Nonlinear Formation of the Three-Dimensional Spectrum of Mesoscale Wind Velocity and Temperature Fluctuations in a Stably Stratified Atmosphere

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(Manuscript received 8 January 2018, in final form 6 July 2018)

ABSTRACT

The theory of formation of the space–time spectrum of the mesoscale fluctuations in the horizontal wind velocity and vertical displacements (or relative temperature fluctuations) in a stably stratified atmosphere is developed. The nonlinear mechanism of the formation of the finescale layered inhomogeneities in the internal wave fields associated with the nonresonant wave–wave and wave–vortical mode interactions is described. The 3D spatial spectra of the layered inhomogeneities are obtained from the approximate solutions of Lagrangian motion equations for internal waves and subsequent transition to the Eulerian coordinate system. Because of such transition, the advection of internal waves by the wind induced by the waves and vortical modes is taken into account. The contributions from the large-scale wind disturbances and finescale layered inhomogeneities to the horizontal wavenumber spectrum of the velocity fluctuations are found. Using an analytic form obtained for the 3D spectrum, the comparison is made between the modeled one-dimensional (1D) wavenumber spectra (vertical and horizontal) of the fluctuations with the observed spectra in the upper troposphere and lower stratosphere. The observed 1D (horizontal and vertical) wavenumber spectra of the horizontal velocity fluctuations with a -3 power-law decay are explained.

1. Introduction

This paper develops the model of formation of a 3D spatial spectrum of mesoscale wind velocity and temperature fluctuations in a stably stratified atmosphere to explain the forms of the observed vertical and horizontal wavenumber spectra of these fluctuations in the upper troposphere and stratosphere (Chunchuzov 2001, 2002; Gurvich and Chunchuzov 2008). The explanation of the shaping mechanism for the 3D spectrum of mesoscale fluctuations is of independent theoretical interest, since at present, a considerable number of works is devoted to modeling and explanation of the one-dimensional (1D; horizontal and vertical) wavenumber spectra of mesoscale fluctuations (Nastrom and Gage 1985; Hostetler and Gardner 1994; Dewan 1979, 1997; Bacmeister et al. 1996; Skamarock et al. 2014; Lindborg 2006, 2007, 2015; Callies et al. 2014; Sun et al. 2016). The knowledge of the 3D spectra of the wind velocity and temperature fluctuations is of practical interest since it makes it possible to find the statistical characteristics of the fluctuations of the parameters (such as an amplitude, phase, and angle of arrival to the

reception point) of infrasonic waves (Chunchuzov 2004; Ostashev et al. 2005) and light (Gurvich and Kan 2003; Kan et al. 2014) propagating in the atmosphere.

There is still a question of how internal gravity waves (IGWs) contribute to the formation of the horizontal wavenumber spectrum of the mesoscale wind velocity and temperature fluctuations in the upper troposphere and lower stratosphere (Lilly and Lester 1974; Vinnichenko et al. 1980; Bacmeister et al. 1996; Dewan 1979, 1997; Hertzog et al. 2002; Callies et al. 2014). According to the observations of Nastrom and Gage (1985), such spectrum decays as $k_x^{-5/3}$ with increasing horizontal wavenumber k_x in the range of horizontal scales from 500 km to several kilometers. At synoptic scales of more than 500 km, the transition is observed to the k_x^{-3} spectrum inherent to the 2D geostrophic turbulence. Such a transition is intensively discussed now in the literature (see, e.g., Callies et al. 2014; Skamarock et al. 2014; Lindborg 2015).

Based on the theory of similarity and dimensionalities, Dewan (1979, 1997) suggested that the portion of the horizontal wavenumber spectrum with a $-5/3$ power-law decay is shaped as a result of the nonlinear wave–wave interactions that cause a cascade-like transfer of wave energy from internal waves of large scales to the

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wave perturbations with small scales. The analysis of numerous aircraft measurements of the horizontal wind velocity in the upper troposphere and lower stratosphere shows the existence of both the horizontal vortical motions (for which the horizontal divergence of the wind velocity field is equal to zero) and wave motions with the nonzero horizontal divergence (Callies et al. 2014; Lindborg 2015).

Lindborg (2015) used a method of Helmholtz decomposition of the wind velocity field onto a rotational component with a zero horizontal divergence and a component with the zero horizontal rotor that includes wave motions. Comparing the ratio of energies for the vortical and wave components of the velocity field in the lower stratosphere and upper troposphere, Lindborg concludes that it exceeds unity in the range of scales for which the $2/3$ law is observed for structure functions (i.e., $-5/3$ for the spectra). Based on this, he concluded that there is an insignificant contribution of internal waves to the wind velocity fluctuations with the $k_x^{-5/3}$ spectrum. A more significant contribution is thought to come from the so-called stratified turbulence.

From the analysis of the aircraft measurements of the wind velocity, Callies et al. (2014) makes a conclusion about the dominance of the contribution of inertia-gravity waves to the formation of a horizontal spectrum of velocity fluctuations in the range of scales from 10 to 500 km. At synoptic scales, over 500 km, the slow (in comparison with wave motions) horizontal vortex motions predominate, obeying geostrophic and static equilibrium. The importance of the wave contribution to the mesoscale fluctuations of wind velocity and temperature was also shown by Skamarock et al. (2014) using a numerical model of atmospheric circulation with high spatial resolution. It is obvious that the estimation of the ratio of energy contributions of the wave and vertical vorticity motions can be strongly influenced both by the errors in calculating these components caused by the chosen numerical model and by the errors of the very method of decomposition of the velocity field into the vortex and wave components.

It is important to pay attention to earlier studies of horizontal spectra of wind velocity and temperature fluctuations obtained from aircraft measurements under stable stratification conditions of the troposphere and the lower stratosphere (Shur 1962; Lilly and Lester 1974; Vinnichenko et al. 1980; Bacmeister et al. 1996). All of them pointed to the presence of a transition in the rate of decay of the spectra with decreasing scales from the $-5/3$ power law to a steeper decay in the range from about 6 km to 300 m, where the exponent ranged from -2.5 to -3 . Above the mountainous terrain, such transition occurred at scales of 15–23 km (Nastrom et al.

1987). It is to this steep part of the horizontal wavenumber spectrum that an explanation will be given in this paper.

As for the observed vertical wavenumber spectra of the temperature and horizontal wind velocity fluctuations in the stratosphere and mesosphere, they follow the power law close to k_z^{-3} in some range of the vertical wavenumbers k_z (Vincent et al. 1997; Fritts and Alexander 2003; Tsuda 2014). In the upper stratosphere, this range corresponds to vertical scales from several kilometers to about 10 m (Gurvich and Chunchuzov 2003).

The nonlinearity of the hydrodynamic equations, including advective nonlinearity, plays an important role in the formation of the space–time spectrum of mesoscale velocity and temperature fluctuations induced by internal waves in stably stratified fluid (Allen and Joseph 1989; Hines 1991a,b, 2001; Bacmeister et al. 1996; Eckermann 1999; Chunchuzov 1996, 2001, 2002; Pinkel 2008). The wave-induced advection of fluid parcels arises because of the presence of the acceleration term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ in the Eulerian motion equations. This component vanishes in the case when the perturbation of a stratified fluid is caused by a single-plane internal wave in the Boussinesq approximation (Hines 1991a), for which the fluid density variation over the wavelength can be neglected. In this approximation, the wave vector is perpendicular to the velocity of the parcel, $(\mathbf{k} \cdot \mathbf{v}) = 0$. Because of the absence of the effect of nonlinear “self action” for one-plane internal wave, there is no restriction on the growth of its amplitude. However, when two or more internal waves with different wavenumbers $\mathbf{k}$ are generated, each of these waves experiences the influence of the advection caused by the total internal wave field. Such advection along with the nonlinearities of the equation of state and of the buoyancy forces leads to the interaction between internal waves.

Considering the interaction of two internal waves with the velocity amplitudes α_1 and α_2 and the wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, Phillips (1967) found the condition $(k_1\alpha_1k_2\alpha_2)^{1/2} \sim N$, where N is the Brunt–Väisälä frequency (BV), at which the generated resonant wave mode with the combinative wavenumber $\mathbf{k}_1 + \mathbf{k}_2$ and the frequency $\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_1 + \mathbf{k}_2)$, satisfying the resonance condition, becomes comparable in amplitude with a nonresonant mode of the wavenumber $\mathbf{k}_1 - \mathbf{k}_2$ and the frequency $\omega(\mathbf{k}_1) - \omega(\mathbf{k}_2) \neq \omega(\mathbf{k}_1 - \mathbf{k}_2)$ that does not obey the dispersion relation.

While the amplitudes of the interacting modes are small, the amplitude of the generated resonance mode grows linearly in time over a time interval small in comparison with the characteristic mode interaction time, reaching a maximum $O(\alpha_1\alpha_2)^{1/2}$. In addition to this mode, there appears also a nonresonance mode of finite amplitude,

periodically varying in time and reaching the maximum values $O\{(k_1 k_2)^{1/2} \alpha_1 \alpha_2 / N\}$. If $(k_1 \alpha_1 k_2 \alpha_2)^{1/2} \ll N$, then the ratio of the amplitude of the nonresonant mode to the amplitude of the resonant mode is small. In this case, the resonance interactions of the wave triads are selective and lead to a periodic energy exchange between the interacting modes so that the characteristic times of their interaction are large in comparison with the periods of the waves. To find the equilibrium spectrum of waves in the wavenumber region of their weak resonant interactions, the Hamiltonian formalism was used by [Voronovich \(1979\)](#), [Lvov and Tabak \(2001\)](#), and [Lvov et al. \(2010\)](#).

If velocity shears induced by the initial modes become sufficiently large, then the generated nonresonant modes become comparable in amplitude with the resonant modes, and the wave interactions themselves cease to be weak. The frequencies of the generated nonresonant modes are not related to their wave vector by the dispersion relation; therefore, these modes are no longer linear internal waves. They can interact with any other wave mode because of the nonselective nature of the nonresonant interactions. Moreover, the characteristic time of such interactions becomes short and comparable to the periods of the original waves. Such fast nonresonant interactions of the modes give rise to the cascade character of energy transfer over the mode scale, which is analogous to the turbulent cascade of energy transfer along the spectrum of turbulent fluctuations.

In the case when the 1D spectrum of the wave modes is continuous and has a spectral energy density $E(k)$, [Phillips \(1967\)](#) has written down the following condition: $\{k^3 E(k)\}^{1/2} \sim N$, under which the nonresonant wave-wave interaction form the spectrum $E(k) \sim N^2 k^{-3}$. Such a spectrum obtained from the assumption of a cascade caused by nonresonant wave-wave interactions coincides with the spectrum of turbulent fluctuations in stably stratified fluid obtained by [Shur \(1962\)](#) and [Lumley \(1964\)](#) for the so-called buoyancy subrange.

Shur and Lumley followed the assumption of a cascade transfer of the kinetic energy of turbulent motions from large scales to small ones, taking into account the transformation of some of the kinetic energy into potential energy due to work performed by turbulent vortex motions against buoyancy forces in a stably stratified medium. It is this work that leads to a more rapid decay (with a slope of -3) of the spectrum of horizontal velocity fluctuations with increasing k in a stratified fluid compared to the $-5/3$ law for turbulence in the inertial range of scales. As [Lumley \(1964\)](#) noted, his theory is applicable in the buoyancy subrange regardless of whether random fluctuations in a stably stratified fluid are caused by wave disturbances or

turbulence. Thus, the cascade energy transfer over scales caused by strong nonresonant interactions of wave disturbances (with a short interaction time) and a turbulent cascade in a stably stratified fluid lead to the same spectrum with a -3 slope in the buoyancy subrange.

The purpose of this paper is to obtain an analytic form for the 3D spectrum of the mesoscale fluctuations of the vertical displacements (or relative temperature fluctuations) and horizontal velocity in stably stratified atmosphere, which will allow us to explain their observed vertical and horizontal wavenumber spectra. To find this form, we will rely on a Lagrangian approach to the description of the shaping mechanism for the internal wave spectrum developed by [Chunchuzov \(1996, 2001, 2002, 2009\)](#), [Eckermann \(1999\)](#), [Hines \(2001\)](#), and on analogous approach developed by [Gurbatov et al. \(1991\)](#) for a high-intensity acoustic noise. Here, we generalize this approach to the case when horizontal vortical motions contribute along with internal waves to the formation of the spectrum of velocity fluctuations. In [section 2](#), a relation between the correlation functions of the random velocity and displacement fields in Lagrangian and Eulerian coordinate systems will be obtained. This relation will be used in [section 3](#) to find the shapes of the 3D spectra of vertical displacements and horizontal velocities in the Eulerian system. Using these forms, the vertical and horizontal wavenumber spectra will be obtained in [section 4](#) and compared with the observed spectra.

2. The relation between correlation functions of the random displacements and velocities in Lagrangian and Eulerian coordinate systems

The small perturbations of the velocity field $\mathbf{u}(x, y, z, t)$ that satisfies the linearized equations of motion of a nonrotating stably stratified medium in the Boussinesq approximation (for which $\text{div } \mathbf{u} = 0$) can be presented as a superposition of horizontal vortical motions with zero horizontal divergence and divergence-free wave motions with zero vertical vorticity whose velocity is determined by the potential $\Phi(x, y, z, t)$ and has a vertical component $w(x, y, z, t)$ (see, e.g., [Lelong and Riley 1991](#)):

$$\mathbf{u} = \mathbf{i}_3 \times \nabla_{\perp} \psi + (\nabla_{\perp} \Phi + w \mathbf{i}_3), \quad (1)$$

where $\psi(x, y, z, t)$ is the streamfunction, $\nabla_{\perp}$ is the horizontal gradient operator, and $\mathbf{i}_3$ is the unit vector in the vertical direction. The linearization in this case is carried out over a small Froude number $\text{Fr} = U/(NL) \ll 1$, where U and L are the characteristic velocity and length

scales, respectively, on which the velocity field varies significantly.

The nonlinearity of the motion equations leads to the interactions of internal waves, internal waves with horizontal vortical motions, and horizontal vortical motions (Lelong and Riley 1991; Embid and Majda 1998). As shown by Lelong and Riley (1991), the interaction of internal waves can generate a vertical vorticity component without violating the potential vorticity conservation law along the trajectory of a fluid particle (the potential vorticity coincides with the vertical vorticity only in the linear approximation).

If at the initial moment a stationary mode of vertical vorticity (with a zero frequency) is absent, and only two internal waves are present, then their weak interaction can generate vertical vorticity modes with sum and difference frequencies and wavenumbers of the primary waves. The forcing of the vertical vorticity arises because of the so-called vortex turning mechanism, when the velocity field of each wave turns the other wave's horizontal vorticity toward the vertical (Lelong and Riley 1991). Such vortical motions do not arise only when internal wave velocities lie in the same vertical plane or two waves have equal frequencies.

The nonresonant wave-wave interactions, the interactions between internal waves and horizontal vortical modes (that do not change the frequencies of primary waves), and horizontal vortical mode interactions (which change only modal wavenumbers under zero frequency) are nonselective, in contrast to the resonant wave-wave interactions. Such interactions can occur many times, generating new waves and horizontal vortex modes. Each of these modes experiences the influence of advection in the velocity field created by all other waves and vortical modes.

To strictly take into account the influence of the advection of fluid particles on the spectrum of velocity perturbations arising in the total field of internal waves and horizontal vortical motions, we use the Lagrangian approach (Chunchuzov 1996, 2001, 2002, 2009; Eckermann 1999; Hines 2001). First, we seek a solution of the nonlinear equations of fluid motion written in the Lagrangian coordinate system (Lamb 1932; Gossard and Hooke 1975) and then take an exact transformation from the Lagrangian to the Eulerian coordinates:

$$\mathbf{x} = \mathbf{r} + \mathbf{S}(\mathbf{r}, t), \quad (2)$$

where $\mathbf{S}(\mathbf{r}, t)$ are the Lagrangian displacements of the fluid parcels having coordinates $\mathbf{r}$ in the unperturbed atmosphere ($S_i = 0$), which is in static equilibrium. The displacement $\mathbf{S}_E(\mathbf{x}, t) = \{S_{E,1}, S_{E,2}, S_{E,3}\}$ at a fixed point

$\mathbf{x}$ and a time t is caused by the instantaneous displacement $\mathbf{S}(\mathbf{r}, t) = \{S_1, S_2, S_3\}$ of the fluid parcel with the coordinate $\mathbf{r}$ to a point $\mathbf{x}$ satisfying (2); therefore, $\mathbf{S}(\mathbf{r}, t) = \mathbf{S}_E(\mathbf{x}, t)$. The latter relation may be rewritten through delta-function $\delta(\mathbf{x})$:

$$S_{E,j}(\mathbf{x}, t) = \int d^3r J S_j(\mathbf{r}, t) \delta(\mathbf{x} - \mathbf{y}),$$

$$\mathbf{y} = \mathbf{r} + \mathbf{S}(\mathbf{r}, t), \quad (j = 1, 2, 3), \quad (3)$$

where

$$J = \sum_{\alpha, \beta, \gamma} \varepsilon_{\alpha\beta\gamma} \left(\delta_{1\alpha} + \frac{\partial S_1}{\partial r_\alpha} \right) \left(\delta_{2\beta} + \frac{\partial S_2}{\partial r_\beta} \right) \times \left(\delta_{3\gamma} + \frac{\partial S_3}{\partial r_\gamma} \right), \quad (\alpha, \beta, \gamma = 1, 2, 3) \quad (4)$$

is the Jacobian of transformation of the variables (5), $\varepsilon_{\alpha\beta\gamma}$ is an antisymmetric unit tensor of third rank, and $\delta_{\alpha\beta}$ is the Kronecker symbol. Substituting $\delta(\mathbf{x} - \mathbf{y}) = (2\pi)^{-3} \int d^3k \exp[i\mathbf{k}(\mathbf{x} - \mathbf{y})]$ into (3), we obtain the following relationship between Eulerian and Lagrangian vertical displacements (Allen and Joseph 1989):

$$S_{E,3}(\mathbf{x}, t) = (2\pi)^{-3} \int d^3r \int d^3k J S_3(\mathbf{r}, t) \times \exp\{i\mathbf{k}[\mathbf{x} - \mathbf{r} - \mathbf{S}(\mathbf{r}, t)]\}. \quad (5)$$

The similar relationship can be obtained for any hydrodynamic field, such as velocity components, density, and pressure. Designating such fields in Lagrangian and Eulerian coordinates as $F(\mathbf{r}, t)$ and $F_E(\mathbf{x}, t)$, respectively, we obtain their relationship:

$$F_E(\mathbf{x}, t) = (2\pi)^{-3} \int d^3r \int d^3k J F(\mathbf{r}, t) \exp\{i\mathbf{k}[\mathbf{x} - \mathbf{r} - \mathbf{S}(\mathbf{r}, t)]\}. \quad (6)$$

Suppose that the random Lagrangian vertical displacements in the atmosphere with a constant value of the Brunt-Väisälä frequency N are caused by an ensemble of upward-propagating internal gravity wave modes:

$$S_3(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\rho_s(0)}{\rho_s(r_3)} \right]^{1/2} \int_{-\infty}^0 dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 A(\mathbf{k}_\perp, m, t) \times \exp\{-i[\omega(m, \mathbf{k}_\perp)t - \mathbf{k}_\perp \mathbf{r}_\perp - mr_3]\} + \text{c.c.}, \quad (7)$$

where $A(\mathbf{k}_\perp, m, t) = |A(\mathbf{k}_\perp, m, t)| e^{i\alpha(\mathbf{k}_\perp, m, t)}$ are the random complex amplitudes of the wave modes with wavenumbers $\mathbf{k} = (\mathbf{k}_\perp, m)$ and frequencies $\omega(m, \mathbf{k}_\perp) = \pm N k_\perp / (k_\perp^2 + m^2)^{1/2}$ and $\rho_s(r_3)$ is the density of the

unperturbed atmosphere $r_3 = z$, which in the Boussinesq approximation varies slightly on the vertical scales m^{-1} of the modes. Because of the nonlinearity of the Lagrangian motion equations, the modal amplitudes $|A(\mathbf{k}_\perp, m, t)|$ and their phases $\alpha(\mathbf{k}_\perp, m, t)$ are assumed to vary slowly with time over modal periods $\sim 2\pi/\omega(m, \mathbf{k}_\perp)$. The model of the atmosphere chosen here may be easily generalized for the case of slowly varying $N(z)$ over vertical scales of wave modes or for two-layer

atmosphere with some constant $N = N_1$ within the lower layer and $N = N_2 \ll N_1$ in the upper infinite layer (Chunchuzov 1996, 2002).

The horizontal displacements induced by wave modes, $S_{1,2}^w(\mathbf{r}, t)$, and those induced by modes with vertical vorticity, $S_{1,2}^v(\mathbf{r}, t)$, contribute to the horizontal displacement components $S_{1,2}(\mathbf{r}, t)$:

$$S_{1,2}(\mathbf{r}, t) = S_{1,2}^w(\mathbf{r}, t) + S_{1,2}^v(\mathbf{r}, t), \quad (8)$$

$$S_{1,2}^w(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\rho_s(0)}{\rho_s(r_3)} \right]^{1/2} \int_{-\infty}^0 dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 [-k_{1,2} m k_\perp^{-2} A(\mathbf{k}_\perp, m, t)] \exp\{-i[\omega(m, \mathbf{k}_\perp)t - \mathbf{k}_\perp \mathbf{r}_\perp - mr_3]\} + \text{c.c.},$$

$$S_{1,2}^v(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\rho_s(0)}{\rho_s(r_3)} \right]^{1/2} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 A_{1,2}^v(\mathbf{k}_\perp, m, t) \exp[i(\mathbf{k}_\perp \mathbf{r}_\perp + mr_3)] + \text{c.c.}, \quad (9)$$

where $A_{1,2}^v(\mathbf{k}_\perp, m, t) = |A_{1,2}^v(\mathbf{k}_\perp, m, t)|e^{i\phi(\mathbf{k}_\perp, m, t)}$ are the random complex amplitudes of horizontal displacements in the modes with a vertical vorticity that are thought to slowly vary in time because of their weak interactions with all other vortical and wave modes and also because of the generation of vertical vorticity modes by internal wave interactions.

We express the components of the Lagrangian velocities $v_{1,2}^v = \partial S_{1,2}^v / \partial t$ in the horizontal vortical motions through the streamfunction $\psi(\mathbf{r}, t)$: $v_1^v = -\partial\psi/\partial r_2$, $v_2^v = \partial\psi/\partial r_1$, presenting ψ in a form analogous to (9), in which the amplitudes of the modes of the streamfunction $\Psi(\mathbf{k}_\perp, m, t) = |\Psi(\mathbf{k}_\perp, m, t)|e^{i\phi(\mathbf{k}_\perp, m, t)}$ are instead of the amplitudes $A_{1,2}^v$. Then the velocities $v_{1,2}^v$ can be expressed from (9) in terms of displacements to obtain the following relationship:

$$-ik_2 \Psi = \partial A_1^v / \partial t, \quad ik_1 \Psi = \partial A_2^v / \partial t. \quad (10)$$

The relation (10) is satisfied if $\partial|A_1^v|/\partial t = \partial|A_2^v|/\partial t = 0$ and $-k_2|\Psi| = |A_1^v|(\partial\phi/\partial t)$, $k_1|\Psi| = |A_2^v|(\partial\phi/\partial t)$, from which we obtain the required relationship between the amplitudes of the displacements and the streamfunction:

$$A_1^v = -k_2 \Psi / \Omega, \quad A_2^v = k_1 \Psi / \Omega, \quad (11)$$

where $\Omega(\mathbf{k}_\perp, m, t) \equiv \partial\phi/\partial t$ is the frequency of time modulation of the phase of the vertical vorticity mode with wavenumbers $\mathbf{k}_\perp$ and m caused by the weak interaction of this mode with gravity wave modes and with other vertical

vorticity modes. Taking into account (6), the space-time correlation function $B_E(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) \equiv \langle F_E(\mathbf{x}_1, t_1) \cdot F_E(\mathbf{x}_2, t_2) \rangle$ of a random Eulerian field $F_E(\mathbf{x}, t)$ can be written in the form

$$B_E(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = (2\pi)^{-6} \int d^3 r' \int d^3 r'' \int d^3 k' \int d^3 k'' M_F(\mathbf{r}', \mathbf{r}'', t_1, t_2) \times \exp[i(\mathbf{k}' \mathbf{x}_1 - \mathbf{k}'' \mathbf{x}_2) - i(\mathbf{k}' \mathbf{r}' - \mathbf{k}'' \mathbf{r}'')], \quad (12)$$

where

$$M_F(\mathbf{r}', \mathbf{r}'', t_1, t_2) = \langle J' J'' F(\mathbf{r}', t_1) F(\mathbf{r}'', t_2) \times \exp\{-i[\mathbf{k}' \mathbf{S}(\mathbf{r}', t_1) - \mathbf{k}'' \mathbf{S}(\mathbf{r}'', t_2)]\} \rangle, \quad (13)$$

where J' and J'' are the values of the Jacobian (4) taken at points $\mathbf{r}', t_1$ and $\mathbf{r}'', t_2$, respectively.

The Jacobian J is the ratio of the air parcel density ρ_i in the undisturbed atmosphere ($S_i = 0$) to the density ρ of the same parcel but displaced along its trajectory; therefore, the deviation of J from 1 equals the relative change in density $\rho_i/\rho - 1$ of the displaced parcel. This change is small for small gradients of the displacement field $|\partial S_i/\partial r_j| \ll 1$ so that the effects of compressibility of the medium can be neglected by taking $J' = J'' \approx 1$ in (13). As the gradients $|\partial S_i/\partial r_j|$ increase with increasing amplitudes of modes, the nonlinear terms in J [(4)] also

contribute to the correlation (13) and can be taken into account (Chunchuzov 2002).

Another assumption simplifying the derivation of the expression for the correlation function (13) was that the components of the displacements and hydrodynamic fields $F(\mathbf{r}, t)$ in the Lagrangian system were considered as random fields with probability density functions close to Gaussian, as well as statistically stationary in time and locally homogeneous along the horizontal and vertical in a certain volume of the medium (Chunchuzov 2002). In the case when the components of the Lagrangian displacements (7)–(9) represent the sums of a large number of almost linear waves and vortical modes with randomly independent amplitudes and phases, the probability distribution of the component of the total Lagrangian displacement will be close to Gaussian in accordance with the central limit theorem. However, in the Eulerian system, the distribution functions deviate from the Gaussian ones because of the nonlinearity of the transformation (4), and this deviation increases with increasing displacement amplitudes.

The assumptions made above allow us to present the correlation function (12) as

$$B_E(\mathbf{X}, \tau, z) = (2\pi)^{-3} \int d^3R \int d^3k' M_F(\mathbf{R}, \tau, \mathbf{k}', z) \times \exp[i\mathbf{k}'(\mathbf{X} - \mathbf{R})], \quad (14)$$

where in the case of vertical displacements, $F = S_3(\mathbf{r}, t)$, the correlation (13) takes the following form [see Chunchuzov 2002, his (B.12)]:

$$M_F(\mathbf{R}, \tau, \mathbf{k}, z) = \left\{ C_{33}(\mathbf{R}, \tau, z) + \sum_{j=1}^3 [D_{3j}(\mathbf{R}, \tau, z)k_j]^2 \right\} \times \exp \left[- \sum_{ij} D_{ij}(\mathbf{R}, \tau, z) k_i k_j \right], \quad (15)$$

where $\mathbf{X} = \mathbf{x}_1 - \mathbf{x}_2$ is the difference between coordinates of two points within a chosen volume, $z = (z_1 + z_2)/2$ is the height of the point above ground $z = 0$, which is in the midpoint between the heights of the points z_1 and z_2 (in the undisturbed atmosphere, the Lagrangian and Eulerian coordinates coincide), $\mathbf{R} = \mathbf{r}' - \mathbf{r}''$, $\tau = t_1 - t_2$, $C_{ij}(\mathbf{R}, \tau, z) = \langle S_i(\mathbf{r}', t_1) S_j(\mathbf{r}'', t_2) \rangle$ are the correlation functions of different components of the Lagrangian displacements with $i, j = 1, 2, 3$, and $D_{ij}(\mathbf{R}, \tau, z) = C_{ij}(\mathbf{R} = 0, \tau = 0, z) - C_{ij}(\mathbf{R}, \tau, z)$ is the half of their structure function.

Similar relation may also be obtained from (11) for the horizontal velocity components $\mathbf{v}_\perp(\mathbf{r}, t) = (v_1, v_2)$ if we take $F = \partial S_i / \partial t = v_i(\mathbf{r}, t)$. Then for the horizontal velocity

component in the Eulerian system, $F_E = v_{E,1}(\mathbf{x}, t)$, the correlation function $B_E(\mathbf{x}, \tau, z)$ takes the form (14), where

$$M_F(\mathbf{R}, \tau, \mathbf{k}, z) = \left\{ - \frac{\partial^2 C_{11}(\mathbf{R}, \tau, z)}{\partial \tau^2} - \left[\sum_{j=1}^3 k_j \frac{\partial C_{1j}(\mathbf{R}, \tau, z)}{\partial \tau} \right]^2 \right\} \times \exp \left[- \sum_{ij} D_{ij}(\mathbf{R}, \tau, z) k_i k_j \right]. \quad (16)$$

3. The spectra of the Eulerian vertical displacements and horizontal velocities

a. Low wavenumbers

The obtained expressions (14)–(16) allow us to express Eulerian spectra by taking Fourier transforms from the correlation functions (14). In particular, the space–time spectrum, $S_{4E}(\mathbf{k}, \omega)$, and spatial spectrum of the vertical displacements, $S_{3E}(\mathbf{k})$, may be presented as

$$S_{4E}(\mathbf{k}, \omega) = (2\pi)^{-4} \int d^3R \int d\tau M_F(\mathbf{R}, \tau, \mathbf{k}, z) \times \exp[-i(\mathbf{k}\mathbf{R} - \omega\tau)], \quad (17)$$

$$S_{3E}(\mathbf{k}) = (2\pi)^{-3} \int d^3R M_F(\mathbf{R}, \tau = 0, \mathbf{k}, z) \times \exp(-i\mathbf{k}\mathbf{R}). \quad (18)$$

Similar expressions may also be written for the horizontal velocity spectra, where $M_F(\mathbf{R}, \tau, \mathbf{k}, z)$ is given by (16).

Let us consider the spectra (17) and (18) in the regions of small and large values of the parameter

$$q^2 \equiv 2(\nu_1^2 k_\perp^2 + \nu_v^2 k_3^2), \quad (19)$$

which is the quadratic form of the wavenumber components, where $\nu_v^2 = \langle S_3(\mathbf{r}, t)^2 \rangle$ and $\nu_1^2 = \langle S_1(\mathbf{r}, t)^2 \rangle = \langle S_2(\mathbf{r}, t)^2 \rangle$ are the variances of the vertical and horizontal displacement components, respectively (the equality of the variances of the horizontal displacement components is due to assumed axial isotropy of their field).

If $q^2 \ll 1$, then we can neglect by small terms of order q^2 in expressions (15) and (16) by taking the exponential term equal to 1 and keeping only the first term on the right-hand side of (15) and (16). In this linear approximation, the spectra of random displacements and velocities in the Eulerian and Lagrangian systems coincide. This also follows directly from the relation $F_E(\mathbf{x}, t) = F(\mathbf{r}, t)$. Indeed, if such a field changes slowly over horizontal scale $l_\perp = k_\perp^{-1}$ and vertical scale $l_v = k_3^{-1}$, then, taking into account (2), we can write down the following relation between the Eulerian field at a point $\mathbf{x}$

at a time t and the Lagrangian field of a fluid parcel with the coordinate $\mathbf{x}$ at a time t :

$$F_E(\mathbf{x}, t) = F[\mathbf{x} - \mathbf{S}(\mathbf{r}, t), t] \approx F(\mathbf{x}, t) - \frac{\partial F}{\partial x_i} S_i + O(k_i^2 S_i^2). \quad (20)$$

In the linear approximation $F_E(\mathbf{x}, t) \approx F(\mathbf{x}, t)$, but as the parameter $k_i^2 S_i^2 \ll 1$ increases in the expansion (17), the nonlinear terms of the first, second, and higher order in this parameter, caused by advection of the medium, become important. If the Lagrangian wave field $F(\mathbf{r}, t) = F_0(\mathbf{r}, t) + F'(\mathbf{r}, t)$ is composed from only two harmonic waves, $F_0(\mathbf{r}, t)$ and $F'(\mathbf{r}, t)$, with the wavenumbers $\mathbf{K}$ and $\mathbf{k}$, which are far from each other, $|\mathbf{K}| \ll |\mathbf{k}|$, then in the Eulerian system, the wave with high-wavenumber $\mathbf{k}$ will undergo phase modulation $\mathbf{k}\mathbf{S}(\mathbf{x}, t)$ [Chunchuzov 2002, his (D5)]; that is, $F_E(\mathbf{x}, t) = F_0(\mathbf{x}, t) + F'(\mathbf{x}, t)e^{-i\mathbf{k}\mathbf{S}(\mathbf{x}, t)}$, where $\mathbf{S}(\mathbf{x}, t) = \mathbf{S}_0(\mathbf{x}, t) + \mathbf{S}'(\mathbf{x}, t)$ is the Lagrangian displacement field induced by two waves (under Boussinesq approximation $\mathbf{k}\mathbf{S}'$ and $\mathbf{K}\mathbf{S}_0$ are equal to 0).

In the more general case, when a large-scale Lagrangian field $S_0(\mathbf{x}, t)$ is specified as a superposition of quasi-harmonic waves and vertical vorticity modes with small wavenumbers ($|\mathbf{K}| \ll |\mathbf{k}|$) and random amplitudes and phases, the phase of a mode with a wavenumber $\mathbf{k}$ will experience in the Eulerian system the random fluctuations $\mathbf{k}\mathbf{S}_0(\mathbf{x}, t)$, which is equivalent to the random deviations of the modal wavenumber components $k_{E,i}(\mathbf{x}, t)$ and frequency $\omega_E(\mathbf{x}, t)$ in the Eulerian system with respect to their values in the Lagrangian system:

$$\begin{aligned} \omega_E(\mathbf{x}, t) - \omega &= \mathbf{k} \frac{\partial \mathbf{S}_0(\mathbf{x}, t)}{\partial t}, \\ k_{E,i}(\mathbf{x}, t) - k_i &= -k_j \frac{\partial S_{0,j}(\mathbf{x}, t)}{\partial x_i}, \quad (i, j = 1, 2, 3). \end{aligned} \quad (21)$$

Such deviations lead to the fact that $\omega_E(\mathbf{x}, t)$ and $k_{E,i}(\mathbf{x}, t)$ do not necessarily obey the dispersion relation for linear internal waves.

While the displacements are caused by vertical vorticity modes and waves of small amplitudes and with small wavenumbers, the space-time spectrum $S_E(\mathbf{k}, \omega)$ of the Eulerian field is localized near the dispersion surfaces $\omega = \omega(\mathbf{k})$ of linear internal waves (Chunchuzov et al. 2005) and near-zero frequency $\omega = 0$ of the stationary vorticity mode. As the wavenumbers and the parameter q^2 increase, an increasing role is played by the nonlinear terms in the expansion of the right-hand sides of (15) and (16) in a power series of q^2 , caused by the advection of the fluid parcels. The advection leads to the interaction of all the Lagrangian modes and the generation of their combinative harmonics with frequencies and wavenumbers

that do not satisfy the linear dispersion relation. As a result, a nonlinear diffusion of the spectral density $S_{4E}(\mathbf{k}, \omega)$ occurs in the frequency-wavenumber space beyond the dispersion surfaces of linear internal waves $\omega = \omega(\mathbf{k})$ and the surface $\omega = 0$ of stationary modes with vertical vorticity.

b. 3D spectra of vertical displacements and horizontal velocities at high wavenumbers ($q^2 \gg 1$)

When wavenumbers approach the ellipsoid surface $q^2 = 1$ in the wavenumber space, the nonlinear diffusion of the spectral density $S_{4E}(\mathbf{k}, \omega)$ beyond the dispersion surfaces is so strong that this spectrum no longer has anything in common with the spectrum of linear internal waves. The wave modes and vertical vorticity modes with such high wavenumbers repeatedly interact with other modes, generating modes with even higher wavenumbers, for which $q^2 \gg 1$. At the same time, such interactions also generate modes with lower wavenumbers, for which $q^2 \ll 1$; therefore, the nonlinear diffusion occurs in the entire frequency-wavenumber space.

The nonlinear spectral tail at $q^2 \gg 1$ generated in the Eulerian system takes a universal form regardless of the form of the Lagrangian spectrum of the random sources of internal waves (Chunchuzov 2002). As shown below, this conclusion remains valid when the vortical modes also contribute to the spectrum.

Assuming the spectral amplitudes of the wave modes $A(\mathbf{k}_\perp, m, t)$ to be randomly independent, statistically stationary in time, and locally homogeneous in space, we can introduce the spectral density $L(\mathbf{k}_\perp, m)$ of the vertical displacement field (7) by the condition

$$\begin{aligned} \langle A(\mathbf{k}_\perp, m, t) A^*(\mathbf{k}'_\perp, m', t) \rangle \\ = L(\mathbf{k}_\perp, m) \delta(\mathbf{k}_\perp - \mathbf{k}'_\perp) \delta(m - m'). \end{aligned} \quad (22)$$

In (22), we took into account the delta correlation of spectral wavenumber amplitudes $A(\mathbf{k}_\perp, m, t)$ due to local homogeneity and the time independence of their correlation due to the stationarity.

Then the correlation function of the Lagrangian vertical displacements $C_{33}(\mathbf{R}, \tau, z) = \langle S_3(\mathbf{r}', t_1) S_3(\mathbf{r}'', t_2) \rangle$ can be expressed in terms of their spectral density $L(\mathbf{k}_\perp, m)$:

$$\begin{aligned} C_{33}(\mathbf{R}, \tau, z) &= \frac{\rho_s(0)}{2\rho_s(z)} \int_{-\infty}^0 dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 L(\mathbf{k}_\perp, m) \\ &\times \exp[-i(\omega\tau - \mathbf{k}_\perp \mathbf{R}_\perp - mR_3)] + \text{c.c.}, \end{aligned} \quad (23)$$

where the correlation $\langle A(\mathbf{k}_\perp, m, t_1) A^*(\mathbf{k}_\perp, m, t_2) \rangle$ is a slow function of $t_2 - t_1$ over modal periods $\sim 2\pi/\omega(m, \mathbf{k}_\perp)$; therefore, it can be replaced by (22).

Assuming that similar conditions are satisfied for the amplitudes of the modes of vertical vorticity $A_{1,2}^v(\mathbf{k}_\perp, m, t)$,

we introduce the spectral densities $\tilde{S}_{ij}(\mathbf{k}_\perp, m)$ of their horizontal displacement components:

$$\begin{aligned} \langle A_i^v(\mathbf{k}_\perp, m, t) A_j^{v*}(\mathbf{k}'_\perp, m', t) \rangle \\ = \tilde{S}_{ij}(\mathbf{k}_\perp, m) \delta(\mathbf{k}_\perp - \mathbf{k}'_\perp) \delta(m - m'), \quad (i, j = 1, 2). \end{aligned} \quad (24)$$

Taking into account the relation (11) between the displacement amplitudes and the streamfunction, we express the spectrum $\tilde{S}_{ij}(\mathbf{k}_\perp, m)$ in terms of the spectral density of the streamfunction $\Phi(\mathbf{k}_\perp, m)$:

$$\tilde{S}_{ij}(\mathbf{k}_\perp, m) = -k_j k_i \Phi(\mathbf{k}_\perp, m) / \Omega^2(\mathbf{k}_\perp, m), \quad (i, j = 1, 2), \quad (25)$$

where $\Omega^2(k_\perp, m) \equiv \langle \Omega(k_\perp, m, t) \Omega(k_\perp, m, t) \rangle$ is the variance of $\Omega(k_\perp, m, t)$, which does not depend on t .

Then the Lagrangian correlation functions of horizontal displacement components can be presented as a sum of wave, $C_{ij}^w(\mathbf{R}, \tau, z)$, and vortical, $C_{ij}^v(\mathbf{R}, z)$, correlation functions:

$$C_{ij}(\mathbf{R}, \tau, z) = C_{ij}^w(\mathbf{R}, \tau, z) + C_{ij}^v(\mathbf{R}, z), \quad (i, j = 1, 2), \quad (26)$$

$$C_{ij}^v(\mathbf{R}, z) = \frac{\rho_s(0)}{2\rho_s(z)} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 [-k_j k_i \Phi(\mathbf{k}_\perp, m) / \Omega^2(\mathbf{k}_\perp, m)] \exp[i(\mathbf{k}_\perp \mathbf{R}_\perp + m R_3)] + \text{c.c.}, \quad (27)$$

$$C_{ij}^w(\mathbf{R}, \tau, z) = \frac{\rho_s(0)}{2\rho_s(z)} \int_{-\infty}^0 dm \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 [k_i k_j m^2 k_\perp^{-4} L(\mathbf{k}_\perp, m)] \exp[-i(\omega\tau - \mathbf{k}_\perp \mathbf{R}_\perp - m R_3)] + \text{c.c.} \quad (28)$$

From the expressions (26)–(28) obtained, it follows that the variance of horizontal displacement components is the sum of the variances of horizontal displacements in internal waves and in modes with vertical vorticity:

$$\begin{aligned} \nu_1^2 &= \langle S_1(\mathbf{r}, t)^2 \rangle = \nu_{1,v}^2 + \nu_{1,w}^2, \\ \nu_{1,v}^2 &\equiv C_{11}^v(\mathbf{R} = 0, z), \quad \nu_{1,w}^2 \equiv C_{11}^w(\mathbf{R} = 0, \tau = 0, z). \end{aligned} \quad (29)$$

As for the dispersion of vertical displacements,

$$\nu_v^2 = \langle S_3(\mathbf{r}, t)^2 \rangle = C_{33}(\mathbf{R} = 0, \tau = 0, z), \quad (30)$$

it is due to the contribution of wave modes, including those that arise as a result of the interaction of wave modes with vortical modes. Because of the assumed presence of axial isotropy of the spectra of vertical displacements and streamfunction, we can write $L(\mathbf{k}_\perp, m) = L(k_\perp, m)$, $\Phi(\mathbf{k}_\perp, m) = \Phi(k_\perp, m)$, $\Omega(\mathbf{k}_\perp, m) = \Omega(k_\perp, m)$.

Suppose that in the Lagrangian system, some equilibrium spectra of the displacements and velocities are formed because of the balance between the energy input to the system of waves and horizontal vortical motions from their random sources at wavenumbers $k_\perp \sim k_0$ and $m \sim m_0$, by nonlinear transfer of this energy toward higher wavenumbers $k_\perp$ and m , and its dissipation for high-enough m and $k_\perp$ caused by wave-breaking processes induced by shear or convective instabilities.

Each component of the Eulerian space–time spectrum (17) with the wavenumber $\mathbf{k}$ and frequency ω is the Fourier transform of the function $M_F(\mathbf{R}, \tau, \mathbf{k})$, which is the two-point correlation of the product $S_3(\mathbf{r}, t) \exp[-i\mathbf{k}\mathbf{S}(\mathbf{r}, t)]$. According to (13), this product is a superposition of modes, each of which in the Eulerian system experiences additional phase modulation $\mathbf{k}\mathbf{S}(\mathbf{r}, t)$ because of advection caused by the total displacement field $\mathbf{S}(\mathbf{r}, t)$.

In the case of high wavenumbers $q^2 \gg 1$, the exponential term in (17) and (18) becomes negligibly small for any R_i and τ except their very small values for which the functions $C_{ij}(\mathbf{R}, \tau, z)$ are close to their values at $R_i = 0$ and $\tau = 0$ (so that the exponential term remains close to 1). In this case, an essential contribution to the spectrum is provided only by the first terms in the expansion of Lagrangian correlation functions $C_{ij}(\mathbf{R}, \tau, z)$ in a series with respect to small values of R_i and τ :

$$\begin{aligned} C_{ij}(\mathbf{R}, \tau) &\approx C_{ij}(\mathbf{R}, 0) + \left(\frac{\partial^2 C_{ij}}{\partial \tau^2} \right)_{\tau=0} \frac{\tau^2}{2}, \\ C_{ij}(\mathbf{R}, 0) &\approx C_{ij}(0, 0) + \sum_{\gamma, \delta} C_{ij\gamma\delta} R_\gamma R_\delta / 2, \end{aligned} \quad (31)$$

where $C_{ij\gamma\delta} \equiv [\partial^2 C_{ij}(\mathbf{R}, 0) / \partial R_\gamma \partial R_\delta]_{R=0, \tau=0}$ [hereafter, we omit a slow dependence $C_{ij}(\mathbf{R}, \tau, z)$ on z]. Because of the assumed axial symmetry of the Lagrangian spectra $\Phi(k_\perp, m)$ and $L(k_\perp, m)$ connected with correlation

functions (26)–(28), there are only a few nonzero coefficients $C_{ij\gamma\delta}$:

$$\begin{aligned} C_{1212} &= C_{1221} = C_{2112} = C_{2121} = C_{1122} = C_{2211}, \\ C_{1313} &= C_{3113} = C_{3131} = C_{2323} = C_{2332} = C_{3232}, \\ C_{3311} &= C_{3322}, \quad C_{1133} = C_{2233}, \quad C_{3333}. \end{aligned} \quad (32)$$

They can be expressed through the following variances of the vertical and horizontal derivatives of the displacement components designated as a_0 , e_0 , and c_0 :

$$\begin{aligned} C_{1122} &= - \left[\left\langle \left(\frac{\partial S_1^w}{\partial r_2} \right)^2 \right\rangle + \left\langle \left(\frac{\partial S_1^v}{\partial r_2} \right)^2 \right\rangle \right] \equiv -2a_0, \\ C_{3311} &= - \left\langle \left(\frac{\partial S_3}{\partial r_1} \right)^2 \right\rangle \equiv -8e_0, \\ C_{1133} &= - \left[\left\langle \left(\frac{\partial S_1^w}{\partial r_3} \right)^2 \right\rangle + \left\langle \left(\frac{\partial S_1^v}{\partial r_3} \right)^2 \right\rangle \right] \equiv -8c_0, \end{aligned} \quad (33)$$

where the displacement components are defined in (7)–(9). The magnitudes of the coefficients (33) characterize the rate of nonlinear diffusion of the 3D spectral density $S_{3E}(k_\perp, k_3)$ with increasing wavenumbers k_3 and $k_\perp$ and the anisotropy of such diffusion in wavenumber space caused by the stable stratification of the medium. They can be estimated using expressions for the correlation functions (23) and (26)–(28):

$$-C_{1122} \sim k_0^2 \nu_1^2, \quad -C_{3311} \sim k_0^2 \nu_v^2, \quad -C_{1133} \sim m_0^2 \nu_1^2, \quad (34)$$

where k_0 and m_0 are the characteristic horizontal and vertical wavenumbers at which the energies of internal waves and horizontal vortices are pumped from their random sources. According to (29), the contributions to the variances of the horizontal and vertical gradients of horizontal displacements, $-C_{1122}$ and $-C_{1133}$, give both internal waves and horizontal vortices, and therefore, both affect the rate of nonlinear spreading of the spectral density in the wavenumber space.

In the Boussinesq approximation, $\nabla \mathbf{S} = 0$, from which it follows that the ratio $\chi^2 = \nu_1^2/\nu_v^2 \approx m_0^2/k_0^2$ is proportional to the square of the ratio of the characteristic horizontal scale of the Lagrangian wave field to its vertical scale (i.e., characterizes the anisotropy of the wave field in the Lagrangian system). Since in stably stratified fluid, the buoyancy forces limit the vertical displacements of the particles, then $\chi^2 \gg 1$. Thus, all the coefficients in (33) and (34) can be expressed only in terms of the parameter $M \equiv \nu_v m_0 \sim k_0 \nu_1 \ll 1$ characterizing the degree of nonlinearity of the displacement field and its anisotropy parameter χ in the Lagrangian system:

$$-C_{1122} \sim M^2, \quad -C_{3311} \sim M^2/\chi^2, \quad -C_{1133} \sim M^2\chi^2. \quad (35)$$

The parameter M being the ratio of the rms vertical (horizontal) displacements to the vertical scale m_0^{-1} (horizontal scale k_0^{-1}) can be expressed in terms of rms velocity pulsations σ if we take into account that $\sigma^2 = \sigma_v^2 + \sigma_w^2$, $\sigma_v^2 = \langle (\partial S_1^v/\partial t)^2 \rangle + \langle (\partial S_2^v/\partial t)^2 \rangle$, and $\sigma_w^2 = \langle (\partial S_1^w/\partial t)^2 \rangle + \langle (\partial S_2^w/\partial t)^2 \rangle + \langle (\partial S_3^w/\partial t)^2 \rangle = N^2 \nu_v^2$. If we assume that the contributions to these pulsations from the wave and vortical modes are quantities of the same order, then $\sigma \sim N \nu_v$; therefore, $M \sim \sigma/[N m_0^{-1}]$. It can also be interpreted as the Froude number.

At small distances between two particles of the medium, $m_0 R_3 \ll 1$ and $k_0 R_{1,2} \ll 1$, the spatial structure function $2D_{ij}(\mathbf{R}, \tau=0) = 2[C_{ij}(0, 0) - C_{ij}(\mathbf{R}, 0)] \approx -\sum_{\gamma\delta} C_{ij\gamma\delta} R_\delta R_\gamma$ depends quadratically on the distances R_j . The quadratic form in the exponential factor in (15) and (16) can be reduced to diagonal form by introducing new variables w_1 , w_2 , and w_3 via linear transformations $R_1 = (k_1 w_1 - k_2 w_2) k_\perp^{-1}$, $R_2 = (k_1 w_2 + k_2 w_1) k_\perp^{-1}$, $R_3 = w_3 + 8a_0 k_1 k_3 w_1 C^{-1}$:

$$\sum_{ij} D_{ij}(\mathbf{R}, 0) k_i k_j = A w_1^2 + B w_2^2 + C w_3^2, \quad (36)$$

where

$$\begin{aligned} C &= 4c_0 k_\perp^2 + 8a_0 k_3^2, \quad A = 3a_0 k_\perp^2 + 4e_0 k_3^2 - 64a_0^2 k_\perp^2 k_3^2 / C, \\ B &= a_0 k_\perp^2 + 4e_0 k_3^2. \end{aligned} \quad (37)$$

The rate of decrease of the correlation function $M_F(\mathbf{R}, \tau=0, \mathbf{k}, z)$ with increasing R_i determines the rate of decrease of the Eulerian 3D spectrum (15) with increasing wavenumbers k_i . The characteristic horizontal correlation radius of $M_F(\mathbf{R}, \tau=0, \mathbf{k}, z)$ is equal to $w_\perp \sim A^{-1/2} \sim B^{-1/2}$, and vertical correlation radius $w_v \sim C^{-1/2}$; therefore, the ratio $w_v/w_\perp \sim A^{1/2}/C^{1/2} \sim \chi^{-1} \ll 1$. The scales $w_\perp$ and w_v decrease rapidly with increasing wavenumbers k_i . Taking into account the expansion of the Lagrangian correlation functions to the terms of order R_i^2 and the quadratic form (36), the integration in (18) gives the following asymptotic expression for the 3D spectrum [Chunchuzov 2002, his (55a)] in the case of high vertical wavenumbers $2k_z^2 \nu_v^2 \gg 1$ and low horizontal wavenumbers $\chi^2 k_\perp^2/k_z^2 \ll 1$:

$$S_{3E}(k_\perp, k_z) = A_0 |k_z|^{-5} \exp \left(-\frac{k_\perp^2}{4e_0 k_z^2} \right) \left[1 + O \left(\frac{\chi^2 k_\perp^2}{k_z^2} \right) \right], \quad (38)$$

where

$$A_0 = \frac{\beta}{8\pi e_0}, \quad \beta = (2\pi)^{-1/2} 2^{-13} a_0^{-5/2} \exp\left(-\frac{1}{32a_0}\right) \quad (39)$$

is a nondimensional coefficient depending on the parameter of nonlinearity M and anisotropy χ of the Lagrangian displacement field. The Eulerian spectrum (38) is localized in the region of high vertical wavenumbers $|k_z| \gg m^*$ and low horizontal wavenumbers $0 < k_\perp \leq w_\perp^{-1} \sim 2\sqrt{e_0}k_z \sim \nu_v k_0 k_z$, where $m^* = 1/(\nu_v \sqrt{2})$ is the characteristic vertical wavenumber. With an increase in the vertical wavenumber k_z and the rms value of the horizontal gradient of the vertical displacements $2\sqrt{e_0}$, where $e_0 \sim M^2/\chi^2 \ll 1$, the nonlinear diffusion of the spectrum (29) leads to an expansion of its localization region $0 < k_\perp \leq w_\perp^{-1}$.

The surfaces of the constant amplitude of the spectrum are strongly elongated along the vertical axis k_z (Fig. 1), indicating a strong anisotropy of this spectrum. On each of these surfaces, the ratio of the components of the wavenumber vector, whose end lies on the surface, remains the same: $k_\perp/k_z = 2\sqrt{e_0} \ll 1$, so these surfaces are of similar shape. A family of wavenumber regions bounded by the surfaces of a constant spectrum corresponds in real space to a family of layered inhomogeneities in the displacement field having vertical scales $l_z = 2\pi/k_z < L^*$, where $L^* = 2\pi/m^* = 2^{3/2}\pi\nu_v$ is the characteristic (outer) vertical scale of the inhomogeneities with the following ratio of the horizontal scale $l_\perp$ to the vertical one:

$$l_\perp/l_z = 1/(2\sqrt{e_0}) \sim \chi/M \gg 1. \quad (40)$$

The amplitude coefficient A_0 grows from zero along with the mean square of the vertical gradient $\langle (\partial S_3/\partial r_3)^2 \rangle = -C_{3333}$ (in Fig. 2, this dependence is shown for the case $\chi = 10$), that is, with an increase in the nonlinearity parameter M . However, beginning from $M = 0.22$, such growth slows down sharply so that a maximum of A_0 is reached when $M \sim 0.3$. Thus, in spite of the growth with height of the variance of vertical displacements ν_v^2 due to the decrease in density $\rho_s(z)$, the growth of the amplitude of the spectral tail (38) is limited when $|C_{3333}|$ reaches the values of 0.17–0.18. Such a nonlinear saturation of the amplitude of the spectral tail is caused by the growing nonlinear diffusion of the spectral density in the 3D wavenumber space with the growing values of $|C_{3333}|$ and k_z . The growth of $|C_{3333}|$ is limited itself to a value on the order of 1 at which convective instability arises; therefore, M^2 does not exceed 0.5. For $q^2 \equiv 2(k_\perp^2 \nu_1^2 + k_z^2 \nu_v^2) \gg 1$ (i.e., outside the characteristic ellipsoid), the 3D spectrum (38) decreases approximately as k^{-5} with increasing wavenumber k_z .

FIG. 1. Contours of constant amplitude of the 3D wavenumber spectrum (shown in the vertical plane $k_y = 0$) for the vertical displacements induced by IGWs in the case $\chi = 10$, $M = 4 \times 10^{-1}$. The contours are shown only in the high-wavenumber region: $2(k_\perp^2 \nu_1^2 + k_z^2 \nu_v^2) \gg 1$, where k_z and $k_\perp$ are vertical and horizontal wavenumbers, respectively; ν_v and ν_1 are the rms values of the vertical and horizontal displacements, respectively. The characteristic ellipsoid $q^2 \equiv 2(k_\perp^2 \nu_1^2 + k_z^2 \nu_v^2) = 1$ and vertical wavenumber m^* are also shown.

In a similar way, using relation (16) and the expansion of the correlation function (41) over small R_i , a 3D Eulerian spectrum $V_{3E}(k_\perp, k_z)$ can be obtained for the horizontal velocity $v = \sqrt{v_x^2 + v_y^2}$ (Chunchuzov 2002):

$$V_{3E}(k_\perp, k_z) = B_0 N^2 |k_z|^{-5} \exp\left(-\frac{k_\perp^2}{4e_0 k_z^2}\right) \left[1 + O(M^2) + O\left(\frac{\chi^2 k_\perp^2}{k_z^2}\right)\right], \quad |k_z| \gg m^*, \quad (41)$$

$$B_0 = \alpha/(8\pi e_0), \quad \alpha = (2\pi)^{-1/2} 2^{-8} a_0^{-3/2} \exp\left(-\frac{1}{32a_0}\right). \quad (42)$$

The important result that follows from the obtained asymptotic spectra (38) and (39) and (41) and (42) at high vertical wavenumbers is that they do not depend on the specific forms of the Lagrangian spectra $\Phi(k_\perp, m)$ and $L(k_\perp, m)$ but depend only on the average spatial gradients of the Lagrangian displacements (35) caused by the random ensemble of waves and horizontal vortices. These gradients characterize the degree of nonlinearity and anisotropy of the displacement field. To calculate them, the Lagrangian spectrum $L(k_\perp, m)$ was chosen by Chunchuzov [2002, his (26)] in the specific form $L(k_\perp, m) = \text{const } k_\perp m \exp[-m^2/(2m_0^2) - k_\perp^2/(2k_0^2)]$.

FIG. 2. (top) The amplitude A_0 of the 3D spectrum as a function of the variance of the vertical gradient of the vertical displacements $-C_{3333}$. (bottom) The coefficients α and β for the vertical wavenumber spectra of the wave-induced horizontal wind velocity fluctuations and vertical displacements as functions of the nonlinear parameter M .

c. *Space–time (four-dimensional) spectra of vertical displacements and horizontal velocities at high wavenumbers ($q^2 \gg 1$)*

For $q^2 \gg 1$, the space–time spectral density $S_{4E}(\mathbf{k}, \omega)$ is obtained by integrating the correlation function (15) in (17), taking into account the expansions (31) and (36) for the correlation functions $C_{ij}(\mathbf{R}, \tau, z)$:

$$S_{4E}(\mathbf{k}, \omega) = S_{3E}(\mathbf{k}) [\pi / (\eta^2 k_{\perp}^2 + \eta_V^2 k_3^2)]^{1/2} \times \exp\{-\omega^2 / [4(\eta^2 k_{\perp}^2 + \eta_V^2 k_3^2)]\}, \quad (43)$$

where $\eta^2 = -2^{-1}(\partial^2 C_{11} / \partial \tau^2)_{\tau=0, R=0} = -2^{-1}(\partial^2 C_{22} / \partial \tau^2)_{\tau=0, R=0}$ and $\eta_V^2 = -2^{-1}(\partial^2 C_{33} / \partial \tau^2)_{\tau=0, R=0}$ are variances of the fluctuations of horizontal and vertical velocity components (up to a factor of $1/2$), respectively, and $S_{3E}(\mathbf{k})$ is the 3D spectrum (38).

It is seen from (43) that nonlinear diffusion of the spectrum $S_{4E}(\mathbf{k}, \omega)$ in the frequency–wavenumber space leads to its localization in the low-frequency region, for which $\omega^2 < [4(\eta^2 k_{\perp}^2 + \eta_V^2 k_3^2)]$, including the value $\omega = 0$,

at which the function $S_{4E}(\mathbf{k}, \omega)$ reaches its maximum. With increasing wavenumbers, this region expands proportionally to the growing variance of the Doppler frequency fluctuations in the Eulerian system, $2(\eta_{\perp}^2 k_{\perp}^2 + \eta_V^2 k_3^2)^{1/2}$, whereas the amplitude of the spectrum $S_{4E}(\mathbf{k}, \omega)$, on the contrary, decreases inversely proportional to this variance.

An expression similar to (43) may be obtained for the space–time spectrum of the horizontal velocity $V_{4E}(\mathbf{k}, \omega)$ by replacing the 3D spectrum $S_{3E}(\mathbf{k})$ by the 3D velocity spectrum (41). Thus, the finescale layered structure formed in the displacement and velocity fields has a zero frequency spectral component, that is, constant in time potential energy component $N^2 S_{4E}(\mathbf{k}, 0)/2$ and kinetic energy component $V_{4E}(\mathbf{k}, 0)/2$.

4. One-dimensional spectra of vertical displacements and horizontal velocities

a. Vertical wavenumber spectra at high wavenumbers

The integration of the 3D spectra (38) and (41) over horizontal wavenumbers k_x and k_y reduces to the integration over the radial wavenumbers $k_{\perp}$ from 0 to ∞ , taking into account the axial isotropy of the wave fields. It leads to the following one-sided vertical wavenumber spectra of the vertical displacements,

$$S_{1E}(k) = \beta k_z^{-3}, \quad m^* \ll k_z < m_c, \quad (44)$$

and horizontal velocity (Chunchuzov 2001, 2002):

$$V_E(k_z) = \alpha N^2 / k_z^3, \quad m^* \ll k_z < m_c, \quad (45)$$

where the dimensionless coefficients β and α are defined by (39) and (42), respectively. The spectra (44) and (45) were also obtained by Hines (2001) using a Lagrangian approach; however, instead of the statistical averaging over the realizations used here, he used the averaging over a spatial volume within which the medium can be assumed to be statistically homogeneous.

As can be seen from (44) and (45), the k_z^{-3} spectral tail is bounded from below by a characteristic vertical wavenumber m^* . Taking into account that $\sigma^2 \sim N^2 \nu_v^2$, we can write

$$m^* = \frac{1}{2^{1/2} \nu_v} \sim \frac{N}{\sigma}. \quad (46)$$

The k_z^{-3} spectral tail is bounded by the upper critical wavenumber

$$m_c = m^* \exp\left(\frac{1}{\beta}\right), \quad (47)$$

at which wave breaking processes caused by convective or shear instabilities switch on in some local regions of space (Hines 1991a,b). It is possible to estimate a maximal vertical size of the anisotropic vortices, $l_c = 2\pi/m_c$, generated as a result of the wave instabilities. To do this, we estimate first the internal wave energy generation rate $\varepsilon_g \sim \sigma^2 \omega_0$, where ω_0 is the characteristic frequency of the internal waves generated by their sources. According to the dispersion relation, $\omega_0 \approx Nk_0/m_0 = N/\chi$; therefore, $\sigma = [\chi(\varepsilon_g/N)]^{1/2}$, which allows us to express from (46) and (47) the wave-breaking-scale $l_c = 2\pi/m_c$:

$$l_c = 2\pi\mu \left(\frac{\varepsilon_g}{N^3} \right)^{1/2}, \quad (48)$$

where $\mu = (2\chi)^{1/2} e^{-1/\beta}$ is the nondimensional coefficient depending on M and χ . For the values $\chi = (10\text{--}100)$ and $\beta \sim 0.2$, the coefficient μ takes the values 0.03–0.1. Since the dissipation rate of the kinetic energy of the generated vortices ε is proportional to ε_g , the wave-breaking-scale (48) is proportional to the Ozmidov scale $l_o = (\varepsilon/N^3)^{1/2}$, which characterizes the maximum possible size of isotropic vortices in a stably stratified medium (Ozmidov 1965). When $\chi \sim 100$, the vertical-scale l_c is on the order of l_o .

The amplitude of the spectral tail (44) reaches a maximum value of 0.22 for $M = m_0/(m^*\sqrt{2}) \approx 0.4$; it does not change with increasing altitude z if the vertical wavenumber $m_0(z)$ decreases inversely proportional to the rms fluctuations of the velocity or displacements: $m_0(z) \approx 0.4/\nu_v(z) \sim 0.4N/\sigma(z)$. The generation of the spectral components with high k_z is equivalent to a nonsinusoidal distortion of the vertical oscillations of the displacement and velocity fields upon transition from the Lagrangian to the Eulerian coordinate system and an increase in the steepness of the crests and troughs of these oscillations (Eckermann 1999; Chunchuzov 2009). To limit the infinite growth of the steepness of the oscillations, we need to take into account the wave dissipation effects caused by turbulent viscosity when approaching the wave-breaking-scale l_c (Sukoriansky et al. 2005; Sukoriansky and Galperin 2013; Sun et al. 2015). It should lead to the smoothing of the spatial gradients of the displacement and velocity fields.

For the lower stratosphere, the typical rms values of horizontal wind velocity fluctuations σ are 2–4 m s^{−1} (Vincent et al. 1997). Taking $\beta \approx 0.2$ and $N = 0.02 \text{ rad m}^{-1}$, it is possible to estimate the maximal (outer) and minimal vertical sales that bound the k_z^{-3} spectral tail (44) and (45): $L^* = 2\pi/m^* = 1.6\text{--}3.0 \text{ km}$, $l_c = 2\pi/m_c = 10\text{--}16 \text{ m}$.

We note that under nonlinear saturation of the spectrum, $M \sim 0.32\text{--}0.40$, the anisotropy of the layered

inhomogeneities with small vertical scales, $l_z \ll L^*$, determined by (40), exceeds M^{-1} times the ratio of the scales $\chi \sim m_0/k_0$ at which the wave energy is pumped from the wave sources. This means that the nonlinear transfer of wave energy from the sources, concentrating within the characteristic ellipsoid ($q^2 < 1$), occurs toward smaller vertical scales and, at the same time, longer horizontal scales than those inherent in the sources themselves.

It is possible to estimate the maximum anisotropy of the inhomogeneities (40) using the dispersion relation $\omega_0 = N/\chi$. Taking into account that $\omega_0 \geq f$, where f is the Coriolis parameter, we obtain from (40)

$$l_1/l_z = \chi/M \sim N/(\omega_0 M) \leq N/(fM). \quad (49)$$

For $N = 0.02 \text{ rad m}^{-1}$, $f \sim 10^{-4} \text{ rad m}^{-1}$, $M = 0.4$, we obtain from (49) $l_1/l_z \leq 500$. The obtained theoretical vertical wavenumber spectra are in good agreement with the observed spectra in the stratosphere (Vincent et al. 1997; Fritts and Alexander 2003; Tsuda 2014), including those obtained from atmospheric sounding data using infrasound from volcanoes and ground explosions [see Fig. 3, reproduced from Chunchuzov et al. (2015)].

The 1D frequency spectrum of the velocity fluctuations predicted by the model is given by $\alpha N^2 \langle w^2 \rangle \omega^{-3}$ at frequencies $\omega > N$, where $\langle w^2 \rangle$ is the variance of the vertical velocity fluctuations. Such a spectrum was obtained by Chunchuzov et al. [2005, their (10)] by integrating a four-dimensional (4D) velocity spectrum over wavenumbers. The ω^{-3} tail at high ω is of the same nonlinear nature as the vertical wavenumber spectrum [(45)], from which it can be obtained by replacement $\omega = k_z(\langle w^2 \rangle)^{1/2}$ meaning rms Doppler shifting due to vertical velocity fluctuations.

As shown by Chunchuzov (2002, 1771–1772), the contribution to the obtained k_z^{-3} spectrum from the nonlinear terms of the Jacobian J [given by (4)], which take into account effects of fluid compressibility, does not change the k_z^{-3} form of the spectrum but gives only a small (of order 10%) correction to the amplitude of the spectrum obtained for $J = 1$. However, the numeric simulation by Klaassen and Sonmor (2006) and Klaassen (2009a,b) of formation of the internal wave spectrum with the kinematic model comprising a number of specified wave modes showed that, for the typical wave amplitudes required to produce the k_z^{-3} spectral tail, the vertical profiles of the Eulerian displacement wave field become multivalued, and the transformation from Lagrangian to Eulerian variables becomes invalid. In this case, the extreme values of J may reach zero values that mean unphysical fluid parcel volume changes.

FIG. 3. (top) Vertical wavenumber spectra of the horizontal wind velocity fluctuations retrieved from the stratospheric returns of the infrasound generated by different volcanoes (Tungurahua and Karymsky) and surface explosion [reproduced from [Chunchuzov et al. \(2015\)](#)]. The horizontal axis is a cyclic vertical wavenumber $k_z/(2\pi)$. The dashed line corresponds to the theoretical spectrum [(45)] with a -3 power-law decay. The vertical bars indicate the 95% confidence intervals for the spectral estimates, and arrows show the vertical wavenumbers m^* and m_c that bound the k_z^{-3} spectrum. (bottom) The multivalued (arrows) arising in the instant vertical profile of the vertical displacements with $\nu_v = 118$ m ($M \sim 0.5$) simulated by [Chunchuzov \(2009\)](#).

The model by [Chunchuzov \(2009\)](#) with only three interacting discrete wave modes also showed the appearance of multivalued and discontinuities in the Eulerian field at some finite wave amplitudes (see the bottom of [Fig. 3](#)). Such unphysical multivalued and discontinuities in a highly nonlinear wave field arise only in the ideal (nonviscous) fluid and within the narrow layers, whose thickness is small compared to the vertical scales, $2\pi/k_z$, typical for the k_z^{-3} tail.

Another well-known example of the wave solutions in the ideal fluid is a plane sinusoidal acoustic wave of finite amplitude, whose wave profile distorts and undergoes nonsinusoidal steepening with increasing distance from a

plane source until it becomes multivalued within a narrow wave front ([Lighthill 1978](#); [Rudenko and Soluyan 1977](#)). By taking into account a molecular viscosity and thermal conductivity within a narrow shock front, one can stabilize the nonlinear steepening of the wave profile and prevent the arising of multivalued.

In case of internal waves propagating through realistic atmosphere, the different types of wave-induced instabilities may prevent the arising of discontinuities in the wave field profile through wave-breaking processes and transferring wave energy into the energy of turbulent eddies. [Klaassen \(2009b, p. 1123\)](#) suggested that “through instability atmospheric wave fields create, and possibly

coexist with, a field of smaller-scale (perhaps turbulent) eddies, which act to exert a force on the mean background flow. In other words, the secondary field of eddies acts as a dissipation mechanism for the larger-scale internal waves, producing momentum deposition and perhaps saturation as the latter continue to propagate.” As shown by [Chunchuzov \(2009\)](#), the smaller-scale eddies are generated within local regions of space whose vertical sizes are smaller than the vertical scales of highly nonlinear wave perturbations shaping the k_z^{-3} tail. Such local regions of space play the role of the sinks of wave energy that balance the nonlinear wave energy transfer from the large-scale sources through the entire k_z^{-3} spectral tail.

In the statistical model developed here for the case of a high number of randomly independent amplitudes of modes, the multivaluedness and unphysical volume changes are reached when the rms value $\Lambda = [(\langle J - 1 \rangle)^2]^{1/2}$ becomes close to 1. As shown by [Chunchuzov \(2002, p. 1771\)](#), $\Lambda = [(\langle J - 1 \rangle)^2]^{1/2} = 3.8M^2$ for the specific form of the Lagrangian spectrum, where $M = m_0\nu_v$ is the parameter of nonlinearity. Thus, Λ becomes close to 1 when M becomes close to 0.5. It is important to note that the nonlinear saturation of the k_z^{-3} spectrum is reached at lower values: $M = 0.32\text{--}0.40$.

b. Horizontal wavenumber spectra in the region of large wavenumbers

We will obtain now a 1D horizontal wavenumber spectrum for the horizontal velocity $V_{E,\perp}(k_x)$:

$$V_{E,\perp}(k_x) = \iint dk_z dk_y V_{3E}(k_\perp, k_z) = \int_{-\infty}^{\infty} dk_y [V_{2E}^{(1)}(k_\perp) + V_{2E}^{(2)}(k_\perp)], \quad (50)$$

where $V_{2E}^{(1)}(k_\perp) = 2 \int_0^{m^*} dk_z V_{3E}(k_\perp, k_z)$ and $V_{2E}^{(2)}(k_\perp) = 2 \int_{m^*}^{\infty} dk_z V_{3E}(k_\perp, k_z)$ are the contributions of the 3D spectrum $V_{3E}(k_\perp, k_z)$ to the spectrum (50) coming from the regions $0 \leq k_z \leq m^*$ and $m^* < k_z \leq \infty$, respectively. When $k_z^2 \gg m^{*2}$, the spectrum $V_{3E}(k_\perp, k_z)$ takes a universal asymptotic form (41), which we extend to the whole region $m^* \leq k_z \leq \infty$, then

$$V_{2E}^{(2)}(k_\perp) \approx 2B_0 N^2 \int_{m^*}^{\infty} dk_z k_z^{-5} \exp\left(-\frac{k_\perp^2}{4e_0 k_z^2}\right). \quad (51)$$

Let us introduce new variables: $p = m^*/k_z$ and $K \equiv k_\perp^2/(4e_0 m^{*2})$, then after integration in (51), we obtain

$$V_{2E}^{(2)}(k_\perp) = \frac{B_0 N_1^2}{m^{*4}} \frac{F(K)}{K^2} = \frac{2\alpha_0 e_0 N_1^2}{\pi} \frac{F(K)}{k_\perp^4}, \quad (52)$$

$$F(K) = 1 - (K + 1) \exp(-K).$$

Taking into account (52), the corresponding contribution to the 1D spectrum (50) takes the form

$$V_{E,\perp}^{(2)}(k_x) = \int_{-\infty}^{\infty} dk_y V_{2E}^{(2)} = \frac{2\alpha_0 e_0 N_1^2}{\pi} \int_{-\infty}^{\infty} dk_y \frac{F(K)}{k_\perp^4} = \frac{4\alpha_0 e_0 N_1^2}{\pi} k^{*-3} \int_0^{\infty} dy' \frac{F(q'^2 + y'^2)}{(q'^2 + y'^2)^2}, \quad (53)$$

where $k^* = 2\sqrt{e_0}m^*$, $q' = k_x/k^*$, $y' = k_y/k^*$. At large horizontal wavenumbers, such that $q'^2 \gg 1$, the exponential term in (52) becomes negligibly small in comparison with 1, so we can put $F \approx 1$ in (53), which after integration gives

$$V_{E,\perp}^{(2)}(k_x) \approx 2e_0 \alpha_0 N_1^2 |k_x|^{-3}, \quad |k_x| \gg k^*. \quad (54)$$

From (54), we also obtain a one-sided horizontal spectrum $\tilde{V}_{E,\perp}^{(2)}(k_x) = 2V_{E,\perp}^{(2)}(k_x)$ for $k_x \geq 0$:

$$\tilde{V}_{E,\perp}^{(2)}(k_x) = 4e_0 \alpha_0 N_1^2 k_x^{-3}. \quad (55)$$

Thus, the layered inhomogeneities with vertical scales $l_z < 2\pi/m^*$ and horizontal scales $l_x < 2\pi/k^*$, where m^* is determined by (46) and $k^* = \sqrt{4e_0}m^*$, have vertical and horizontal wavenumber spectra with a -3 power-law decay. Moreover, for the vertical and horizontal wavenumbers connected by relation $k_x = \sqrt{4e_0}k_z$, the horizontal spectral density (55) in the interval $(k_x, k_x + dk_x)$ is equal to the vertical wavenumber spectral density (45) contained in the interval $(k_z, k_z + dk_z)$. In other words, in the interval of vertical scales $2\pi/m_c < l_z < 2\pi/m^*$, there are similar layered inhomogeneities with the same scale ratio $l_x/l_z = 1/\sqrt{4e_0}$. Their maximum (or external) vertical scale is $L^* = 2\pi/m^*$, and the corresponding horizontal scale is $L_\perp = 2\pi/k^*$.

As seen from (55), the horizontal wavenumber spectrum depends on the anisotropy of the inhomogeneities $(4e_0)^{-1/2}$, while the vertical wavenumber spectrum (45) does not depend on it. The tendency of the nonlinear transfer of energy of internal waves and horizontal vortical motions from their sources to the velocity fluctuations with small vertical scales ($l_z < 2\pi/m^*$) and large horizontal scales leads to the localization of the horizontal wavenumber spectrum at large horizontal scales $l_x > 2\pi/k^*$. If $\sigma = 4 \text{ ms}^{-1}$, $N = 0.02 \text{ rad m}^{-1}$, $\chi = 10$, and $M = 0.4$, then $e_0 \approx M^2/(8\chi^2) = 0.0002$, $\alpha = 0.12$, $m^* \sim N/\sigma = 0.005 \text{ rad m}^{-1}$ (the corresponding vertical-scale $L^* = 2\pi/m^* = 1.26 \text{ km}$), and $k^* = 2\sqrt{e_0}m^* = 1.4 \times 10^{-4} \text{ rad m}^{-1}$ (the corresponding horizontal scale is $L_\perp = 2\pi/k^* \approx 44 \text{ km}$).

With the parameters chosen above, the result of calculating the contribution (53) from the inhomogeneities with small vertical scales, $m^* \leq k_z \leq \infty$, to the one-sided

FIG. 4. The horizontal wavenumber spectrum of the horizontal velocity fluctuations (thick solid line) representing the sum of contributions to this spectrum from layered inhomogeneities of the velocity field with small vertical scales (thin continuous) and wave perturbations with large vertical scales (dashed line indicated as $-5/3$) determined by (53) and (54) and (65) and (66), respectively. The label k_1 is the horizontal wavenumber above which the influence of Earth's rotation on internal waves can be neglected, and k^* is the wavenumber above which the wave-induced advection of waves becomes significant, leading to a -3 power-law decay of the spectrum.

horizontal wavenumber spectrum $\tilde{V}_{E,\perp}^{(2)}(k_x) = 2V_{E,\perp}^{(2)}(k_x)$ for $k_x \geq 0$ is shown in Fig. 4 (thin solid line). We should also add a contribution $V_E^{(1)}(k_x)$ to the spectrum (50) from the 3D perturbations of the velocity field with large vertical scales corresponding to the region $0 \leq k_z \leq m^*$. It is in this region that internal waves and horizontal vortices are supposed to be generated by their random sources. These include, in particular, the continuous adaptation of meteorological fields to the state of quasi-geostrophic and quasi-static equilibrium, accompanied by generation of both the 3D acoustic-gravity waves and 2D waves propagating horizontally under the influence of Earth's rotation and gravity (Obukhov 1988). The meteorological fronts are also the sources of IGWs with the horizontal wavelengths on the order of 100 km (Gossard and Hooke 1975; Chunchuzov et al. 2017). In mountainous regions, there arises an additional source caused by interaction of vortical motions with mountains that leads to the generation of nonstationary mountain waves (Chunchuzov 1994).

The large-scale IGWs and horizontal vortical motions with $q^2 \ll 1$ that lie inside the ellipsoid $q^2 = 1$ cause almost linear velocity perturbations; therefore, they interact weakly with each other. Their vertical and horizontal wavelengths significantly exceed the rms values of the corresponding vertical and horizontal displacements, v_v and v_1 . However, the multiple interactions of large-scale

internal waves and vortices generate their high-frequency harmonics with $q^2 > 1$ lying outside the ellipsoid $q^2 = 1$. The nonlinear diffusion of the spectral density in the wavenumber space caused by these interactions forms a universal 3D spectrum (41) of the velocity perturbations regardless of the shape of the spectrum of their random sources.

c. The model of 3D spectrum at small wavenumbers ($q^2 < 1$)

We assume that the sources of IGWs and vortical motions are localized at small wavenumbers, $q^2 \ll 1$, for which their interactions are weak. In this wavenumber region, the Eulerian and Lagrangian spectra of the velocity perturbations coincide.

As q^2 increases, the source spectrum of the perturbations is supposed to continuously transfer into its nonlinear spectral tail at $q^2 \gg 1$. To clarify the question about the spectrum of random sources, we have to take into account that the experimental horizontal spectra decrease as k_x^{-p} with increasing k_x in the range of horizontal scales from 500 to 2–10 km, where $p = 5/3$ according to Nastrom and Gage (1985) and $p = 2$ according to Bacmeister et al. (1996). The mechanism of formation of the $k_x^{-5/3}$ power-law decay in the mesoscale range is still debated (Callies et al. 2014; Skamarock et al. 2014; Lindborg 2006, 2007, 2015; Sun et al. 2016). Both the IGWs with long horizontal wavelengths and horizontal vortical (divergence free) motions contribute to this part of the spectrum.

Considering the observed $k_x^{-5/3}$ spectrum as a part of the source spectrum at long horizontal wavelengths ($q^2 \ll 1$), we choose a 3D Lagrangian spectrum $L(k_\perp, m)$ in some range of horizontal wavenumbers $k_1 \leq k_\perp \leq k_0$ in the following form:

$$L(k, m) = Cmk_\perp^{-8/3} \exp[-m^2/(2m_0^2) - k_\perp^2/(2k_0^2)], \quad (56)$$

where the ratio $y_1 = k_1/k_0 \ll 1$ and C is a coefficient of proportionality. Within such a wide wavenumber interval, $L \sim k_\perp^{-8/3}$ and the corresponding 1D spectrum along the x axis decreases as $k_x^{-5/3}$ with increasing k_x . A similar form is taken for the 3D streamfunction spectrum $\Phi(k_\perp, m)$.

For each of internal wave modes [(8)] with long horizontal wavelengths, for which $k_\perp^2 \ll m^2$, the influence of the Coriolis force can be neglected if the mode frequencies $\omega \approx Nk_\perp/m \gg f$, where f is the inertial frequency. The effect of rotation becomes significant when the frequency becomes close to f ; that is, for the horizontal wavenumber $k_\perp \approx m|f|/N$, which is the inverse Rossby deformation radius a^{-1} for the mode with vertical wavenumber m (Gill 1982, section 7),

$$a = c/|f| = N/(m|f|), \quad (57)$$

where $c = N/m$ is the phase velocity of the mode. For modes with the horizontal scales $\sim a$, the rotation effects associated with the action of the Coriolis force become as important as buoyancy effects, and at larger horizontal scales, they become dominant. Since the main energy injection into the system of waves and vortices occurs at $m \sim m_0$, the influence of Earth's rotation on their spectrum can be neglected for $k_\perp \gg k_1$, where

$$k_1 = m_0|f|/N. \quad (58)$$

$$\begin{aligned} I_1 &= 2\pi \int_{k_1}^{\infty} dk_\perp k_\perp^{-5/3} \exp\left(-\frac{k_\perp^2}{2k_0^2}\right) = 2\pi k_0^{-2/3} \int_{y_1}^{\infty} dy y^{-5/3} \exp\left(-\frac{y^2}{2}\right) \\ &\approx 2\pi k_0^{-2/3} \int_{y_1}^1 dy y^{-5/3} \approx 2\pi k_0^{-2/3} (3y_1^{-2/3}/2) = 3\pi k_1^{-2/3}. \end{aligned} \quad (59)$$

Taking (59) into account, we obtain the following relation for the ratio C_{3311}/C_{3333} :

$$e_0/(2a_0) = k_0^2 y_1^{2/3} / (6m_0^2). \quad (60)$$

If we take $k_0 \approx m_0$, then the ratio $y_1 = k_1/k_0 \approx f/N = 0.005$. It is close to the observed ratio of the wavenumbers bounding the $k_x^{-5/3}$ spectrum (Nastrom and Gage 1985; Bacmeister et al. 1996). From (60), we obtain

$$e_0 \approx (a_0/3)(f/N_1)^{2/3}, \quad a_0 = M^2/8. \quad (61)$$

Substituting $\sigma = 3 \text{ ms}^{-1}$, $N = 0.02 \text{ rad s}^{-1}$, $f = 10^{-4} \text{ rad s}^{-1}$ in (46) and taking into account that, under saturation of the spectrum, the nonlinear parameter $M = m_0/(\sqrt{2}m^*) \approx 0.4$, we find that $m_0 = 0.0027 \text{ rad m}^{-1}$ (the corresponding vertical scale is 2.3 km), the outer vertical scale of the layered inhomogeneities $L^* = 2\pi/m^* = 1.33 \text{ km}$ and $k_1 = Mf/\sigma = 1.33 \times 10^{-5} \text{ rad m}^{-1}$, which corresponds to the horizontal-scale $L_1 = 2\pi/k_1 = 471.25 \text{ km}$. From (61), we obtain

$$a_0 = 0.02, \quad e_0 = 1.95 \times 10^{-4}. \quad (62)$$

We now proceed directly to calculate the contribution $V_{2E}^{(1)}(k_\perp)$ to the horizontal wavenumber spectrum (50) from the perturbations of the velocity field with large vertical scales lying in the region $0 \leq k_z \leq m^*$. If the vertical scales of the perturbations significantly exceed the outer vertical scale of the layered inhomogeneities, $k_z \ll m^*$, and the horizontal wavenumbers do not go beyond the ellipsoid

The choice of the Lagrangian spectrum in the form (56) allows us to estimate the variance of the horizontal gradient $|C_{3311}| = 8e_0$ that defines the amplitude of the 3D spectrum [(41)] at $q^2 \gg 1$ and its anisotropy $2\sqrt{e_0}$. We substitute (56) in the expression for correlation functions of displacements [(23)] and calculate the ratio C_{3311}/C_{3333} .

When calculating C_{3311} , the integration over horizontal wavenumbers gives $I = \pi \int_{k_1}^{\infty} dk_\perp k_\perp^{1/3} \exp[-k_\perp^2/(2k_0^2)] = \pi k_0^{4/3} \int_{y_1}^{\infty} dy y^{1/3} \exp(-y^2/2) \approx \pi k_0^{4/3}$, where it is taken into account that $y_1 = k_1/k_0 \ll 1$. We can also estimate the integral arising in calculating C_{3333} :

$q^2 = 1$, that is, $k_\perp^2 \leq (1/2 - k_z^2 \nu_v^2) \nu_1^{-2}$, then the Eulerian spectrum is very close to the Lagrangian spectrum [(56)]. With increasing q^2 , the advection leads to a nonlinear diffusion of the spectrum in wavenumber space.

When transiting from the Lagrangian to the Eulerian system, the nonlinear diffusion changes the vertical and horizontal scales of the spatial variation of the wave field. Denoting the Eulerian 3D spectrum $V_{3E}(k_\perp, k_z)$ in the region $0 \leq k_z < m^*$ through $V_{3E}^{(1)}(k_\perp, k_z)$, we present it in the following form:

$$\begin{aligned} V_{3E}^{(1)}(k_\perp, k_z) &= C(z) N^2 \frac{\rho_s(0)}{\rho_s(z)} |k_z| \exp[-k_z^2/(2m_0^2)] k_\perp^{-8/3} \\ &\times \exp(-k_\perp^2/k^{*2}), \quad k_\perp \geq k_1, \end{aligned} \quad (63)$$

where $C(z)$ is unknown function. In (63), we took into account that for sufficiently large horizontal wavenumbers, satisfying the condition $k_\perp^2 > 1/(2\nu_1^2) \gg k^{*2}$, that is, lying outside the ellipsoid $q^2 > 1$, the horizontal scales of velocity perturbations become comparable to the rms horizontal displacements ν_1 , and this leads to a strong interaction of such perturbations.

If $k_\perp^2 \gg k^{*2}$, then a strong nonlinear diffusion of the spectral density in the wavenumber space leads to a change in the $k_\perp^{-8/3}$ power-law decay of the spectrum with increasing $k_\perp$ by a faster decay of its spectral tail. To find $C(z)$, we integrate (63) over $k_\perp$ using the condition of continuity at $k_z = m^*$ for the obtained vertical wavenumber spectrum as it passes into the spectral tail (45). Then we obtain $C(z) = [\rho_s(z)/\rho_s(0)][\alpha/(2I_1 m^{*4})] \exp[m^{*2}/(2m_0^2)]$,

whose substitution in (63) gives the 3D velocity spectrum in the region $0 \leq k_z < m^*$ and $k_\perp \geq k_1$:

$$V_{3E}^{(1)}(k_\perp, k_z) = \frac{\alpha N^2 k_1^{2/3}}{6\pi m^{*4}} |k_z| \exp[-(k_z^2 - m^{*2})/(2m_0^2)] k_\perp^{-8/3} \times \exp(-k_\perp^2/k^{*2}). \quad (64)$$

Substituting (64) into (50) and integrating over k_z , and then over k_y [see (A4) in the appendix, where $2k_0^2 \equiv k^{*2}$], we finally obtain the following contribution to the one-sided horizontal wavenumber spectrum $\tilde{V}_{E,\perp}(k_x)$ from the wave perturbations with large vertical scales:

$$\begin{aligned} \tilde{V}_{E,\perp}^{(1)}(k_x) &\approx \frac{2\alpha_0 N^2 m_0^2}{3\pi m^{*4}} \exp\left(\frac{m^{*2}}{2m_0^2}\right) k_1^{2/3} k_x^{-5/3} \exp(-k_x^2/k^{*2}) \\ &= C_1^{2/3} k_x^{-5/3} \exp(-k_x^2/k^{*2}), \end{aligned} \quad (65)$$

where

$$\begin{aligned} C_1 &= \gamma \varepsilon_g, \quad \varepsilon_g \equiv \sigma^2 f, \\ \gamma &= \left(\frac{4}{3\pi}\right)^{3/2} M \left[2\alpha_0 M^2 \exp\left(\frac{1}{4M^2}\right)\right]^{3/2}. \end{aligned} \quad (66)$$

As seen from (65) and (66), the large-scale part of the spectrum $\tilde{V}_{E,\perp}(k_x)$ depends on the rate of energy generation ε_g by the sources of internal waves and vortical motions, where ε_g is determined by the product of the rms velocity of the fluid parcel σ on the parcel's acceleration σf . The dimensionless coefficient γ depends only on the nonlinearity parameter under saturation, $M = 0.32$ – 0.40 , and lies in the range 0.008 – 0.012 . For the parameters chosen ($\sigma = 3 \text{ m s}^{-1}$, $f = 10^{-4} \text{ rad m}^{-1}$), we obtain $\varepsilon_g = 9 \times 10^{-4} \text{ m}^2 \text{ s}^{-3}$, $C_1 \approx 6.8 \times 10^{-6} \text{ m}^2 \text{ s}^{-3}$. The horizontal wavenumber spectrum $\tilde{V}_{E,\perp}(k_x)$ for the horizontal velocity fluctuations is shown in Fig. 4 (solid line). It represents the sum of the contributions from the layered inhomogeneities of the velocity field with small vertical scales (Fig. 4, thin line) and wave perturbations with large vertical scales (Fig. 4, dotted line) determined by the approximate expressions (53) and (54) and (65) and (66), respectively.

d. Comparison of the theoretical and observed horizontal wavenumber spectra

The aircraft measurements of the wind velocity fluctuations show that the variances of the fluctuations over a mountainous terrain is several times higher than over a flat terrain or ocean (Lilly and Lester 1974; Nastrom et al. 1987). The horizontal wavenumber spectra of the velocity fluctuations obtained in these studies are shown in Fig. 5

along with the spectra calculated from our model. Lilly and Lester (1974) analyzed in detail the horizontal wind velocity and temperature fields over the mountain ridge of southern Colorado. These fields were measured along 150–200-km tracks in the altitude range from 13 to 20 km with a vertical separation of 600 m. The most intense wave disturbances with characteristic horizontal scales of 20–30 km, alternating with the regions of small-scale turbulence, were detected in the lee zone of the ridge. Along with wave disturbances, the alternating layers with strong and weak winds and strong and weak vertical gradients of potential temperature were found.

At the same time, the variances of the horizontal wind velocity fluctuations were high enough, $\sigma^2 = 30$ – $68 \text{ m}^2 \text{ s}^{-2}$, whereas the variances of the vertical velocity fluctuations $\sigma_w^2 = 0.9$ – $7.8 \text{ m}^2 \text{ s}^{-2}$ (see Table 1 in Lilly and Lester 1974). The ratio $\chi \sim \sigma/\sigma_w$ was in the range from 4 to 20, and $N^2 = (5.1$ – $6.2) \times 10^{-4} \text{ rad}^2 \text{ s}^{-2}$.

We note that the values of σ were comparable in magnitude to the horizontal phase speeds of the observed waves, which should cause a strong influence of wave advection by inhomogeneous wind velocity field generated by the waves themselves. The spectra of the wind velocity components relative to the moving aircraft obtained by Lilly and Lester (1974) are reproduced in Fig. 5b. These spectra are related by an approximate relation $(2\pi/U)V_x(k_x)$ to the horizontal wavenumber spectra of the longitudinal velocity component $V_x(k_x)$, where $k_x = 2\pi f/U$. The observed spectra show on average a power-law decay $A_x N^2 k_x^{-3}$ with a decrease in the horizontal scales $2\pi/k_x$ from 20 to 3 km, where the numerical coefficient A_x is in the range $(10^{-4}$ – $10^{-3})$.

The model spectra $(2\pi/U)[\tilde{V}_\perp^{(1)}(k_x) + \tilde{V}_\perp^{(2)}(k_x)]/2$ are also shown in Fig. 5b, where $\tilde{V}_\perp^{(1)}(k_x)$ and $\tilde{V}_\perp^{(2)}(k_x)$ were calculated using formulas (53) and (54) and (65) and (66), respectively, for $\sigma = 7 \text{ m s}^{-1}$, $\chi = 5$, $N = 0.025 \text{ rad s}^{-1}$, $M = 0.32$. Then $m^* = N/\sigma = 0.0036 \text{ rad s}^{-1}$, $e_0 \sim a_0/\chi^2 = M^2/(8\chi^2) = 5.1 \times 10^{-4}$, $k^* = 2\sqrt{e_0}m^* = 1.6 \times 10^{-4} \text{ rad m}^{-1}$; hence, the anisotropy of the inhomogeneities $l_x/l_z = 1/\sqrt{4e_0} \sim 22$, and the amplitude of the spectrum (55) is $A_x \sim 4e_0\alpha \approx 0.00023$, that is, on the same order with the observed values of A_x .

Using expressions (65) and (66) for the $k_x^{-5/3}$ spectrum, we can also estimate energy generation rate and the amplitude of the spectrum: $\varepsilon_g = 0.0049 \text{ m}^2 \text{ s}^{-3}$, and $C_1 = 5.7 \times 10^{-5} \text{ m}^2 \text{ s}^{-3}$. The estimate of the dissipation rate of the kinetic energy of turbulence, $\varepsilon \sim 30 \text{ cm}^2/\text{c}^3$, obtained by Lilly and Lester (1974) from the experimental data, agrees in order of magnitude with our estimate of ε_g . It should be taken into account that part of the generated wave energy is used for maintaining the potential energy of the finescale structure of the atmosphere with vertical scales $l_z \ll 2\pi/m^*$. As can be seen from Fig. 5b, the

FIG. 5. (a) Average horizontal wavenumber spectra of the meridional wind velocity fluctuations obtained over 5 years with a large number of flights in the upper troposphere and lower stratosphere [reproduced from [Nastrom et al. \(1987\)](#)]. The tropospheric spectrum is shifted a decade downward with respect to the stratospheric spectrum. The model spectrum (dots) is shown for $\sigma = 3 \text{ m s}^{-1}$, $\chi = 10$, $N_1 = 2 \times 10^{-2} \text{ rad s}^{-1}$, $f = 10^{-4} \text{ rad s}^{-1}$, $M = 0.32$. (b) Frequency spectra of the longitudinal wind velocity component relative to the flight direction [reproduced from [Lilly and Lester \(1974\)](#)]. The horizontal axis shows linear frequency f (Hz) relative to the moving aircraft with the speed $U \sim 200 \text{ m s}^{-1}$, and the corresponding wavelength $\lambda = U/f$ (m). The vertical axis represents the spectral density of the horizontal velocity ($\text{m}^2 \text{ s}^{-2} \text{ Hz}^{-1}$). Theoretical spectrum (squares) is calculated for $\sigma = 8 \text{ m s}^{-1}$, $\chi = 5$, $N_1 = 0.025 \text{ rad s}^{-1}$, $f = 10^{-4} \text{ rad s}^{-1}$, $M = 0.32$.

calculated spectra are in a good agreement with the experimental spectra.

Figure 5a shows the average horizontal spectra of meridional wind velocity fluctuations obtained by [Nastrom et al. \(1987\)](#) in the upper troposphere and lower stratosphere (the tropospheric spectrum is shifted down on one decade with respect to the stratospheric spectrum). The flight routes were divided into 64- and 256-km-long segments, for which there was an increase in the variance of the velocity fluctuations σ^2 with increasing segment length, from about 1.5 to 3.5–10 $\text{m}^2 \text{ s}^{-2}$. The spectrum calculated for $\sigma = 3 \text{ m s}^{-1}$, $\chi = 10$, $f = 10^{-4} \text{ rad s}^{-1}$, $N = 2 \times 10^{-2} \text{ rad s}^{-1}$ is shown in Fig. 5a and, as seen, agrees well with the observed spectra.

5. Conclusions

In this paper, the analytic form for the 3D spatial spectrum of mesoscale fluctuations of the horizontal wind velocity and vertical displacements in the model of a stably stratified atmospheric layer with constant buoyancy frequency N and in the absence of mean wind shear was obtained. It was assumed that during some observational periods, the random displacements of fluid parcels in

Lagrangian frame caused by an ensemble of weakly interacting internal gravity modes and horizontal vortical motions may be considered as a stationary random and homogeneous Gaussian process within some local volume of the atmosphere. The random wind velocity and displacement spatial gradients were supposed to be induced by the waves and horizontal vortical motions themselves.

The nonlinear transfer to the Eulerian frame leads to the deviation of random displacements from the Gaussian distribution. In this frame, the 3D spectrum of the wind velocity fluctuations was obtained, using which the observed vertical wavenumber and horizontal wavenumber spectra of the fluctuations were explained. The mechanism proposed for the formation of such a spectrum is in the nonlinear transfer of energy, coming from the sources of internal waves and horizontal vortical motions at some vertical and horizontal scales, toward velocity fluctuations with smaller vertical scales and, at the same time, larger horizontal scales. It was shown that such energy transfer caused by strong wave-wave interactions, the interactions between waves and vertical vorticity modes, and between vertical vorticity modes leads to the formation of finescale layered inhomogeneities in the velocity and displacement fields.

It was also shown that the frequency–wavenumber (4D) spectrum of the layered inhomogeneities is localized at low frequencies, including zero frequency, high vertical wavenumbers, and low horizontal wavenumbers. Because of the strong advection of wave modes and vortical modes by the wind induced by all these modes, the obtained 3D and 4D spectra of the layered inhomogeneities take universal forms regardless of the forms of the Lagrangian spectra of the random sources of these modes.

The 3D spectrum was shown to depend only on the mean-square gradients of the displacement field, which determine the rate of nonlinear diffusion with increasing wavenumbers of the spectral density in the 3D wavenumber space and the anisotropy of such diffusion. The horizontal vortices contribute to the rms gradients of the displacement and velocity fields along with internal waves, thereby increasing the rate of nonlinear spreading of the 3D spectrum in comparison with the case when the displacements are caused by internal waves only.

The obtained 1D (vertical and horizontal) wavenumber spectra of the horizontal velocity and vertical displacements fall off in accordance with the -3 power law. The ranges of the vertical and horizontal wavenumbers were found for which the anisotropy of the inhomogeneities (the ratio of their horizontal scales to the vertical ones) remains constant. In the region of large scales, in comparison with the rms displacements of the air parcels, the 3D spectrum of wave perturbations was chosen so that the corresponding 1D horizontal wavenumber spectrum was close to the observed $k_x^{-5/3}$ spectrum, and the vertical wavenumber spectrum would continuously transform with decreasing vertical scales into the obtained universal k_z^{-3} spectral tail.

The horizontal wavenumber spectrum of the velocity fluctuations was shown to contain the contributions from the large-scale perturbations caused by internal waves and horizontal vortical motions and from fine-scale layered inhomogeneities. The horizontal scale is found at which there is a transition from the $k_x^{-5/3}$ power-law decay of the horizontal wavenumber spectrum to the steeper k_x^{-3} power law.

The theoretical spectra obtained were compared with the observed vertical and horizontal spectra of the mesoscale horizontal wind velocity fluctuations in the upper troposphere and stratosphere. It was shown that the main parameters on which these spectra depend are the rms fluctuations of the horizontal velocity during the observation time, their ratio to the rms fluctuations of the vertical velocity (i.e., the anisotropy parameter of the wave sources), the Brunt–Väisälä frequency, and the Coriolis parameter.

Acknowledgments. This work was supported by Grants RFBR 16-05-00438 and 15-05-3461 (sections 2 and 3) and Grant RSF 14-47-00049 (section 4). I thank G. Golitsyn for discussion of my work.

APPENDIX

Contribution to the 1D Spectrum from Large Vertical Scales

The contribution to the 1D horizontal wavenumber spectrum from the wave perturbations with large vertical scales ($|k_z| < m^*$) can be obtained by substituting (52) into (39). The integration over k_y reduces to the calculation of the integral:

$$\begin{aligned} R &= \int_{-\infty}^{\infty} dk_y (k_x^2 + k_y^2)^{-4/3} \exp[-(k_x^2 + k_y^2)/(2k_0^2)] \\ &= 2k_x^{-5/3} \exp[-k_x^2/(2k_0^2)] \int_0^{\infty} d\xi (1 + \xi^2)^{-4/3} \exp[-k_x^2 \xi^2/(2k_0^2)], \quad |k_x| \geq k_1, \end{aligned} \quad (\text{A1})$$

For $k_1 \leq |k_x| \leq \sqrt{2}k_0$, the main contribution to the integral in (A1) gives the integration over ξ from 0 to 2, for which the influence of the exponential function on the result of integration can be neglected; therefore, this integral can be replaced by its approximate expression:

$$\int_0^2 d\xi (1 + \xi^2)^{-4/3} = \int_0^{\infty} d\xi (1 + \xi^2)^{-4/3} - \int_2^{\infty} d\xi (1 + \xi^2)^{-4/3}. \quad (\text{A2})$$

The first integral on the right-hand side of (A2) after replacement $t = \xi^2$ is reduced to the tabular integral:

$$\frac{1}{2} \int_0^{\infty} dt t^{-1/2} (1 + t)^{-4/3} = \frac{1}{2} B\left(\frac{1}{2}, \frac{5}{6}\right) = \frac{1}{2} \frac{\Gamma(1/2)\Gamma(5/6)}{\Gamma(8/6)} \approx 1.1, \quad (\text{A3})$$

where $B(p, q) = \int_0^{\infty} dt t^{p-1} (1 + t)^{-(p+q)}$ is the Beta function. As for the last integral in (A2), it does not exceed the value $3 \times 5^{-1} \times 2^{-5/3} = 0.19$; therefore, expression

(A3) can be set equal to 1 to within 20%, and from (A1), we get

$$R \approx 2k_x^{-5/3} \exp[-k_x^2/(2k_0^2)]. \quad (\text{A4})$$

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