Weakly or Strongly Nonlinear Mesoscale Dynamics Close to the Tropopause?

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ABSTRACT

Recently, it has been discussed whether the mesoscale energy spectra in the upper troposphere and lower stratosphere are generated by weakly or strongly nonlinear dynamics. A necessary condition for weak nonlinearity is that the Rossby number $Ro = |\zeta|/f \ll 1$, where $\zeta$ is the vertical vorticity and $f$ is the Coriolis parameter. First, it is shown that $Ro$ can be estimated by integration of the rotational wavenumber energy spectrum $E_r$. Then divergence and rotational energy spectra and their ratio, $R = E_d/E_r$, are calculated from the Measurement of Ozone and Water Vapor by Airbus In-Service Aircraft (MOZAIC) dataset, and it is shown that at least 1000 flight segments are needed to obtain converged results. It is found that $R < 1$ in the upper troposphere, ruling out the hypothesis that the spectra are produced by inertia–gravity waves with frequencies larger than $f$. In the lower stratosphere $R$ is slightly larger than unity. An analysis separating between land and ocean data shows that $E_d$ and temperature spectra have somewhat larger magnitude over land compared to ocean in the upper troposphere—a signature of orographically or convectively forced gravity waves. No such effect is seen in the lower stratosphere. At midlatitudes the Rossby number is on the order of unity and at low latitudes it is larger than unity, indicating that strong nonlinearities are prevalent. Also the temperature spectra, when converted into potential energy spectra, have larger magnitude than predicted by the weakly nonlinear wave hypothesis.

1. Introduction

The wavenumber kinetic energy spectrum at atmospheric mesoscales (wavelengths between 10 km and a couple of hundred kilometers) exhibits an approximate $k^{-5/3}$ dependence (Nastrom and Gage 1985), where $k$ is the wavenumber. Most investigators have tried to explain this observation by cascade hypotheses, falling into two categories: the upscale energy cascade hypothesis (Gage 1979; Lilly 1983) and the downscale energy cascade hypothesis (Dewan 1979). There is strong observational and numerical evidence (Cho and Lindborg 2001; Augier and Lindborg 2013; Deusebio et al. 2014) supporting the downscale energy cascade hypothesis. The question is if a downscale energy cascade is produced by interacting gravity waves (Dewan 1979), strongly stratified turbulence (Lindborg 2006), or quasigeostrophic turbulence (Tung and Orlando 2003). An important tool to discriminate between different hypotheses is to make a Helmholtz decomposition of the energy spectrum: $E(k) = E_d(k) + E_r(k)$, where $E_d$ is the spectrum associated with the divergent component of the horizontal velocity field, $u_d = \nabla \chi$, and $E_r$ is associated with the rotational component, $u_r = -\nabla \times (e_z \Psi)$, where $\chi$ is the velocity potential, $\Psi$ is the streamfunction, and $e_z$ is the vertical unity vector. For quasigeostrophic dynamics, $E_d$ should be very small compared to $E_r$. For linear inertia–gravity waves we have

\[ R = \frac{E_d}{E_r} = \frac{\omega^2}{f^2}, \tag{1} \]

where $\omega$ is the intrinsic frequency. The relation (1) follows directly from the linear vertical vorticity equation and holds for each Fourier mode of a wave field, irrespectively of the vertical structure of the field. For waves with $\omega \gg f$ we should thus have $E_d \gg E_r$. For waves with frequencies very close to $f$ we should have $E_r \approx E_d$. For stratified turbulence it has been shown that $E_d \sim E_r$ (Lindborg and Brethouwer 2007).

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Lindborg (2007) used a method to differentiate second-order longitudinal and transverse horizontal structure functions calculated using data taken from a great number of commercial flights to estimate the relative magnitude of $E_d$ and $E_r$. The second-order structure function is the second-order statistical moment of a velocity difference between two points separated by a distance $r$. The longitudinal velocity component is in the direction of the separation vector $\mathbf{r}$ between the two points and the transverse velocity component is perpendicular to $\mathbf{r}$. The ratio $R = E_d/E_r$ was estimated to be around 0.5 at mesoscales. From this observation it was argued that gravity waves cannot be the dynamic agent behind the observed spectrum.

Results from general circulation models (GCMs) show quite different results. Hamilton et al. (2008) found that rotational energy was about 4 times larger than divergent energy at mesoscales near the tropopause, while Skamarock et al. (2014) obtained rotational and divergent energy of the same order in the upper troposphere (8.5–10.5 km). Divergent energy was more than 5 times larger in the stratosphere (16–18 km). Brune and Becker (2013) found that divergent energy was greater than rotational energy in the troposphere as well as in the stratosphere at the very highest resolved wavenumbers. Based on simulations of baroclinic waves Waite and Snyder (2013) argued that the $k^{-5/3}$ spectrum is produced by a downslope energy cascade produced by interactions of inertia–gravity waves with larger-scale straining flows or rotational modes as well as interacting gravity waves, in accordance with the theory of Bartello (1995).

Callies et al. (2014) argued that the mesoscale spectra are generated by weakly nonlinear inertia–gravity waves. However, they did not introduce any assumption of an energy cascade, in contrast to Dewan (1979), who first introduced the interacting wave hypothesis, explicitly assuming that wave interactions are associated with a downslope energy cascade giving rise to a $k^{-5/3}$ spectrum of the Kolmogorov type. Callies et al. (2014) made a rotational–divergent decomposition of spectra calculated from aircraft data, using formulas derived by Bühler et al. (2014), relating the rotational and divergent spectra to the longitudinal and transverse spectra, $E_l$ and $E_t$, respectively, which can be directly calculated from the data. The formulas were derived under the assumption of statistical axisymmetry, which means that the one-dimensional spectra are invariant under rotations with respect to vertical axes. They used data from 458 aircraft data segments from the upper troposphere and lower stratosphere reported in the Measurement of Ozone and Water Vapor by Airbus In-Service Aircraft (MOZAIC) dataset (Marenco et al. 1998), where each segment was 6000 km long. They found that $R$ was slightly larger than unity at wavenumbers corresponding to wavelengths between 20 and 200 km and used the theory of linear inertia–gravity waves to interpret this observation. From the observation that $R$ was just slightly larger than unity at mesoscales, Callies et al. (2014) drew the conclusion that the mesoscale spectrum is produced by linear or weakly nonlinear inertia–gravity waves with frequencies very close to $f$, in accordance with relation (1).

Lindborg (2015) showed that the relation between, on the one hand, the rotational and divergent spectra, and, on the other hand, the longitudinal and the transverse spectra, under the assumption of axisymmetry can be written in the closed form:

$$E_r = E_l + \frac{1}{k} \int_k^\infty (E_l - E_t) dk,$$

(2)

$$E_d = E_l - \frac{1}{k} \int_k^\infty (E_l - E_t) dk.$$  

(3)

Corresponding relations for structure functions were also derived and longitudinal and transverse structure functions calculated by Cho and Lindborg (2001) were used to estimate the ratio between divergent and rotational energy at separation smaller than $r = 20$ km, where the structure function show an approximate $r^{2/3}$ dependence. It was found that the ratio between the divergent and the rotational structure function is close to unity in the lower stratosphere and smaller than 0.5 in the upper troposphere at these separations. From this observation Lindborg (2015) concluded that inertia–gravity waves cannot explain the mesoscale dynamics. An obvious drawback of the structure function analysis is that the approximate $r^{2/3}$ range is much narrower than the corresponding shallow mesoscale energy spectrum.

Using data from a GCM Bierdel et al. (2016) have recently carried out an extensive study in which (2) and (3) and the mathematically equivalent formulas derived by Bühler et al. (2014) were tested. It was concluded that these formulas give quite accurate results, with typical relative errors of 10% or less. The errors were also found to decrease with decreasing scale or increasing wavenumber. Strong support was thus provided for the use of these formulas in calculating divergent and rotational energy spectra at mesoscales.

Recently, Callies et al. (2016) carried out a renewed spectral analysis of wind and temperature data from the MOZAIC and the 2008 Stratosphere–Troposphere Analyses of Regional Transport (START08) datasets, using altitude to discriminate between stratospheric and tropospheric data. They found that $R$ is slightly larger than unity in the lower stratosphere, while different results were obtained from the two datasets in the upper
troposphere. From the MOZAIC data it was found that $R \approx 0.5$ in the upper troposphere, consistent with the result by Lindborg (2015), while $R$ was found to be slightly larger than unity when the START08 data was used. Despite the fact that only 15 flight segments were used from the START08 data, Callies et al. (2016) argued that the result from this dataset is likely to be more reliable because of better instrumentation. In analyzing the aircraft data Callies et al. (2014) and Callies et al. (2016) use linear wave theory (e.g., Gill 1982) of inertia-gravity waves according to which the minimum frequency is equal to $f$, which follows from the linear dispersion relation derived under the assumption of a constant $N$. Callies et al. (2016) developed a simple model in which $N$ is constant in the upper troposphere as well as in the lower stratosphere but has a singular jump over the tropopause. The dispersion relation is supposed to be valid in each region but with different values of $N$. From the observation $R \approx 0.5$ it follows that $\omega \approx \sqrt{0.5f} < f$ and we must conclude that their model is not applicable. However, it may be the case that a more advanced linear wave model could be consistent with observational data. Therefore, it would be useful to formulate another condition for weak nonlinearity that is not dependent on any assumption of the vertical structure of the flow. In this paper we formulate such a condition and carry out the test. To do this, rotational and divergent spectra in the upper troposphere and lower stratosphere are calculated using the MOZAIC data. We also calculate the temperature spectra. A condition based on ozone concentration and turbulence spectra is supposed to be valid—from a strictly mathematical point of view—that there may be systematic cancellations among them that are unlikely, in appendix A we show that no such cancellations will occur for a statistically homogeneous and axisymmetric field with Gaussian statistics and we derive a stronger condition from which (6) follows. It should also be pointed out that (6) is a necessary but not sufficient condition for strong nonlinearity. There are cases when the condition is fulfilled even when the dynamics is strongly nonlinear, such as quasigeostrophic dynamics on an $f$ plane (Charney 1971).

To be able to estimate $|\zeta|$ from observations we consider a homogeneous and statistically axisymmetric (or horizontally isotropic) random hydrodynamic field in which many modes are excited. We define the two-point correlation functions of vorticity and rotational velocity as

$$Q(r) = \langle \zeta(x) \zeta(x+r) \rangle \quad \text{and} \quad R(r) = \langle u(x) \cdot u(x+r) \rangle,$$

where the primed quantities are measured a distance $r$ from the unprimed quantities, both at the same horizontal plane; the angle brackets represent a space average; and $Q$ and $R$ are related (Lindborg 2015) as follows:

$$\frac{\partial^2 \zeta}{\partial t^2} = \mathbf{e}_z \cdot \{ \nabla \times [ \mathbf{u} \times (\zeta + f \mathbf{e}_z) ] \},$$

where we have neglected the baroclinic term, $\mathbf{e}_z \cdot (\nabla \rho \times \nabla p)/\rho^2$, which is small if horizontal density gradients are small. The form of (5) immediately suggests that a necessary condition for weak nonlinearity can be formulated as

$$\text{Ro} \ll 1,$$

where $\text{Ro} = |\zeta|/f$ is the vorticity-based Rossby number. However, after having written out all nonlinear terms of (5) in component form (see appendix A) it can be argued—from a strictly mathematical point of view—that such cancellations among them that will make (6) a too-strong condition. Although this seems unlikely, in appendix we show that no such cancellations will occur for a statistically homogeneous and axisymmetric field with Gaussian statistics and we derive a stronger condition from which (6) follows. It should also be pointed out that (6) is a necessary but not sufficient condition for strong linearity. Therefore, it would be useful to formulate another condition for weak nonlinearity that is not dependent on any assumption of the vertical structure of the flow. In this paper we formulate such a condition and carry out the test. To do this, rotational and divergent spectra in the upper troposphere and lower stratosphere are calculated using the MOZAIC data. We also calculate the temperature spectra. A condition based on ozone concentration and turbulence spectra is supposed to be valid—from a strictly mathematical point of view—that there may be systematic cancellations among them that are unlikely, in appendix A we show that no such cancellations will occur for a statistically homogeneous and axisymmetric field with Gaussian statistics and we derive a stronger condition from which (6) follows. It should also be pointed out that (6) is a necessary but not sufficient condition for strong linearity. Therefore, it would be useful to formulate another condition for weak nonlinearity.
\[ Q = -\nabla^2 R. \]  

We define the two-dimensional enstrophy and rotational energy spectra as

\[ \Omega^{2D}(k) = k \int_0^\infty r J_0(rk) \frac{1}{2} Q(r) \, dr, \]

\[ E_r^{2D}(k) = k \int_0^\infty r J_0(rk) \frac{1}{2} R(r) \, dr. \]

Evidently, \( \Omega^{2D}(k) = k^2 E_r^{2D}(k) \), in virtue of relation (8). The characteristic magnitude of vorticity associated with a wavenumber range, \( k \in [k_{\text{min}}, k_{\text{max}}] \), can be estimated as

\[ |\xi| = \left( \int_{k_{\text{min}}}^{k_{\text{max}}} 2\Omega^{2D}(k) \, dk \right)^{1/2}, \]

a quantity that can be calculated using the rotational energy spectrum that we determine from the aircraft data. To do this, we must realize that the two-dimensional rotational energy spectrum in (10) is not identical to the one-dimensional rotational spectrum \( E_r(k) \). The reason why we use the two-dimensional enstrophy spectrum, rather than the one-dimensional spectrum \( \Omega^{1D} \) to estimate \( |\xi| \), is that the dominant contributions from \( \Omega^{1D}(k) \) when expressed in terms of \( E_r(k) \) come from wavenumbers that are much larger than \( k \). If we instead use \( \Omega^{2D}(k) \) there is no such problem, which is shown in appendix B. If \( E_r(k) = ak^{-5/3} \) in a broad range of wavenumbers, then it can be shown (see appendix B) that \( E_r^{2D}(k) \approx 1.40ak^{-5/3} \) in the corresponding range. Analyzing simulations of stratified turbulence Lindborg and Brethouwer (2007) found that this relation is very well satisfied if the \( k^{-5/3} \) range is one decade. If \( k_{\text{max}} \gg k_{\text{min}} \), the characteristic vorticity scale corresponding to such a range can be thus estimated as

\[ |\xi| = \left( \frac{k_{\text{max}}^{1/3}}{k_{\text{min}}^{1/3}} \right) 2.8ak^{1/3} \text{dk} \equiv 1.4a^{1/2}k_{\text{max}}^{2/3}, \]

and the condition for weak nonlinearity can be written as

\[ \text{Ro} = \frac{1.4a^{1/2}k_{\text{max}}^{2/3}}{f} \ll 1. \]

3. Data analysis

The aircraft data used in the study are reported in the MOZAIC dataset (Marenco et al. 1998), ranging from August 1994 to December 2010, containing data from 25,922 flights. The flight levels are restricted to the altitude range from 9.4 to 12 km. Each segment that we use contains 2048 data points corresponding to a flight distance of approximately 2000 km. The local velocity components parallel and perpendicular to the flight track are calculated at each point. The two orthogonal components are defined in the following way: \( u_r(x) = u(x) \cdot n \) and \( u_t(x) = u(x) \cdot t \), where \( n = (x' - x)/|x' - x| \), \( x \), and \( x' \) are the coordinates of two points along the flight track, and \( t \) is a unit vector that is perpendicular to \( n \), lying in the horizontal plane. Contrary to other spectral studies using aircraft data, we carry out the Fourier transform directly on the given time signal that is sampled with a frequency of 0.25 Hz. Longitudinal and transverse frequency spectra are computed by applying Welch’s method using a Hanning window with 50% overlap. For each segment, the frequency spectra are then transformed to wavenumber spectra, using the average flight speed over the segment. Using (2) and (3), \( E_r \) and \( E_t \) are calculated from \( E_r \) and \( E_t \), where the integration is performed using the trapezoidal rule. The final spectra are then linearly interpolated to equally spaced wavenumbers, \( k = 2\pi/\lambda \), where \( \lambda \in [4, 1000] \) km. Apart from divergent and rotational energy spectra, we also calculate temperature spectra. First we present a global analysis in which we use a strong condition on ozone concentration to separate between tropospheric and stratospheric data. Then we present a global analysis separating between land and ocean. In this analysis we use a weaker condition to separate between stratospheric and tropospheric data in order to obtain a sufficient number of segments. Finally, we present an analysis of data collected at low latitudes, between 20°S and 20°N.

a. Global analysis of tropospheric and stratospheric data

To separate between stratospheric and tropospheric data we use a condition based on ozone concentration as in Cho and Lindborg (2001). To classify as stratospheric the ozone concentration has to be larger than 200 ppbv and to classify as tropospheric it has to be smaller than 100 ppbv at all data points in the segments. The condition that we use in this analysis is considerably stronger than the condition used by Cho and Lindborg (2001), since we require that all points of a whole flight segment should meet the threshold condition, while Cho and Lindborg applied the threshold condition only to each pair of points they used in the structure function calculation. The advantage of using a stronger condition is that data taken very close to the tropopause will be excluded, while the drawback is that less data can be used. In this study we also exclude flights near the equator, between 20°S and 20°N. From the total database we identify 5274 tropospheric data segments and 5553
stratospheric segments. In Fig. 1 we see the distribution of ozone concentration for the whole MOZAIC dataset as well as the distribution in the two subsets that we used. In Figs. 2 and 3 we see the pressure altitude distribution and the latitude distribution, respectively, of both the tropospheric and stratospheric that we have used.

In Fig. 4 we see the divergent and rotational energy spectra in the troposphere and stratosphere. The spectra are premultiplied by $k^{5/3}$ and are plotted as functions of inverse wavelength $\lambda^{-1}$. All compensated spectra are almost flat at mesoscales, indicating that the spectral dependence is indeed very close to $k^{-5/3}$. Such a clean $k^{-5/3}$ dependence has not been demonstrated in previous studies showing uncompensated spectra in log–log plots rather than compensated spectra in lin–log plots. It should be pointed out that plotting the compensated spectra in lin–log plots is an extremely sensitive test of the $k^{-5/3}$ power-law dependence. The $k^{-5/3}$ dependence is visible already at $\lambda \approx 500$ km in the stratosphere, as compared to $\lambda \approx 200$ km in the troposphere. The transition to a very steep spectrum at $\lambda \approx 500$ km in the stratosphere is an artifact of the finite length of the data segments that we have used. It may very well be that the $k^{-5/3}$ range extends to even larger scales in the stratosphere. The difference between the troposphere and the stratosphere is even more pronounced when we look at the compensated temperature spectra, depicted in Fig. 5. First, we observe the remarkably clean $k^{-5/3}$ range, $\lambda \in [10, 500]$ km, of the stratospheric temperature spectrum. Here, it is even more obvious that the transition to a steeper spectrum at $\lambda \approx 500$ km is an artifact of the finite size of the data segments. Second, we also observe a very clean $k^{-5/3}$ range, $\lambda \in [4, 100]$ km, in the troposphere. In this case, the transition to a steeper spectrum at $\lambda \approx 100$ km, cannot be explained by the finite length of the data segments, but is a significant observation. An interesting observation is that the $k^{-5/3}$ range extends to the cutoff wavenumber $\lambda = 4$ km in the troposphere, while it ends at $\lambda = 10$ km in the stratosphere, where we see a transition to a steeper spectrum. The transition is also seen in the divergent and rotational energy spectra in the stratosphere. A similar transition was observed by Bacmeister et al. (1996) and Callies et al. (2016) when they used the START08 data. In the analysis of the MOZIAC data, however, they used a larger cutoff wavelength, $\lambda = 20$ km, as compared to $\lambda = 4$ km that we use, and could therefore not detect this transition. In all likelihood the observation is significant.

The clean $k^{-5/3}$ dependence is obtained thanks to the strict condition on ozone concentration excluding data close to the tropopause. In Fig. 6 we see rotational and divergent spectra as well as temperature spectra from data taken close to the tropopause. The near-tropopause spectra are mean values over 6721 segments, where each segment has a mean concentration of ozone that is larger...
than 120 ppbv and smaller than 180 ppbv. While the rotational spectrum is still close to showing a $k^{-5/3}$ dependence close to the tropopause, the divergent spectrum is steeper, $\sim k^{-1.9}$, as is the temperature spectrum.

In Fig. 4 we see that the rotational spectrum has the form $E_r(k) = a k^{-5/3}$ in a broad range of wavenumbers and $a \approx 5.5 \times 10^{-4} \text{m}^{4/3} \text{s}^{-2}$. Approximately the same value for the parameter $a$ is obtained in the troposphere as in the stratosphere. With $f = 10^{-4} \text{s}^{-1}$ and $k_{\text{max}} = 2 \pi / \lambda_{\text{min}}$, $\lambda_{\text{min}} = 10 \text{km}$, we find that

$$R_o = \frac{1.4 a^{1/2} k_{\text{max}}^{2/3}}{f} \approx 2.4,$$  

(14)

suggesting that the dynamics is strongly nonlinear. Of course, there is a certain degree of arbitrariness in the choice of $k_{\text{max}}$, equivalently the choice of $\lambda_{\text{min}}$. Had we instead chosen $\lambda_{\text{min}} = 100 \text{km}$, which excludes the major part of the $k^{-5/3}$ spectrum range, we would still have $R_o \approx 0.5$, which also violates the condition for weak nonlinearity. Another way of defining a scale-dependent Rossby number would be to base it on a measured structure function $\delta \mathbf{u} \cdot \delta \mathbf{u}$, where $\delta \mathbf{u}$ is the difference between the velocities measured at two points separated by a distance $\lambda$. Such a scale-dependent Rossby number may be defined as

$$R_{os} = \frac{\sqrt{\langle \delta \mathbf{u} \cdot \delta \mathbf{u} \rangle}}{f \lambda}.$$  

(15)

In Fig. 7 we have plotted the two Rossby numbers defined by (14) and (15) as functions of $\lambda$, where we have used the structure function calculated from aircraft data by Cho and Lindborg (2001) in the lower stratosphere to estimate (15).

There is only a minor difference between the two Rossby numbers for $\lambda < 100 \text{ km}$. In the range $\lambda \in [10, 100] \text{ km}$ they are both on the order of unity and for $\lambda < 10 \text{ km}$ they rapidly increase to values larger than unity.

The temperature power spectrum $E_T$ can be transformed into a spectrum of available potential energy (Lorenz 1955) as follows:

$$E_p = 0.5 g^2 E_T / (T_0 N)^2,$$  

(16)

where $g$ is the acceleration due to gravity, $T_0$ is the ambient temperature, and $N$ is the Brunt–Väisälä frequency. Assuming that $T_0 = 220 \text{ K}$, $N = 0.02 \text{ s}^{-1}$ in the stratosphere, and $N = 0.01 \text{ s}^{-1}$ in the troposphere, we obtain a ratio between kinetic and potential energy that
is close to 2 at mesoscales in both the troposphere and the stratosphere.

In Fig. 8 we see the average ratio $R$ between divergent and rotational energy. The lines are the averages over all data segments and the gray shaded area indicates the maximum deviation from the total mean among mean values calculated from five randomly generated exclusive subsets each containing 1000 segments. In the troposphere, $R$ is smaller than unity for all wavenumbers, with a peak, $R \approx 0.6$, at $\lambda \approx 100$ km. Although $R$ is smaller than unity, we can conclude that divergent energy is on the same order of magnitude as rotational energy at mesoscales. In the stratosphere, on the other hand, $R$ is somewhat larger than unity in most of the mesoscale range. It has a peak, $R \approx 1.3$, at $\lambda \approx 200$ km, and then falls off to approach unity at smaller wavenumbers, reflecting the small deviations from a perfect $k^{-5/3}$ dependence of both $E_r$ and $E_d$, which are visible in Fig. 4. As indicated by the gray shaded area, there is obviously some statistical spread in our results, suggesting that at least 1000 flight segments are needed in order to obtain reasonably converged results.

To make a more thorough analysis of the statistical distribution of the estimated values of $R$ we should first decide the minimum number of segments that is reasonable to use when calculating $R$. Since the determination of $E_d$ and $E_r$, using (2) and (3), is based on assumptions of axisymmetry and homogeneity that are not fulfilled unless the dataset is large enough, a single segment is not enough. In fact, single flight segments may give negative values of either $E_d$ or $E_r$, which is clearly not meaningful. After some trial and error we found that at least 10 segments that are 2000 km long are needed in order to avoid most negative values. Thus, we define 10 segments as one sample. The amount of data contained in such a sample is larger than the amount of data in the set of 15 flight segments used by Callies et al. (2016), since the average length of their segments was considerably shorter than 2000 km.¹ For the tropospheric data we use $n = 527$ samples and for the stratospheric data we use $n = 555$ samples and calculate $R$ at $\lambda = 10$ km for each of them. One way to quantify the typical deviation from the mean $R, \overline{R}$, is to use $R_{\text{rms}}$. We

¹The average length of the 15 tropospheric segments from the START08 data used by Callies et al. (2016) was 195 km (J. Callies 2017, personal communication).
found that $R_{\text{rms}} = 1.06$ and $R_{\text{rms}} = 1.16$, for the troposphere and stratosphere, respectively, which corresponds to $\sim 200\%$ and $\sim 100\%$ relative deviations in the two cases, respectively. However, since $R$ by definition is positive definite and the distribution is far from symmetric with respect to the mean we plot the distribution of $\ln(R/R)$ (Fig. 9) instead of the distribution of $R$. For comparison we have in each case also added a normal distribution with the directly calculated standard deviation, $\sigma$, of $\ln(R/R)$. For the tropospheric data we found that $\sigma = 0.85$ and for the stratospheric data we found that $\sigma = 0.68$. The relative deviation of $R$ from the mean corresponding to one standard deviation can be estimated as $\exp(\sigma) - 1$ and is equal to 134% and 97% in the two cases, respectively. The standard error, $s = \sigma/\sqrt{n}$, of the mean is found to be $s = 0.037$ and $s = 0.030$, for the troposphere and stratosphere, with corresponding relative deviations of $R$ of 4% and 3%, respectively. With $n = 100$, that is, 1000 flight segments, the corresponding values are 9% and 7%. If we assume that $\ln(R/R)$ is normally distributed and we require a 95% confidence interval, these numbers should be multiplied by 1.96. A corresponding analysis for $\lambda = 100$ km gave comparable numbers. For the stratospheric data the deviation from the mean seen in Fig. 8 is consistent with these estimates, while the corresponding deviation for the tropospheric data seems to be somewhat larger. We conclude that at least 1000 flight segments are needed to determine $\overline{R}$ within some acceptable limits at an acceptable level of confidence.

### b. Global analysis separating between land and ocean

In analyzing the Global Atmospheric Sampling Program (GASP) data Nastrom and Gage (1985) made a separation between data taken over land and ocean and found that the magnitude of mesoscale spectra taken over land exceeded the magnitude of spectra taken over ocean by about 25%. Spectra from an area over
the western United States showed 2–4-times-larger magnitudes than global averaged. These observations indicate that upward-propagating gravity waves, induced by orography or convection, may contribute to energizing mesoscale motions close to the tropopause. On the other hand, Callies et al. (2016) reported that they found no difference in the ratio $R$ between areas with low and high orographic forcing.

Upward-propagating gravity waves are expected to have frequencies considerably larger than $f$, since the group velocity of inertia gravity waves with frequencies close to $f$ is almost horizontal (Gill 1982). Upward-propagating waves are therefore expected to give rise to a larger magnitude of divergent and temperature spectra but not of the rotational spectrum. To investigate this we identify a great number of segments where all points in each segment are over either land or ocean. Given the large difference between tropospheric and stratospheric data that we saw in the previous section it would be preferable to separate between the troposphere and stratosphere also in this case. However, in order to obtain a sufficiently large number of segments we have to relax the condition on ozone concentration that we use to distinguish between tropospheric and stratospheric data. In this case we use a condition on mean concentration. To classify as tropospheric the mean concentration of ozone over a segment has to be lower than 120 ppbv and to classify as stratospheric it has to be larger than 180 ppbv.

In Fig. 10 we see the ratios between spectra calculated over land and ocean for the tropospheric data and stratospheric data. Overall, there is no dramatic difference between land and ocean. In the upper troposphere, the land–ocean ratio is around 1.4 for the divergent and the temperature spectra and close to unity for the rotational spectrum at $\lambda \leq 100$ km. In all likelihood, this is a signature of direct forcing by upward-propagating gravity waves. In the lower stratosphere the ratio is close to unity for all three spectra and there is no signature of enhanced gravity wave activity at $\lambda \leq 100$ km.

c. Analysis of low-latitude data

We identify 2800 segments for which all data points lie between $20^\circ$S and $20^\circ$N. In this case there is no need to distinguish between tropospheric and stratospheric data using a condition on ozone concentration, since all low-latitude data are tropospheric. In Fig. 11 we see rotational and divergent spectra and temperature spectra averaged over all low-latitude segments. The spectra are similar to the tropospheric spectra seen in Figs. 4 and 5. The rotational spectrum displays a relatively clean $k^{-5/3}$ dependence at $\lambda \in [10, 100]$ km, while the divergent spectrum is somewhat steeper. The temperature spectrum displays a clean $k^{-5/3}$ dependence at $\lambda < 100$ km. We find that $R = 0.5$, or even somewhat smaller, similar to the averaged result for troposphere at higher latitudes. The magnitude of the rotational spectrum is comparable, but somewhat larger, than we found previously. Since the low-latitude spectra are very similar to the higher-latitude spectra the Rossby number is considerably larger. If we use the low-latitude mean value of $f$, we find that Ro is about 4 times larger for the
low-latitude data (i.e., $\text{Ro} \approx 10$ at $\lambda = 10\text{ km}$ and $\text{Ro} \approx 2$ at $\lambda = 100\text{ km}$).

4. Conclusions

We have found that the ratio $R$ between divergent and rotational energy is smaller than unity, $R \approx 0.5$, at mesoscales in the upper troposphere, consistent with the result of Callies et al. (2016) when they used the MOZAIC dataset. As we discussed in the introduction, from this observation we conclude that linear theory of inertia–gravity waves is not likely to hold. A convergence test shows that at least 1000 flight segments are needed in order to obtain reasonably converged results, while 100 is not enough. Thus, 15 segments, as used by Callies et al. (2016) in analyzing the START08 tropospheric data, is far from enough. In the stratosphere, we obtain a ratio that is somewhat larger than unity, around 1.3, at $\lambda = 200\text{ km}$, and then decreases with decreasing wavelength, to reach a value around unity at $\lambda = 10\text{ km}$. Our result for the stratosphere is also rather similar to the result of Callies et al. (2014) and Callies et al. (2016).

Based on the condition (13), it can be argued that weakly nonlinear wave theory cannot provide a general explanation of the mesoscale energy spectra in the upper troposphere and lower stratosphere. Globally, the Rossby number associated with mesoscale motions is on the order of unity and at low latitudes it is considerably larger than unity. The similarity between the low-latitude spectra and the globally averaged spectra in the upper troposphere suggests that system rotation is not a crucial factor in the generation of the spectra. As pointed out by one of the reviewers, the Rossby number corresponding to scales larger than 100 km is smaller than unity and $R > 1$ for the lower-stratospheric data at these scales. It is therefore possible that linear waves may play a role in the generation of the $k^{-5/3}$ spectrum observed in the stratosphere at these scales.

The strong condition we used on ozone concentration to distinguish between tropospheric and stratospheric data has revealed that there are significant differences between mesoscale spectra in the upper troposphere and lower stratosphere. In the lower stratosphere the $k^{-5/3}$ range extends from wavenumbers with corresponding wave lengths of $\lambda \in [10, 500]\text{ km}$, and possibly to even larger wave lengths. In the upper troposphere, on the other hand, the corresponding range is $\lambda \in [4, 200]\text{ km}$, where the lower limit may be even smaller. Divergent spectra as well as temperature spectra calculated from data collected close to the tropopause are steeper than the spectra collected away from the tropopause. Our results give no support for the hypothesis that the $k^{-5/3}$ mesoscale spectra are produced by a dynamics that is specific to the tropopause, as suggested by Tulloch and Smith (2006). On the contrary, the cleanest $k^{-5/3}$ spectra are observed when data close to the tropopause are excluded.

When we used standard values of $N$ to convert temperature spectra to average potential energy (APE) spectra, we found that $\text{APE}/\text{KE} \approx 0.5$, both in the troposphere and the stratosphere. This result is also consistent with the results of Callies et al. (2016). However, as shown by Bühler et al. (2014), the energy of inertia–gravity waves, conforming to the assumptions of linear theory, is partitioned as

$$E_p + E_r = E_d \Rightarrow \frac{\text{APE}}{\text{KE}} = \frac{R - 1}{R + 1}. \quad (17)$$

In the troposphere, linear wave theory is clearly not applicable, since $R < 1$. In the stratosphere, we found that $R \in [1, 1.3]$ with the corresponding range
Augier and Lindborg (2013) who found that there was no net energy propagation through the tropopause at $\lambda < 100 \text{ km}$ over land as compared to ocean. Although there is some uncertainty regarding nature. The condition of Gaussianity can only be explained by the cascade hypothesis if it is not related to an energy flux?

GCMs are still not capable of resolving the whole mesoscale range in the horizontal and the vertical resolution is still not sufficiently fine to fully simulate stratified turbulence. Different GCMs (e.g., Hamilton et al. 2008; Skamarock et al. 2014) give quite different results for $R$. However, a common feature of the results from different GCMs and observational results is that $R$ is generally larger in the stratosphere as compared to the troposphere. This result may be interpreted in light of the forward-cascade hypothesis. In the troposphere it must be assumed that baroclinic instability is the main forcing mechanism, generating motions at the 1000-km scale carrying vertical vorticity. In the stratosphere, on the other hand, the forcing mechanism is waves propagating from the troposphere (Augier and Lindborg 2013), also at the 1000-km scale. From the 1000-km scales, energy is transferred to smaller scales by strong interactions involving both rotational and divergent modes. During this process, a state develops in which rotational and divergent energy are on the same order of magnitude. However, at the 100-km scale there is still a footprint of the forcing, giving an $R$ which is larger than unity in the stratosphere and smaller than unity in the troposphere. It is only a matter of time before high-resolution GCMs will tell us if this interpretation is correct.

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APPENDIX A

Derivation of Condition for Weak Nonlinearity for a Random Field

In this appendix we derive a global condition for weak nonlinearity for a random axisymmetric (or horizontally isotropic), homogeneous, incompressible field with Gaussian statistical distribution. The condition that we derive is stronger than (6). By axisymmetry we mean invariance under rotations with respect to vertical axes and reflections in vertical planes (parity invariance). That the derivation is carried out for such idealized conditions should not be mistaken for an assumption regarding nature. The condition of Gaussianity can only

$$E(k) = bk^{-5/3},$$

over a wide range, where $b$ is a parameter having the dimension $L^{4/3} T^{-2}$. One may ask: What is the significance of this parameter according to the wave hypothesis if it is not related to an energy flux?
be warranted in the limit of very weak nonlinearity, since deviations from linearity will lead to deviations from Gaussianity. Neither can it be taken for granted that reflectional invariance with respect to vertical planes is realistic in presence of strong system rotation. We may consider the condition that we derive as a global condition on an initial random field in a numerical simulation that we would like to be marginally influenced by nonlinear effects in its future evolution. The analysis is similar to the analysis of Lindborg and Riley (2007), who derived a condition on the average Richardson number for weak nonlinearity of internal gravity waves in the ocean.

The condition of invariance under rotations implies invariance under

\[ x \mapsto y, \quad y \mapsto -x, \quad u \mapsto v, \quad v \mapsto -u, \]  
\[ x \mapsto -y, \quad y \mapsto x, \quad u \mapsto -v, \quad v \mapsto u, \]  
where \( u \) and \( v \) are the velocity components in the \( x \) and \( y \) directions, respectively. We note that \( \zeta \) is invariant under both rotations, while \( \{ \xi_x \mapsto \xi_y, \xi_y \mapsto -\xi_x \} \) under (A1) and \( \{ \xi_x \mapsto -\xi_x, \xi_y \mapsto \xi_y \} \) under (A2). Reflectional invariance with respect to vertical planes implies invariance under

\[ x \mapsto -x, \quad u \mapsto -u, \]  
\[ y \mapsto -y, \quad v \mapsto -v. \]  
We note that \( \zeta_z \mapsto -\zeta_z \) under any of these reflections, while \( \{ \xi_x \mapsto \xi_x, \xi_y \mapsto -\xi_y \} \) under (A3) and \( \{ \xi_x \mapsto -\xi_x, \xi_y \mapsto \xi_y \} \) under (A4). The assumptions of Gaussianity will be used to factorize fourth-order statistical moments into second-order moments in the following way:

\[ \langle abcd \rangle = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle bd \rangle + \langle ad \rangle \langle bc \rangle, \]  
where \( a, b, c, \) and \( d \) represent velocity components or their derivatives. A proof that fourth-order statistical moments of velocities and velocity derivatives can be factorized in this way if they have jointly Gaussian statistical distributions is given by Frisch (1995). The assumption of statistical homogeneity will be used to eliminate spatial derivatives of statistical moments, for example,

\[ \left\langle u \frac{\partial u}{\partial x} \right\rangle = \left\langle \frac{1}{2} \left( \frac{\partial}{\partial x} \right)^{1/2} u^2 \right\rangle = \frac{1}{2} \frac{\partial}{\partial x} \langle u^2 \rangle = 0. \]  

For completeness we also write out the incompressibility condition:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \]  

We now consider the incompressible vertical vorticity equation written in the following form:

\[ \frac{\partial \zeta_z}{\partial t} + u \frac{\partial \zeta_z}{\partial x} + v \frac{\partial \zeta_z}{\partial y} + w \frac{\partial \zeta_z}{\partial z} \]
\[ + \xi_z \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \xi_x \frac{\partial w}{\partial x} - \xi_y \frac{\partial w}{\partial y} \]
\[ = -f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \]  

Taking the square of the sum of all nonlinear terms on the left-hand side of (A8), averaging and comparing it with the corresponding linear term on the right-hand side, we can formulate a global condition for weak nonlinearity as

\[ \left\langle \left( u \frac{\partial \zeta_z}{\partial x} + v \frac{\partial \zeta_z}{\partial y} + w \frac{\partial \zeta_z}{\partial z} + \xi_z \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \xi_x \frac{\partial w}{\partial x} - \xi_y \frac{\partial w}{\partial y} \right)^2 \right\rangle \ll f^2 \langle dd \rangle, \]  

where we have introduced the notation

\[ d = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}. \]  

for the horizontal divergence. We note that the divergence is invariant under rotation and reflection. The left-hand side of (A9) consists of a square of a sum of six terms, which we name \( t_1, t_2, \ldots, t_6 \), in the same order as they appear. A cross correlation between two terms \( t_j \) and \( t_k \), \( j \neq k \), is defined as \( \langle t_j t_k \rangle \). We will now show that all cross correlation are zero, except \( \langle t_1 t_2 \rangle \) and \( \langle t_3 t_4 \rangle \). We start by considering all cross correlations involving the vortex stretching term \( t_4 = \zeta_z d \). As an example we make the expansion

\[ \langle t_1 t_4 \rangle = \left\langle u \frac{\partial \zeta_z}{\partial x} \frac{\partial \zeta_z}{\partial x} \right\rangle \]
\[ = \left\langle u \frac{\partial \zeta_z}{\partial x} \right\rangle \left\langle \frac{\partial \zeta_z}{\partial x} \right\rangle + \langle ud \rangle \left\langle \frac{\partial \zeta_z}{\partial x} \right\rangle \]
\[ + \langle u \zeta_z \rangle \left\langle \frac{\partial \zeta_z}{\partial x} \right\rangle. \]  

The first term cancels by (A3) or (A4), the second by (A1), and the last by (A4). As another example, we make the following expansion:

\[ \langle t_3 t_4 \rangle = \left\langle w \frac{\partial \zeta_z}{\partial z} \right\rangle \]
\[ = \left\langle w \frac{\partial \zeta_z}{\partial z} \right\rangle \left\langle \frac{\partial \zeta_z}{\partial z} \right\rangle + \langle wd \rangle \left\langle \frac{\partial \zeta_z}{\partial z} \right\rangle \]
\[ + \langle w \zeta_z \rangle \left\langle \frac{\partial \zeta_z}{\partial z} \right\rangle. \]  

(A11)
The first and the last terms cancel by (A3) or (A4) and the second term cancels by homogeneity [(A6)]. We leave it to the reader to confirm that all other cross correlations involving \( t_4 \) and \( t_5 \) are zero. The term \( t_1 \) is negatively correlated with \( t_2 \) and uncorrelated with all other terms. To show that \( \langle t_1 t_2 \rangle \leq 0 \) we make the following expansion:

\[
\langle t_1 t_2 \rangle = \left\langle u \frac{\partial \xi}{\partial y} v \frac{\partial \xi}{\partial y} \right\rangle = \left\langle u \frac{\partial \xi}{\partial x} v \frac{\partial \xi}{\partial x} \right\rangle + \langle uv \rangle \left\langle \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial x} \right\rangle + \left\langle u \frac{\partial \xi}{\partial y} \frac{\partial \xi}{\partial y} \right\rangle.
\]

(A13)

The first two terms cancel by (A3) or (A4). The third term can be rewritten as

\[
\left\langle u \frac{\partial \xi}{\partial y} \right\rangle \left\langle v \frac{\partial \xi}{\partial x} \right\rangle = -\left\langle u \frac{\partial \xi}{\partial y} \right\rangle^2 \leq 0,
\]

by (A1). It is a little bit tricky to show that \( \langle t_1 t_2 \rangle = \langle t_1 t_6 \rangle = 0 \).

The first of these two correlations can be expanded as

\[
\langle t_1 t_5 \rangle = \left\langle u \frac{\partial \xi}{\partial x} \right\rangle \left\langle \frac{\partial w}{\partial x} \right\rangle = -\left\langle u \frac{\partial \xi}{\partial x} \right\rangle \left\langle \frac{\partial w}{\partial x} \right\rangle - \langle uw \rangle \left\langle \frac{\partial \xi}{\partial x} \frac{\partial w}{\partial x} \right\rangle
\]

\[
- \left\langle \frac{\partial w}{\partial x} \right\rangle \left\langle \frac{\partial \xi}{\partial x} \right\rangle.
\]

(A15)

The two first terms cancel by (A4) and the first factor of the last term cancels by incompressibility [(A7)], homogeneity [(A6)], and rotational invariance [(A1)]:

\[
\left\langle \frac{\partial w}{\partial x} \right\rangle = \frac{\partial}{\partial x} \langle uw \rangle = \left\langle \frac{\partial u}{\partial x} w \right\rangle,
\]

\[
= \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial u}{\partial x} w \right),
\]

\[
= \frac{1}{2} \left( \frac{\partial w}{\partial y} \right) = \frac{1}{4} \frac{\partial}{\partial z} \langle w^2 \rangle = 0.
\]

(A16)

That \( \langle t_1 t_6 \rangle = 0 \) can be shown in a similar way. The term \( t_2 \) has the same properties as \( t_1 \) since \( t_1 \rightarrow t_2 \) under rotation and therefore we do not have to treat it separately. We have now treated all cross correlations except \( \langle t_5 t_6 \rangle \), which is less than or equal to zero:

\[
\langle t_5 t_6 \rangle = \left\langle \xi \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right\rangle = \left\langle \xi \frac{\partial w}{\partial x} \right\rangle \left\langle \xi \frac{\partial w}{\partial y} \right\rangle + \left\langle \xi \frac{\partial w}{\partial x} \right\rangle \left\langle \xi \frac{\partial w}{\partial y} \right\rangle.
\]

The first two terms cancel by (A3) or (A4) and the third term can be rewritten as

\[
\left\langle \xi \frac{\partial w}{\partial y} \right\rangle \left\langle \xi \frac{\partial w}{\partial x} \right\rangle = -\left\langle \xi \frac{\partial w}{\partial y} \right\rangle^2 \leq 0
\]

(A17)

by (A2). The two autocorrelations \( \langle t_5 t_5 \rangle \) and \( \langle t_4 t_4 \rangle \) can be simplified as

\[
\langle t_5 t_5 \rangle = \langle \xi \xi w \rangle \left\langle \xi \frac{\partial \xi}{\partial z} \frac{\partial \xi}{\partial z} \right\rangle,
\]

\[
\langle t_4 t_4 \rangle = \langle \xi \xi \rangle \langle dd \rangle.
\]

(A18)

by (A3). The condition for weak nonlinearity can now be written as

\[
\langle \xi \xi \rangle \langle dd \rangle + \langle \xi \xi \rangle \langle \xi \xi \rangle + \left( \left\langle \xi \xi \right\rangle + \left\langle \xi \xi \right\rangle \right)^2 \ll f^2 \langle dd \rangle
\]

(A19)

The condition (6) that we use in this paper is obtained by only taking the first term on the left-hand side of (A20) into account. However, there is no reason to believe that any of the other terms should be smaller. For a single Fourier mode, for example, the second term is exactly equal to the first term. We can quite safely conclude that (6) is not a too-strong condition.

APPENDIX B

One-Dimensional and Two-Dimensional Enstrophy Spectra

In this appendix we show how the one-dimensional enstrophy spectrum \( \Omega_{1D}^\text{ID}(k) \) is related to the one-dimensional rotational energy spectrum \( E_r(k) \) and explain why we use the two-dimensional spectrum rather than the one-dimensional spectrum when we estimate the characteristic magnitude of vorticity |\( \zeta | \) corresponding to a wavenumber range [\( k_{\text{min}}, k_{\text{max}} \)]. We let \( R(r) = \langle u_r \cdot u_r \rangle \) and \( Q(r) = \langle \xi \xi \rangle \) be the two-point correlations of \( u_r \) and \( \zeta \), respectively, where a primed quantity is measured at a point separated by a distance \( r \) from the point where the unprimed quantity is measured and the two points are in the same horizontal plane. That the correlations are only functions of \( r \) is a consequence of axisymmetry, sometimes referred to as horizontal isotropy. The two correlations are related by
The one-dimensional rotational spectrum is defined as the one-dimensional Fourier transform of $R$,

$$E_r(k) = \hat{R} = \frac{2}{\pi} \int_0^\infty R(x) \exp(ikx) \, dx,$$  \hspace{1cm} (B2)

where we have changed the radial coordinate name from $r$ to $x$ to make it clear that (B2) is the one-dimensional transform. The one-dimensional vorticity spectrum $\Omega^{1D}(k)$ is defined as the one-dimensional Fourier transform of $Q$.

Integrating (B4) we find

$$\Omega^{1D}(k) = k^2 E_r(k) + \int_k^\infty k'E_r(k') \, dk'.$$  \hspace{1cm} (B5)

If $E_r(k) \sim k^{-5/3}$ the integral on the right-hand side does not converge. In other words, the dominant contribution to $\Omega^{1D}$ comes from the high wavenumber end of the $k^{-5/3}$ range, where $E_r$ starts to fall off exponentially. This happens at wavenumbers corresponding to scales that are not much larger than the Kolmogorov scale, in the atmosphere at centimeter or millimeter scales. The one-dimensional enstrophy spectrum can therefore not be used to estimate the magnitude of vorticity $|\zeta|$ corresponding to a finite wavenumber range $[k_{\text{min}}, k_{\text{max}}]$. There is no similar problem in using the two-dimensional enstrophy spectrum, $\Omega^{2D}(k) = k^2 E_r^{2D}(k)$, where

$$E_r^{2D}(k) = 2\pi k \hat{R},$$  \hspace{1cm} (B6)

and $\hat{R}$ is the two-dimensional Fourier transform of $R$. It is straightforward to show that the relation between the one-dimensional and two-dimensional rotational spectra can be written as

$$E_r(k) = \frac{2}{\pi} \int_k^\infty \frac{E_r^{2D}(k')}{\sqrt{k'^2 - k^2}} \, dk'.$$  \hspace{1cm} (B7)

The integral on the right-hand side is convergent if $E_r^{2D} \sim k^{-5/3}$ and the dominant contribution come from wavenumbers that are not very much larger than $k$.

If $E_r(k) = ak^{-5/3}$ and $E_r^{2D}(k) = bk^{-5/3}$ the relation between $a$ and $b$ is

$$a = \frac{16b}{3\pi} \int_1^\infty \eta^{-11/3} \sqrt{\eta^2 - 1} \, d\eta \approx 0.71b,$$  \hspace{1cm} (B8)

where we have calculated the integral numerically. As a matter of fact, an analytical relation expressed in terms of the gamma function can be derived by another method (Lindborg 1999). The two-dimensional enstrophy spectrum can now be written as

$$\Omega^{2D}(k) \approx 1.4ak^{13/3}.$$  \hspace{1cm} (B9)

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