams (Figs. 17, 18).

All five weather events (Table 3) are summarized in Fig. 19, where all area-averaged, hourly output, 4-km wind, and $z = 12$-km bandpassed wave drag values are combined into scatterplots. Overall, the roughness-mode drag exceeds the volume-mode drag by roughly a factor of 2. In general, wave drag is small until the wind speed at 4 km exceeds $12 \text{ m s}^{-1}$. Beyond this speed, however, the drag magnitude is chaotic with little correlation with wind speed. A similar chaotic variation was reported by S16 on the basis of the aircraft data alone.

One can convert these area-average drag values (Figs. 17–19) to drag values per ridge length (Fig. 5) by multiplying by the averaging area ($\sim 7 \times 10^{11} \text{ m}^2$) and dividing by the Southern Alps ridge length ($\sim 7 \times 10^5 \text{ m}$). Thus, an area-average WRF-simulated drag of 100 mPa becomes $D = 1 \times 10^5 \text{ N m}^{-1}$. For comparison, somewhat larger values of $D = 4 \times 10^5 \text{ N m}^{-1}$ are seen in Fig. 5 at a comparable wind speed. The discrepancy arises in part because the typical New Zealand terrain is not all as high or as rough as supposed in Fig. 5 and because the wave generation process is not linear.

![Spectral analysis of 2-km WRF-simulated airflow at $z = 12$ km over Mt. Cook on 24 Jun 2014 (RF09 case): (a) $w$ power, (b) MFx, (c) $u$ power, (d) $p$ power, and (e) $T$ power. The vertical reference line is the buoyancy cutoff. A 1200-km transect is used to capture long waves. See flow field in Fig. 13.](image)
8. Saturation momentum flux with broad spectra

The concept of saturation momentum flux (Lindzen 1981, 1988, Palmer et al. 1986; McFarlane 1987) is widely used in wave drag parameterization. It is based on the hypothesis that, for a given environment, there is a maximum wave amplitude and corresponding MF value just before wave breaking begins. Once breaking begins, MF is reduced and then maintained at the minimum saturation value previously encountered by the wave. The saturation condition can be derived for monochromatic mountain waves by combining the 2D continuity equation \( \frac{\partial u}{\partial t} + \frac{\partial}{\partial z}(u^2) = 0 \) and a saturation condition on \( u \) power \( \langle u^2 \rangle = C^2 U^2 \). The tunable parameter \( C \) prescribes the degree of flow stagnation required for wave breaking. Combining these expressions gives

\[
\text{MF}_{\text{SAT}} = (\rho \langle u'w' \rangle)_{\text{SAT}} = -\rho C^2 k_0 U^3 / N. \tag{22}
\]

We repeat the derivation for a broad wave spectrum arising from terrain \([13]\). Using the roughness mode drag in \((14b)\) and the volume mode \( u \) power in \((15)\) and eliminating \( h_m \), we obtain

\[
\text{MF}_{\text{SAT}} \approx -\left( \frac{\rho C^2 U^3}{dN} \right) \frac{\pi n b^2}{2(1 - \beta)^2} \tag{23}
\]

where \( d \) is the half-width of the parabola and \( n \) and \( \beta \) describe the roughness. These two MF formulas \((22) \text{ and } (23)\) can be compared if the dominant wavenumber is chosen to represent the volume mode \( (k_0 = 1/d) \). Then the square brackets in \((23)\) display the effect of the terrain roughness. With \( n = 11 \) and \( \beta = 0.3 \) or \( \beta = 0.4 \), the value in the square brackets ranges from 3 to 7. This increase in saturated MF arises because the roughness adds to the MF without increasing \( u \) power.

The WRF event runs \((\text{Table 3} \text{ and Fig. 19)}\) also support the hypothesis that wave fields dominated by the roughness mode can transport more momentum than wave fields dominated by the volume mode. We use parameters \( \rho = 1.0 \text{ kg m}^{-3} \), \( k_0 = 1 \times 10^{-5} \text{ m}^{-1} \), and \( N = 0.01 \text{ s}^{-1} \) and the fitting parameter \( C = 0.07 \) so that \((22)\) matches the maximum volume mode drag \( (i.e., \text{ controlling the } u \text{ power}) \) in Fig. 19a. When the roughness mode is added \((\text{Figs. 19b,c})\), the maximum total drag is increased by a factor of 2–3.
in rough agreement with the enhancement estimate in the square brackets above. Unfortunately, this test has several uncertainties, and much more work would be required to prove the hypothesis. For instance, we have not demonstrated that the wave drag in Fig. 19 is limited by a wave saturation process. Still, the comparison illustrates that the volume mode that controls the $u$ power carries only a fraction of the momentum flux.

9. Conclusions

The goal of this paper is to combine theory, aircraft data, and numerical simulation to explain the striking difference between the spectra of different physical variables in New Zealand mountain waves. We used three wavenumber weightings ($k_0$, $k_1$, and $k_2$) to identify important short- and long-wave components. According to linear hydrostatic theory, these particular weightings (i.e., types K0, K1, and K2) correspond to physically important spectra, such as $u$ power, $p$ power, $T$ power, and horizontal energy flux (type K0); zonal momentum flux and vertical energy flux (type K1); and $w$ power (type K2). These spectra have different shapes, with K0 quantities often being red, while K1 is white and K2 is blue.

We showed that an idealized complex terrain constructed from a parabola with $n$ embedded cosine waves generates a broad wave spectrum composed of two modes: the volume mode and roughness mode. The volume mode represents smooth flow rising and descending over the main mountain range. The roughness mode represents airflow into and out of interior valleys. With $n \geq 3$, the spectrum satisfies our criterion for a broad spectrum. In this case, as their spectra do not overlap, the contributions of the two modes to variances and covariances become additive. In this ideal terrain, the K0 quantities are dominated by the volume mode, while the K2 quantity (i.e., $w$ power) is dominated by the roughness mode. The momentum flux and vertical

![Fig. 16. Spectral analysis of 2-km WRF-simulated airflow at $z = 12$ km over Mt. Cook on 4 Jul 2014 (RF16 case): (a) $w$ power, (b) MFx, (c) $u$ power, (d) $p$ power, and (e) $T$ power. The vertical reference line is the buoyancy cutoff. A 1200-km transect is used to capture long waves. See flow field in Fig. 15.](image-url)
energy fluxes (type K1) typically have contributions from both modes: the volume mode with strong $u'$ and weak $w'$ and the roughness mode with weak $u'$ and strong $w'$. Nonhydrostatic waves near the buoyancy cutoff wavelength ($\lambda_C = 2\pi N/U$) have a peculiar structure. With the small vertical wavenumber (i.e., $m \approx 0$), the streamlines move up and down together, maintaining constant vertical spacing so there is no horizontal wind speed variation. This motion penetrates vertically and may dominate the $w$ power in the lower stratosphere. There is little $u$ power, however, and thus it shows little tendency to transport momentum, stagnate the flow, trigger instability, or cause wave breaking. This mode is generated by the approximate $\lambda \approx 10$-km scale structure of the complex terrain. Waves shorter than the cutoff are not seen at flight level as a result of evanescence.

The analysis of selected DEEPWAVE aircraft data verified most of the above predictions. Notable is the strong roughness mode of $w$ power from $\lambda = 8$ to $40$ km. The shorter part of this range is nearly centered on the buoyancy cutoff wavelength (i.e., $\lambda = 8$–$15$ km). This part has little $u$ power or momentum flux. The longer part of this range (i.e., $\lambda = 15$–$40$ km) carries most of the momentum flux. Using $u$-power, $p$-power, and $T$-power spectra, we identified the volume mode with $\lambda = 200$–$400$ km, caused by airflow rising over the whole Southern Alps massif. While not dominant, it does contribute to the total momentum flux. Statistics from 92 DEEPWAVE legs confirmed the relative contributions of the volume and roughness modes.

Evidence of broad wave spectra was seen in five 2-km-resolution WRF event simulations. Similar to the aircraft data, they show a strong band of $w$ power that we interpret as the roughness mode. The shorter portion of this band had no $u$ power and no momentum flux. The longer portion of the roughness mode carried most of the momentum flux. In one event only (case RF09), the volume mode dominated the flux. In the other four events, including the strongest flux event (case RF16),

![Figure 17](image1.png)

![Figure 18](image2.png)
the roughness mode dominated the MF. This variation in mode dominance from event to event probably arises from changes in wind direction, moist stability, and valley-air heat budgets controlling the upstream blocking, moist convection, large-scale plunging, and valley flushing.

In one respect, our numerical results are consistent with a case study of mountain waves over New Zealand by Lane et al. (2000). With a nested single-sounding model run, they found an area-averaged $MF_x = -20\text{ mPa}$ in the lower stratosphere carried mostly by vertically propagating roughness waves with $\lambda \approx 25\text{ km}$. They did not attempt to identify the volume mode.

The short scale of the dominant roughness mode puts a constraint on numerical simulation of waves and wave drag. While it may be permissible to truncate the nonhydrostatic $w$-power peak at $\lambda = 8-15\text{ km}$, the MF peak at $\lambda = 15-40\text{ km}$ should be accurately represented in numerical models. We found a grid spacing of 2 km was satisfactory, but spacing of 5 km or larger may not be. The model should also be skillful at estimating airflow blocking and valley penetration in order to accurately predict the volume and roughness modes.

The current results concerning broad wave spectra pose a challenge for observing mountain waves. Measurements of $T$ power, such as those from satellite passive IR or Rayleigh lidar, are seeing the K0 spectra dominated by the volume mode. The use of (12) to estimate momentum flux will give poor results. The same is true for measurements of $u$ power, such as radiosondes, constant-level balloons, and conical scanning Doppler lidar. In a broad-spectrum wave field, these instruments too will mostly see the volume mode, while the roughness mode may carry the majority of the momentum flux. These problems are aggravated for satellite remote sensing systems that may not fully resolve the wave structures.

The prediction and parameterization of wave breakdown and momentum deposition are profoundly influenced by the breadth of the mountain-wave spectrum. With a broad spectrum, the usual monochromatic gravity wave relationships between different variables (e.g., $u$, $v$, $w$, $p$, and $T$) are not even approximately satisfied because these different variables are controlled by different parts of the spectrum. In general, it is the $u$ power that controls wave breakdown by flow stagnation, shear instability, or other mechanisms. It is dominated by the volume mode. The momentum flux typically gets contributions from a broad spectrum, including roughness and volume modes. We see a slight dominance of the roughness-mode contribution to MF in the aircraft data and the WRF Model output.

Here is the dilemma then: the long waves control wave breaking, while the shorter waves carry the momentum and vertical energy fluxes. The common monochromatic assumption used to derive saturation conditions for wave drag parameterization does not capture this important complexity. In the future, if satellite data interpretation and wave drag parameterization are to be quantitatively useful, we may have to drop the popular “monochromatic” simplification in favor of a binned or continuous wave spectrum. For New Zealand, we have introduced a “binned” approach, where we divide the spectrum into two categories: a volume mode and a roughness mode. Interior continental terrains may require a spectral continuum.

Acknowledgments. The grant to Yale University was NSF-AGS-1338655. The NSF/NCAR Gulfstream V aircraft was operated by the National Center for Atmospheric Research. The high quality of the airborne data...
was essential for this project. Additional collaboration came from the Naval Research Laboratory, German DLR, the New Zealand MetService, and NIWA. The WRF computing was performed on the Yellowstone supercomputer provided by NCAR’s Computational and Information Systems Laboratory sponsored by the NSF. Special thanks to the DEEPWAVE project PIs, Dave Fritts, James Doyle, Steve Eckerman, Mike Taylor, Andreas Dörnbrack, Michael Udstrom, and the Yale team: Alison D. Nugent, Campbell Watson, Azusa Takeishi, Christine Tsai, Larry Bonneau, and Sigrid R.-P. Smith. Conversations with DLR students Martina Bramberger and Tanja Portele are appreciated. Discussions with Fuying Zhang and Ulrich Schumann regarding short waves in the atmosphere were valuable. Three anonymous reviewers made helpful comments.

APPENDIX

Mathematical Properties of the Parabola–Cosine Hill

We define a parabolic hill with height $h_m$, half-width $d$, and embedded waves of relative amplitude $\beta$ and wavenumber $k_m$:

$$h(x) = h_m \left[ 1 - \left( \frac{x}{d} \right)^2 \right] [1 - \beta + \beta \cos(k_m x)]$$

for $-d < x < d$. (A1)

To simplify (A1) [(13) in the text], we use the quantization condition $k_m = \pi n/d$ with $n = 1, 3, 5, 7, \ldots$ so that an odd integer number of peaks fits into the terrain. This choice reduces the slope discontinuity at $|x| = d$. The example terrains in Figs. 1 and 2 have $h_m = 1800 \text{ m}$, $d = 100 \text{ km}$, $n = 11$, and $\beta = 1$ and $\beta = 0.3$. The parabola envelope can be recovered by setting $\beta = 0$ or $n = 0$. A parabola–cosine with valley bottoms at $z = 0$ can be generated by setting $\beta = 1/2$. When $\beta > 1/2$, negative terrain appears. When $\beta = 1$, there is no mean terrain elevation. The cosine oscillates around the reference level, and the volume is small.

The volume per unit length of ridge for (A1) is

$$\text{Volume} = \int_{-\infty}^{\infty} h(x) \, dx = \left( \frac{4h_m d}{3} \right) \left[ 1 - \beta + \frac{3\beta}{(k_m d)^2} \right]$$

$$\approx \left( \frac{4h_m d}{3} \right) (1 - \beta)$$

(A2)

for $n \geq 3$. The variances of $h(x)$ and $dh/dx$ are, for $n \geq 3$,

$$\text{Var}(h) = \int_{-\infty}^{\infty} h^2 \, dx \approx h_m^2 \left( \frac{16}{15} \right) d \left[ 1 - \beta^2 + \frac{2\beta^2}{d^2} \right]$$

(A3)

and

$$\text{Var} \left( \frac{dh}{dx} \right) = \int_{-\infty}^{\infty} \left( \frac{dh}{dx} \right)^2 \, dx \approx h_m^2 \left( \frac{8}{3d} \right) \left[ 1 - \beta^2 + \frac{2\beta^2}{d^2} \right]$$

(A4)

In (A3) and (A4), the first term in the square brackets is the volume mode, and the second term is the roughness mode.

As (A1) is an even real function, its Fourier transform [see (1)] is also real and even. It is

$$\tilde{h}(k) = 2h_m[A(k) \sin(kd) + B(k) \cos(kd)]/C(k),$$

(A5)

where

$$A(d, \beta, k_m; k) = [2(\beta - 1)\Delta^3] + \beta k^4 (6k_m^2 + 2k^2).$$

$$B(d, \beta, k_m; k) = 2\beta dk^4 (k^4 - k_m^4) - 2(\beta - 1)kd\Delta^3,$$

and

$$C(d, k_m; k) = d^2 k^2 (k - k_m)^3 (k + k_m)^3,$$

where $\Delta = k_m^2 - k^2$ and $(k - k_m)^3 (k + k_m)^3 = -\Delta^3$.

One can verify from (A5) using L'Hôpital’s rule that $\tilde{h}(k = 0) = \text{Volume}$, given by (A2). Note that the zeroes of $C(k)$ at $k = \pm k_m$ tend to form singularities in $\tilde{h}(k)$, but the zeroes of $A(k)$ and $B(k)$ regularize them into finite smooth peaks. Because of these singularities, 8-byte precision or higher may be needed when computing (A5). When $\beta = 0$, (A1) reduces to a simple parabola and the Fourier transform in (A5) reduces to

$$\tilde{h}(k) = \frac{4h_m d}{(kd)^3} \sin(kd) - kd \cos(kd).$$

(A6)

Using (4) and (A6), the hydrostatic drag coefficient of the pure parabola is

$$C_D = 4/\pi.$$  

(A7)

The first and dominant spectral drag peak is at $k \approx 1.525/d$ or $\lambda = 2\pi/k \approx 4.12d$. The first peak of the $w$ variance in (3) occurs at a slightly shorter wavelength: $k \approx 2.08/d$ or $\lambda = 2\pi/k \approx 3.02d$.

When $\beta \neq 0$ and $n = k_m d/\pi \geq 3$, (A5) separates into the volume and roughness spectral modes that make independent contributions to variances and covariances. As the roughness drag is nearly monochromatic with $k \approx \pm k_m$, the factor $k$ comes outside the integral in (4) and
Drag = \(-\frac{\rhoNU}{2\pi} \int_{-\infty}^{\infty} |k|^4 \hat{h}(k) \hat{h}(k)^* dk\) 

\[= -\rhoNUk_m \text{Var}(h). \quad (A8)\]

Using the roughness part of \(\text{Var}(h)\) in \((A3)\),

\[\text{Drag} = -\rhoNUk_m h_m^2 (8/15)\frac{d\beta}{\beta^2} \quad \text{so that}\]

\[C_D = \left(\frac{8\pi}{15}\right) n\beta^2. \quad (A9)\]

As the volume and roughness modal spectra do not overlap in \(k\) space, the two drag contributions \([(A7)\) and \((A9)\)] are additive. Furthermore, additional roughness modes could be factored into \((13)\), each with their own \(\beta\) and \(n\). Their drags will be additive too, as long as their spectra do not overlap. As seen in Fig. 4, even roughness patterns with adjacent odd \(n\) values (e.g., \(n = 9\) and \(n = 11\)) have little spectral overlap. Thus, \((13)\) and \((19)\) can be easily generalized to a broad family of rough terrains.

The variances and covariances determined above using the analytical \(\hat{h}(k)\) expression in \((A5)\) can also be found numerically, using forward and inverse fast Fourier transforms (e.g., Smith 1989). Most of the results given herein have been verified using that method.

\(\text{Figure 6}\) was constructed using the fast Fourier transform method.

REFERENCES


