A Gravity Wave Drag Matrix for Complex Terrain

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(Manuscript received 11 December 2017, in final form 1 March 2018)

ABSTRACT

We propose a simplified scheme to predict mountain wave drag over complex terrain using only the regional-average low-level wind components U and V . The scheme is tuned and tested on data from the South Island of New Zealand, a rough and highly anisotropic terrain. The effect of terrain anisotropy is captured with a hydrostatically computed, 2×2 positive-definite wave drag matrix. The wave drag vector is the product of the wind vector and the drag matrix. The nonlinearity in wave generation is captured using a Gaussian terrain smoothing inversely proportional to wind speed. Wind speeds of $|\mathbf{U}| = 10, 20$, and 30 m s^{-1} give smoothing scales of $L = 54, 27$, and 18 km , respectively. This smoothing treatment of nonlinearity is consistent with recent aircraft data and high-resolution numerical modeling of waves over New Zealand, indicating that the momentum flux spectra shift to shorter waves during high-drag conditions. The drag matrix model is tested against a 3-month time series of realistic full-physics wave-resolving flow calculations. Correlation coefficients approach 0.9 for both zonal and meridional drag components.

1. Introduction

Mountain wave drag is a horizontal pressure force acting on the terrain associated with the generation of gravity waves. The equal and opposite reaction force on the atmosphere is applied at some higher altitude where the gravity waves dissipate their energy. Since the recognition that mountain wave drag could influence the general circulation of the atmosphere (Sawyer 1959; Blumen 1965; Bretherton 1969; Lilly 1972), there have been numerous theoretical attempts to predict this wave drag using regional environmental parameters.

The development of wave drag theory for complex terrain has a long history. Wurtele (1957), Crapper (1962), Blumen and McGregor (1976), and Smith (1980) examined mountain waves from ideally shaped isolated hills using linear theory. Phillips (1984) derived the wave drag on a smooth elliptical hill at various angles to the wind vector. He developed the idea of “transverse drag”: the component of wave drag perpendicular to the wind direction. Drags for random terrains were analyzed by Blumen (1965), Bretherton (1969), Bannon and Yuhás (1990).

In parallel with these theoretical studies, surface observations of pressure drag (Smith 1978) and aircraft observations of wave momentum flux (e.g., Lilly and

Kennedy 1973; Bougeault et al. 1990; Smith et al. 2016) provided the motivation to include wave drag schemes in general circulation models (GCMs).

Interactive wave drag–predicting schemes (i.e., parameterizations) in GCMs were first introduced by Palmer et al. (1986) and McFarlane (1987). These methods were improved by Miller et al. (1989), Lott and Miller (1997), Gregory et al. (1998), and more recently by Scinocca and McFarlane (2000), Webster et al. (2003), Kim and Doyle (2005), and Vosper et al. (2016). The subject is reviewed by Kim et al. (2003), Geller et al. (2013), and Teixeira (2014). As our understanding of wave drag has advanced, two problems have been identified as the most challenging and important: terrain anisotropy and dynamical nonlinearity.

Palmer et al. (1986) used an isotropic linear monochromatic drag formulation that we rewrite for a finite obstacle as

$$\mathbf{D} = (K\rho N \times \text{Var})\mathbf{U}, \quad (1)$$

where K is a typical terrain wavenumber, ρ is the air density, N is the buoyancy frequency, Var is the variance of the terrain (m^4), and $\mathbf{U}$ is the low-level wind vector. The drag vector $\mathbf{D}$ (N) is assumed to be parallel to the wind vector $\mathbf{U}$. The wavenumber parameter K could also be used as a tuning factor. McFarlane (1987) used a similar formula.

Miller et al. (1989) extended the wave drag formulation by including a nonlinear correction for flow blocking

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and used Phillips' elliptical-mountain approach to anisotropy. Baines and Palmer (1990), Lott and Miller (1997), Gregory et al. (1998), Scinocca and McFarlane (2000), and Teixeira and Miranda (2006) followed a similar methodology with a nonlinear correction and Phillips' "equivalent ellipses" approach to anisotropy.

Kim and Arakawa (1995) and Kim and Doyle (2005) took a new approach using numerical mesoscale models over a variety of hill shapes to derive wave drag laws. Anisotropy was treated by using a range of wind directions over specified hill shapes.

In the gravity wave "parameterization" literature, the authors worked not only to predict the gravity wave drag but also to predict how the associated momentum flux would be applied to the atmosphere above the rough terrain (e.g., Shutts and Gadian 1999; Teixeira and Miranda 2009; Teixeira and Yu 2014; Xu et al. 2012, 2017). In this paper, we undertake a simpler task. We look only at wave generation and wave drag on the terrain.

2. Uncertainties in wave drag prediction

The motivations for this paper are three uncertainties in existing wave drag prediction schemes: (i) the role of the terrain spectrum, (ii) the generality and efficiency of the Phillips anisotropy formula, and (iii) the best way to correct for nonlinearity.

Regarding the first issue, linear theory predicts that for waves of a given displacement amplitude, the momentum flux increases with the horizontal wavenumber as in (1). As real terrain typically has a broad power spectrum, it is inaccurate to use a monochromatic representation. Likewise, the terrain variance does not provide sufficient spectral information. Recent aircraft observations have suggested that high-drag states may result from a shift in the spectral peak rather than a change in average wave amplitude (Smith and Kruse 2017). The use of total terrain variance and a single wavenumber, as in (1), may not capture this sensitivity to the terrain spectrum.

The second issue relates to the generality of the Phillips formula for transverse drag. Some authors have proceeded as if the drag component formulas derived for a smooth elliptical hill may be valid only for that simple shape. To preserve validity, they imagined the terrain as composed of "equivalent ellipses." We will show that the effect of terrain anisotropy can be treated for any complex terrain $h(x, y)$ using a 2×2 symmetric drag matrix $\mathbf{B}$. The drag vector $\mathbf{D}$ is the product of the horizontal wind vector $\mathbf{U}$ and the drag matrix $\mathbf{B}$; that is, $\mathbf{D} = \mathbf{U} \cdot \mathbf{B}$. A similar approach was suggested by Garner (2005).

The third uncertainty regards the nonlinear correction. The importance of wave drag nonlinearity is widely accepted in the drag parameterization literature (e.g.,

Lott and Miller 1997; Gregory et al. 1998; Scinocca and McFarlane 2000; Webster et al. 2003; Garner 2005). The standard representation is an "airflow-blocking algorithm." The concept of mountain airflow blocking due to density stratification has been well developed by Kao (1965), Hunt et al. (1979), Snyder et al. (1985), Spangler (1987), Smith (1989), Miranda and James (1992), Smith and Grønås (1993), Ólafsson and Bougeault (1996), Baines (1997), Reinecke and Durran (2008), and others. These studies all found that when the nondimensional mountain height (hN/U) is large, the lower air layers will stagnate, split, and flow around isolated hills. Only the layers just below the hilltop will be able to rise over the top. There may also be an upstream blocking wedge of cold air that will produce a broader and gentler upslope ascent (Pierrehumbert and Wyman 1985).

In a standard blocking algorithm, it is assumed that in slow stratified flow, the air is unable to lift more than $\Delta z = |\mathbf{U}|/N$, where the wind speed $|\mathbf{U}| = (U^2 + V^2)^{1/2}$ and N is the stability frequency. If the wind speed is high and Δz exceeds the tallest hill, so $\Delta z > h_{\text{MAX}}$, the air will flow over the terrain. Otherwise, if $h_{\text{MAX}} > \Delta z$, a slow layer of "blocked" air will exist near the ground and pass around the hills. The blocked layer depth is $H = h_{\text{MAX}} - \Delta z$, while the effective peak height is $h_{\text{EFF}} \approx \Delta z = |\mathbf{U}|/N$. If the variance in (1) is proportional to the square of h_{EFF} , (1) gives a cubic relationship between drag and wind speed.

While this conceptual model is clear enough for simple isolated hills, it is difficult to implement quantitatively in complex terrain (Lott and Miller 1997). The difficulty arises because the depth of the blocked layer depends in the terrain height. If mountain peaks with $h = 1, 2$, and 3 km exist in close proximity and $\Delta z = 1$ km, the blocked layer has depths of $D = 0, 1$, and 2 km in the same region. This pattern of flow is difficult to visualize.

There are at least three additional problems with this popular conceptual model of blocking. First is the lack of explicit treatment of the lateral extent of the blocked layer. Unless the blocked layer extends very far, it must taper off, producing some large-scale "effective" terrain. This effective terrain will still generate waves, albeit with larger horizontal wavelength. These longer waves cannot be ignored. Unless they exceeded a wavelength of 1000 km or so, their wave drag could be significant.

A second problem is that the blocking algorithm is not well designed to account for cold air trapped in valleys (e.g., Whiteman et al. 1999; Sheridan et al. 2014; Kiefer and Zhong 2015). While strong ambient winds may flush out this cold air, its depth and extent may not scale with regional variables.

The third problem with the blocking algorithm is its influence on the wave spectrum. During slow-wind events, with the lower part of the terrain removed, the blocking algorithm is likely to shift the spectra to shorter waves with smaller horizontal scales, as the retained higher parts of the terrain are usually sharp and rugged. The smoother lower valleys would be removed. This problematic tendency for shorter waves under slow winds goes opposite to the findings of the recent Deep Propagating Gravity Wave Experiment (DEEPWAVE) project over New Zealand in 2014 (e.g., [Smith et al. 2016](#); [Smith and Kruse 2017](#)). From more than 90 transects across New Zealand, they found a tendency for the high-wind and high-drag events to have wave spectra shifted toward short waves. They concluded that it is the shift in spectral peak rather than an increase in wave amplitude that causes the increased wave drag.

A good example of this shift in spectral peak is the New Zealand wave event on 21 June 2014, which was observed during DEEPWAVE research flight 16 (RF16; [Smith et al. 2016](#)). This event had the strongest winds of the DEEPWAVE project and the highest wave drag. It was dominated by short wavelengths in the range of 20–40 km as opposed to weaker events with dominant wavelengths of 60–300 km. Both these wavelengths were in the hydrostatic range, so it is not the ability of the waves to propagate that matters; it is their generation.

In this paper, we focus on the terrain of the South Island of New Zealand. This terrain is high, rugged, highly anisotropic, and misaligned with the cardinal directions. It is surrounded by the featureless Tasmanian Sea and the Southern Ocean. It was the site of the 2014 DEEPWAVE project ([Bossert et al. 2015](#); [Fritts et al. 2016](#); [Gisinger et al. 2017](#)). The DEEPWAVE project provides a unique set of aircraft data ([Smith et al. 2016](#); [Smith and Kruse 2017](#)) and a well validated set of high-resolution Weather Research and Forecasting (WRF) Model simulations ([Kruse et al. 2016](#)). These resources allow us to propose and test a new hypothesis regarding wave drag on complex terrain.

3. Describing New Zealand's terrain

a. Volume and variance

To analyze the South Island of New Zealand, we use the standard global 30-arc-s elevation (GTOPO30) (~1-km terrain) dataset transformed to a local Cartesian coordinate system (x, y) using the New Zealand map grid (NZMG) conformal transformation. The highest peak in the Southern Alps is Mount Cook (Aoraki). The actual Mount Cook peak height of 3724 m

is only 3173 m in the original 1-km dataset. With the smoothing shown in [Table 1](#), the maximum height is further reduced. The terrain volume is

$$\text{Volume} = \iint h(x, y) dx dy \approx 94\,139 \text{ km}^3. \quad (2)$$

The height variance referenced to sea level is

$$\text{Variance} = \iint h^2(x, y) dx dy, \quad (3)$$

with units of m^4 . From the original 1-km dataset, Variance $\approx 92\,800 \text{ km}^4$. Values from (3) could be used in (1). To estimate a typical mountain height, we divide the variance by the volume:

$$h_T = \text{Variance}/\text{Volume}. \quad (4)$$

The typical hill height in (4) also decreases with smoothing, remaining at about half the maximum hill height ([Table 1](#)).

b. Anisotropy

The orientation and anisotropy of a mountain range are important as airflow parallel to a ridge will be less disturbed and will create less drag than airflow across a ridge. We examine two ways to represent terrain anisotropy: “second moments” and “slope variance.” For the second moment method, the centroid coordinates are computed from the first moments,

$$\bar{x} = \iint xh(x, y) dx dy / \text{Volume}, \quad (5a)$$

$$\bar{y} = \iint yh(x, y) dx dy / \text{Volume}, \quad (5b)$$

and the second moments are

$$A_{XX} = \iint (x - \bar{x})^2 h(x, y) dx dy, \quad (6a)$$

$$A_{XY} = A_{YX} = \iint (x - \bar{x})(y - \bar{y})h(x, y) dx dy, \quad (6b)$$

$$A_{YY} = \iint (y - \bar{y})^2 h(x, y) dx dy. \quad (6c)$$

The matrix of second moments,

$$\mathbf{A} = \begin{bmatrix} A_{XX} & A_{XY} \\ A_{YX} & A_{YY} \end{bmatrix}, \quad (7)$$

is symmetric; thus, it has real eigenvalues and orthogonal eigenvectors. Note the similarity of (5) and (6) to the “moment of inertia” calculations in mechanics.

TABLE 1. Characteristics of South Island terrain with various smoothings.

L (km)	Max height (m)	Variance (km^4)	Typical height (m)	$R_{\mathbf{A}}$	$\theta_{\mathbf{A}}$ ($^\circ$)	$R_{\mathbf{S}}$	$\theta_{\mathbf{S}}$ ($^\circ$)	$T_{\mathbf{S}}$
1	3173	92 821	986	16.3	-41.7°	1.4	-18.7°	45.8
10	1699	68 439	727	15.9	-41.7°	2.84	-44.5°	6.2
20	1389	61 096	649	14.7	-40.1°	4.45	-45.6°	4.4
30	1161	54 601	580	13.1	-41.7°	5.01	-44.6°	3.4
40	981	48 670	517	11.4	-41.6°	5.09	-43.7°	2.8
60	818	39 068	415	8.5	-41.4°	4.78	-42.9°	2.0

To determine the amount of anisotropy and its orientation, the eigenvalue equation

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad (8)$$

can be solved. In (8), λ is the eigenvalue associated with the eigenvector $\mathbf{v}$;

$$\lambda_{1,2} = \frac{1}{2}(A_{XX} + A_{YY}) \pm \frac{1}{2}[(A_{XX} - A_{YY})^2 + 4A_{XY}^2]^{1/2}. \quad (9)$$

These two eigenvalues $A_{11} = \lambda_1$ and $A_{22} = \lambda_2$ are the second moments of the terrain rotated into principal coordinates described by $\mathbf{v}$. The ratio of these second moments

$$R_{\mathbf{A}} = A_{11}/A_{22} \quad (10)$$

is a measure of the anisotropy of the terrain.

The angle of rotation to bring the terrain principal coordinates into the cardinal directions is given by the standard formula

$$\theta_T = \frac{1}{2} \text{atan}\left(\frac{2A_{XY}}{A_{XX} - A_{YY}}\right), \quad (11)$$

with some ambiguity regarding octant (Lubarda and Liu 2011).

An alternative measure of terrain anisotropy is the slope variance matrix proposed by Baines and Palmer (1990) and Lott and Miller (1997):

$$\mathbf{S} = \begin{bmatrix} S_{XX} & S_{XY} \\ S_{YX} & S_{YY} \end{bmatrix}, \quad (12)$$

with elements

$$S_{ij} = \iint \left(\frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} \right) dx dy, \quad \text{or} \\ S_{XX} = \iint \left(\frac{\partial h}{\partial x} \right)^2 dx dy, \quad S_{XY} = S_{YX} = \iint \left(\frac{\partial h}{\partial x} \right) \left(\frac{\partial h}{\partial y} \right) dx dy, \quad S_{YY} = \iint \left(\frac{\partial h}{\partial y} \right)^2 dx dy. \quad (13)$$

This matrix describes the variance and covariance of the east–west and north–south terrain slopes. It can also be computed in Fourier space using Parseval’s theorem. As this matrix is also symmetric, it will have real eigenvalues S_{11} and S_{22} from (9) and orthogonal eigenvectors. The first principal direction $\theta_{\mathbf{S}}$ is given by (11) using the elements in (13). A measure of the anisotropy is $R_{\mathbf{S}} = S_{11}/S_{22}$. An invariant measure of terrain roughness is the square root of the determinant of $\mathbf{S}$; that is, $T_{\mathbf{S}} = [\det(\mathbf{S})]^{1/2} = \sqrt{S_{11}S_{22}}$ (km^2). Note that in addition to being symmetric, matrices $\mathbf{A}$ and $\mathbf{S}$ are positive definite and thus have positive eigenvalues, so both $R_{\mathbf{A}}$ and $R_{\mathbf{S}}$ are positive, and $T_{\mathbf{S}}$ is real. These values for South Island, New Zealand (NZ), are given in Table 1 for various smoothings. Matrix $\mathbf{A}$ describes the large-scale terrain, while matrix $\mathbf{S}$ describes the small-scale terrain.

While the scalars volume and variance and matrices $\mathbf{A}$ and $\mathbf{S}$ compactly describe aspects of the South Island terrain, they are not expected to determine the wave drag as they do not characterize the terrain spectra. In a later section, we will use the terrain spectra (Young and Pielke 1983), denoted herein as $\hat{h}(k, l)$, to compute a wave drag matrix.

c. Roughness and smoothing

The roughness of terrain is important for wave drag. A rough terrain may generate gravity waves with shorter scales and more momentum transport (Smith and Kruse 2017).

Terrain smoothing may be done for several reasons. A terrain dataset may be smoothed and subsampled to reduce the size of the dataset. It may be smoothed to match the grid size of a numerical model. Herein, we

smooth to represent the nonlinear effect of stratification, weak winds, low-level blocking, and flow around steep terrain.

A convenient smoother computes the convolution of the actual terrain $h(x, y)$ with an isotropic Gaussian weighting function to give a smoothed terrain $h_G(x, y)$, so

$$h_G(x, y) = (2\pi L^2)^{-1} \iint h(x', y') \exp\left(-\frac{d^2}{2L^2}\right) dx' dy', \quad (14)$$

where $d^2 = (x - x')^2 + (y - y')^2$ and L is the smoothing length scale. Equivalently, the smoothing can be done with a Gaussian filter in wavenumber space. The original Fourier transformed terrain is

$$\hat{h}(k, l) = \iint h(x, y) \exp(-ikx - ily) dx dy, \quad (15)$$

and the transform of the smoothed terrain is

$$\widehat{h}_G(k, l) = \hat{h}(k, l) \exp[-(k^2 + l^2)L^2]. \quad (16)$$

The smoothed terrain is recovered from

$$h_G(x, y) = (2\pi)^{-2} \iint \widehat{h}_G(k, l) \exp(ikx + ily) dk dl. \quad (17)$$

The computation of (14) is fast because of its symmetry, but (15)–(17) can be combined with FFT wave computations to be even faster. An example of Gaussian smoothing is given in Fig. 1, using the FFT method.

The effect of smoothing is to spread out the terrain onto the sea, decreasing the mountain peaks and filling in the valleys. The terrain volume in (2) is unchanged by smoothing. Other effects of Gaussian smoothing are given in Table 1. Generally, smoothing has little impact on the terrain orientation (θ_A and θ_S). The smoothing decreases the large-scale anisotropy R_A as the smoothing kernel [(14) or (16)] is isotropic. With very strong smoothing, the terrain would approach the isotropic Gaussian smoothing kernel. The smoothing increases the small-scale anisotropy R_S as the smallest-scale terrain is rather isotropic. A small patch of the Southern Alps appears random and isotropic. As these small scales are removed by smoothing, the larger-scale anisotropy is revealed. The variance and roughness T_S are quickly reduced by smoothing. Note that while we have included statistics for the original terrain data in Table 1 (i.e., $L = 1$ km), we doubt that the wave field sees this finescale terrain. Even if it did, those short-wavelength waves would be nonhydrostatic and carry little momentum flux (Smith and Kruse 2017).

FIG. 1. South Island, NZ, terrain (m) from the 1-km GTOPO30 dataset (a) without smoothing and (b) smoothed with $L = 10$ km.

4. The WRF wave drag dataset for New Zealand

a. Data quality

The observational basis for the current study is a continuous full-physics WRF Model simulation of airflow over New Zealand, done for the DEEPWAVE project from June through August 2014. The mesoscale simulation was carried out with 6-km resolution with boundary conditions from MERRA, version 2 (MERRA-2), global simulations. This run was compared carefully with a prior run using the operational ECMWF analysis as boundary conditions (Kruse et al. 2016). The two runs compared very well. A detailed comparison with observations was carried out, including aircraft leg winds and temperatures and vertically smoothed balloon soundings. The agreement was excellent. The general meteorological conditions during this period are described by Gisinger et al. (2017).

From the complex WRF fields, wave momentum flux components were computed at various altitudes every 3 h from

$$\begin{aligned} \text{MF}_X &= \rho \iint u'w' dx dy \quad \text{and} \\ \text{MF}_Y &= \rho \iint v'w' dx dy \end{aligned} \quad (18)$$

over the entire South Island and a portion of the adjacent seas. Prior to this calculation, the velocity fields were spatially high-pass filtered to isolate gravity waves (Kruse and Smith 2015). Similarly, gravity wave momentum fluxes were computed along flight tracks and compared with aircraft data for each of 97 cross-mountain transects. Leg-by-leg fluxes compared moderately well between aircraft and WRF, while event-average fluxes agreed quantitatively (Kruse et al. 2016). A few additional WRF runs with 2-km resolution gave slightly larger momentum fluxes, but the comparison with data was not improved.

Detailed analysis of flight-level data for this period and location (Smith et al. 2016) showed that the observed flight-level disturbances had the characteristics of mountain waves. First, the fluxes were more than a factor of 10 higher over the mountains than over the sea. Second, all the zonal momentum fluxes were negative, while all of the vertical energy fluxes were positive. Third, the horizontal energy flux was oriented into the wind, as expected for stationary mountain waves. Fourth, the observed and modeled wave fields satisfied the theoretical relationship between momentum and vertical energy flux (Eliassen and Palm 1961; Smith et al. 2016);

$$\text{EF}_z = -\mathbf{U} \cdot \mathbf{MF} = -(UMF_X + VMF_Y). \quad (19)$$

In summary, this 3-month WRF DEEPWAVE simulation is the best-verified model-generated mountain wave dataset to date. Using this WRF dataset, we ask whether the gravity wave drag over NZ can be predicted from the regional wind vector only using a simple wave drag coefficient.

b. Wave drag anisotropy

A key question is whether the WRF drag vectors in (18) show anisotropy arising from the anisotropy of the New Zealand terrain. Our WRF dataset has 735 regional wind and wave drag vectors, with values every 3 h for a 3-month period (Figs. 2 and 3). Values of wind and wave drag are computed at $z = 4$ km, averaged and summed over the South Island and surrounding waters, respectively.

The wind directions over New Zealand are shown in Fig. 2 by plotting U versus V . The looping character of this scatter diagram indicates the passage of frontal

cyclones past NZ bringing a sequence of wind directions. The zonal wind component is usually positive (i.e., eastward), but the meridional component has both signs with nearly equal probability. Average values are $\bar{U} = 7.6 \text{ m s}^{-1}$ and $\bar{V} = 1.2 \text{ m s}^{-1}$, suggesting a slight northeastward preference (Gisinger et al. 2017). The Pearson correlation coefficient (CC) between U and V is slightly positive (i.e., $\text{CC} = 0.0265$), indicating a slight tendency for a northeastward prevailing wind. If the drag was in the same direction as the wind (e.g., drag law 1), the two drag components (D_X and D_Y) would have a similar small positive CC value. Instead, the WRF drag vectors in Fig. 3 show a preference for drag in the southeastward direction, perpendicular to the terrain orientation; D_X and D_Y are strongly negatively correlated, with $\text{CC} = -0.7276$. This negative correlation suggests a strong anisotropy in the WRF drag data.

c. Wave drag nonlinearity

Another key question is whether the WRF drag dataset from (18) exhibits nonlinearity. For this question, we plot the southeastward-oriented drag D_{SE} and wind component U_{SE} against each other (Fig. 4). This direction is very near the principal direction for this terrain (Table 1). The departure from linearity (i.e., drag law 1) is striking. A good fit to these data is a quadratic drag law

$$D_{SE}(U_{SE}) = C|U_{SE}|U_{SE}, \quad (20)$$

with $C = 0.2 \times 10^9 \text{ N m}^{-2} \text{ s}^2$.

5. Linear theory drag matrix and anisotropy

According to linear Boussinesq hydrostatic non-rotating theory of steady mountain waves, the vertical velocity field $w(x, y, z)$ above terrain $h(x, y)$ with background wind (U, V) can be written in Fourier space as (e.g., Smith 1980)

$$\hat{w}(k, l, z) = \hat{w}(k, l, z=0) \exp(imz). \quad (21)$$

In (21), the hydrostatic vertical wavenumber is

$$m = \frac{N\sqrt{(k^2 + l^2)}}{\sigma}, \quad (22)$$

the intrinsic frequency is $\sigma = Uk + Vl$, and the lower boundary condition is

$$\hat{w}(k, l, z=0) = (i\sigma)\hat{h}(k, l) = i(Uk + Vl)\hat{h}(k, l). \quad (23)$$

From (21) and (22) the transformed perturbation horizontal winds can be written using the conditions of continuity and zero vertical vorticity as

FIG. 2. Scatterplot between WRF 3-hourly wind components (U , V) over South Island for the period June–August 2014. Each point is a regional-average wind at altitude $z = 4$ km. The WRF simulation is forced from MERRA-2.

$$\hat{u} = -\left(\frac{mk}{k^2 + l^2}\right)\hat{w} \quad \text{and} \quad \hat{v} = -\left(\frac{ml}{k^2 + l^2}\right)\hat{w}. \quad (24)$$

The wave drag components (D_X and D_Y) on the terrain $h(x, y)$ can be written as the negative of the momentum flux. According to Parseval's theorem, we can integrate in Fourier space to obtain

$$D_X(U, V) = -MF_X = -\rho \iint \hat{u}\hat{w}^* dk dl = UB_{XX} + VB_{XY}, \quad (25a)$$

and

$$D_Y(U, V) = -MF_Y = -\rho \iint \hat{v}\hat{w}^* dk dl = UB_{XY} + VB_{YY}, \quad (25b)$$

where

$$\begin{aligned} B_{XX} &= \rho N \iint \frac{k^2 P(k, l)}{(k^2 + l^2)^{1/2}} dk dl, \\ B_{XY} &= B_{YX} = \rho N \iint \frac{k l P(k, l)}{(k^2 + l^2)^{1/2}} dk dl, \\ B_{YY} &= \rho N \iint \frac{l^2 P(k, l)}{(k^2 + l^2)^{1/2}} dk dl, \end{aligned} \quad (26)$$

and the terrain power spectrum is $P(k, l) = \hat{h}\hat{h}^* > 0$ using $\hat{h}(k, l)$ from (15).

These wave drag components can be computed using the 2D fast Fourier transform for any wind direction (Smith 1980). Equations (23)–(26) are similar to (6.2) in

FIG. 3. Scatterplot for the WRF 3-hourly wave drag components (D_X , D_Y) over South Island for the period June–August 2014. Each value is a regionally summed wave momentum flux at $z = 4$ km from a 6-km WRF run forced with MERRA-2 boundary conditions.

Phillips (1984), (18) in Gregory et al. (1998), and (19) in Teixeira and Miranda (2006).

To illustrate the properties of linear theory drag for complex terrain, we show in Fig. 5 the two components of wave drag for South Island as a function of wind direction (WD), with $L = 10$ km and $|U| = 10$ m s⁻¹. Starting with the eastward wind (WD = 0°), there is a positive D_X and negative D_Y . The drag vector is oriented to the southeast, approximately perpendicular to the terrain axis. With a northward wind (WD = 90°), both signs have changed: D_X is negative, and D_Y is positive. As the wind swings through 360°, both components oscillate sinusoidally. When the wind reverses, (i.e., a wind shift of 180°), both drag components exactly reverse. The drag vector magnitude $|D|$ reaches a maximum when the vector wind direction is approximately perpendicular to the terrain axis (i.e., WD ≈ 120°–150° and WD ≈ 300°–330°). The smooth sinusoidal behavior for the drag components is similar to the angular dependence derived by Phillips (1984) for a hill with elliptical contours. Note that the transverse drag is symmetrical in the following way (Fig. 5). An eastward wind ($U = 10$, $V = 0$) gives a meridional $D_Y = -40$ giganewtons (GN), while a northward wind ($U = 0$, $V = 10$) gives an identical zonal $D_X = -40$ GN.

The drag computation is greatly simplified by noting that (25) takes the form of a vector matrix multiplication. This occurs because U and V factor out of the intrinsic frequency in the Fourier integrals. Accordingly, we define the 2×2 drag matrix **B**:

$$\mathbf{B} = \begin{bmatrix} B_{XX} & B_{XY} \\ B_{YX} & B_{YY} \end{bmatrix} \quad (27)$$

FIG. 4. Southeastward wave drag on South Island vs southeastward wind speed from a WRF run for June–August 2014. The data are fitted with a quadratic drag law with coefficient $C = 0.2 \times 10^9 \text{ N m}^{-2} \text{ s}^2$.

so that the product of the wind vector $\mathbf{U} = (U, V)$ and the drag matrix $\mathbf{B}$ will give the drag vector $\mathbf{D} = (D_x, D_y)$:

$$\mathbf{U} \cdot \mathbf{B} = \mathbf{D}. \quad (28)$$

If $\mathbf{U}$ is a row vector, column vector $\mathbf{D}$ has elements composed of the dot product of $\mathbf{U}$ with the columns of $\mathbf{B}$. The ordered dot product notation in (28) is ambiguous, but with symmetric $\mathbf{B}$, it reduces to (25) whether vector $\mathbf{U}$ pre- or postmultiplies $\mathbf{B}$. The elements of $\mathbf{B}$ require only two FFT calculations for each terrain considered: one with eastward and one with northward wind. This calculation takes full account of the terrain spectrum.

A similar drag matrix approach was suggested by Garner (2005). Garner computed a velocity potential using Fourier methods but computed the matrix elements in physical space, not using Parseval's theorem [e.g., (26)]. His method would fail for hills with finite volume as the velocity potential integral does not converge.

The diagonal elements of $\mathbf{B}$ from (27), B_{XX} and B_{YY} , are positive as the terrain power spectrum $P(k, l) > 0$ for all (k, l) . The off-diagonal elements of $\mathbf{B}$ (i.e., B_{XY}) can be of either sign. They represent the “transverse drag”: the drag component perpendicular to the wind vector. For example, if the wind were eastward $\mathbf{U} = (U, 0)$, the transverse drag is the component toward the north or south D_y . Evidence that matrix $\mathbf{B}$ is symmetric can be seen in Fig. 5 and is clear from the form of B_{XY} and B_{YX} in (26). Because of this symmetry, only three numbers are needed to specify matrix $\mathbf{B}$ for a given terrain. Other properties of the wave drag matrix are discussed in the appendix.

Another important property of $\mathbf{B}$ is that it is “positive definite.” This property requires

FIG. 5. The zonal (D_x) and meridional (D_y) wave drag components and the drag magnitude $|D|$ for South Island as a function of wind direction with $L = 10 \text{ km}$ and $|\mathbf{U}| = 10 \text{ m s}^{-1}$. Drag components are computed from linear hydrostatic theory using a 2D FFT routine. Wind direction is counterclockwise from eastward. A northwesterly wind has $\text{WD} = 315^\circ$.

$$B_{XX} > 0 \quad \text{and} \quad \det(\mathbf{B}) = B_{XX}B_{YY} - B_{XY}^2 > 0 \quad (29)$$

and guarantees that the dot product of the wind and drag vectors is positive for any wind direction. This dot product is related to the vertical energy flux in the wave field in (19):

$$\mathbf{U} \cdot \mathbf{D} = \text{EF}_z = \iint p'w' dx dy > 0 \quad (30)$$

(Eliassen and Palm 1961; Smith et al. 2008, 2016). A positive EF_z is the causality condition for vertically propagating mountain waves. In practice, this constraint [(29)] limits the magnitude of the off-diagonal drag matrix elements B_{XY} . Conditions (28) and (30) (i.e., scalar $\mathbf{x}^T \mathbf{B} \mathbf{x} > 0$ for any nonzero column vector $\mathbf{x}$) compose the “energy based” definition of a positive-definite matrix (Strang 2016).

As matrix $\mathbf{B}$ is symmetric (27), it has real eigenvalues and orthogonal eigenvectors. They are found from

$$\mathbf{B} \mathbf{v} = \lambda \mathbf{v}, \quad (31)$$

as was done for the terrain analysis in (9). The eigenvectors define the principal directions for which the drag vector is parallel to the wind vector. The eigenvalues are the drag coefficients for airflow along the principal directions [see (11)]. For positive-definite $\mathbf{B}$, the eigenvalues are positive. We define the principal drag coefficients as $B_{11} = \lambda_1$ and $B_{22} = \lambda_2$. The ratio of the principal drag elements,

$$R_B = \frac{B_{11}}{B_{22}}, \quad (32)$$

TABLE 2. The elements of the drag matrix for South Island, its eigenvalues, and other metrics with different smoothings ($N = 0.01 \text{ s}^{-1}$ and $\rho = 1.2 \text{ kg m}^{-3}$). Units of $\mathbf{B}$ are GN s m^{-1} . The first row represents reference values.

L (km)	B_{XX}	B_{XY}	B_{YY}	B_{11}	B_{22}	$R_{\mathbf{B}}$	$\theta_{\mathbf{B}}(^{\circ})$	B_{ISO}
10	6.52	-4.10	6.69	10.7	2.50	4.3	-44.8	5.17
20	4.1	-2.7	4.375	6.95	1.54	4.5	-43.4	3.27
30	2.80	-1.82	3.09	4.77	1.11	4.3	-42.7	2.30
40	1.99	-1.25	2.22	3.36	0.85	4.0	-42.4	1.69
60	1.07	-0.62	1.20	1.75	0.52	3.4	-41.8	0.95

is a measure of the drag anisotropy. The determinant of the drag matrix is unchanged by rotation into the principal coordinates, so $\det(\mathbf{B}) = B_{11}B_{22}$. The trace of $\mathbf{B}$ is also unchanged by rotation, so $B_{XX} + B_{YY} = B_{11} + B_{22}$. The angle of the first principal direction $\theta_{\mathbf{B}}$ with respect to the cardinal direction is given by (11). The maximum and minimum $|D|$ in Fig. 5 can be computed using the eigenvalues B_{11} and B_{22} in Table 2.

For South Island, the elements of the drag matrix are shown in Table 2. We do not show computations with smoothing scale less than $L = 10 \text{ km}$ as those terrains may generate nonhydrostatic waves, which violate our hydrostatic assumption. The off-diagonal B_{XY} elements are nearly as large in magnitude as the diagonal components (B_{XX} and B_{YY}), but the matrix is still positive definite [(29)]. The large off-diagonal elements indicate that the terrain is very anisotropic and that the principal directions are very different than the eastward and northward cardinal directions. The negative sign of B_{XY} arises from the northeast–southwest orientation of the terrain.

The drag anisotropy $R_{\mathbf{B}}$ is large (i.e., 3.4–4.5) and similar to the slope variance anisotropy $R_{\mathbf{S}}$ in Table 1. The first principal directions are in the range -41° to -45° (i.e., wind toward the southeast). It differs from the principal directions of the terrain in Table 1 by only a few degrees.

The last column in Table 2 gives the diagonal elements of a “scalar” drag matrix for an imagined isotropic version of the South Island with the same determinant as (27). This drag matrix is

$$\mathbf{B}_{\text{ISO}} = \begin{bmatrix} B_{\text{ISO}} & 0 \\ 0 & B_{\text{ISO}} \end{bmatrix}, \quad (33)$$

where the diagonal elements are $B_{\text{ISO}} = \sqrt{\det(\mathbf{B})} = \sqrt{B_{11}B_{22}}$.

Table 2 also shows how the drag matrix elements in (26) change with terrain smoothing. The matrix elements decrease strongly with smoothing, as smoothing eliminates the smaller terrain scales that contribute strongly to the total drag (Smith and Kruse 2017). The drag anisotropy $R_{\mathbf{B}}$ decreases only slightly, and the orientation of the

FIG. 6. The principal drag matrix elements B_{11} and B_{22} computed from (9) and (26) for South Island as a function of smoothing scale L (m). Values with units of $\text{N m}^{-1} \text{ s}$ represent the ratio of drag to wind speed for wind direction along the principal axes. The drag decreases as smoothing removes the short-scale rough terrain. Approximate fits (dashed) are shown with drag inversely proportional to L and referenced to $L = 10 \text{ km}$.

principal axes barely changes at all with smoothing. To illustrate the change in drag with smoothing, we plot the principal matrix elements B_{11} and B_{22} versus the smoothing length L in Fig. 6. Both principal elements are approximately inversely proportional to L .

For convenience, we assume that all the drag matrix elements follow the same inverse scaling with respect to L . Defining $\tilde{L} = 10 \text{ km}$ as the reference value of L , we write

$$B_{ij}(L) \approx \tilde{B}_{ij} \left(\frac{\tilde{L}}{L} \right), \quad (34)$$

where values $\tilde{B}_{ij}$ are given in the top row of Table 2. For example, the first principal element $\tilde{B}_{11} = 10.7 \times 10^9 \text{ GN m}^{-1} \text{ s}$, so smoothings $L = 20, 30, 40$, and 60 km give $B_{11} = 5.4, 3.6, 2.7$, and $1.8 \times 10^9 \text{ N m}^{-1} \text{ s}$, respectively. With this common scaling, the drag anisotropy is constant with $R_{\mathbf{B}} = 4.3$, and the angle of the principal coordinates is constant at $\theta_{\mathbf{B}} = -44.8^{\circ}$.

We could probably find a better fit than (34) for New Zealand, but it would not be universal. The form of (34) depends on the terrain power spectrum. Terrains with less small-scale roughness will have a slower decrease than inverse L .

6. Wave drag nonlinearity and smoothing

The importance of nonlinearity is widely accepted in the wave drag literature and supported by (20) for the

WRF drag dataset. To avoid the problematic “airflow-blocking algorithm,” we implement a new variable smoothing approach. Stronger terrain smoothing is used at low wind speeds. At low wind speeds, upstream blocking and flow around isolated peaks tend to smooth the effective terrain shape. Cold air trapped in valleys has a similar smoothing effect. With stronger winds, the airflow is able to follow the rough terrain more accurately. It overcomes the smoothing effects of stratification. Winds can flow up steep slopes and flush out deep valleys. The smoothing algorithm is easy to apply and avoids any untreated blocked layer (section 2). Utilizing smoothing to implement drag nonlinearity finds support in aircraft and modeling data that show the largest drag events are those with lots of short gravity waves with wavelengths in the range from 20 to 60 km (Smith and Kruse 2017).

To quantify this proposal, we test the idea that the smoothing scale L is inversely proportional to the wind speed. Introducing a constant factor F ($\text{m}^2 \text{s}^{-1}$), this is

$$L = F/|\mathbf{U}|. \quad (35)$$

Equation (35) together with (34) will recover the quadratic drag law $B_{ij} \sim |\mathbf{U}|$ seen in Fig. 4 and (20). To determine F , we assume that the southeastward direction is close to the first principal direction. Then using (20), (34), and $\tilde{B}_{11}$ from Table 2, we have

$$B_{11} = C|\mathbf{U}| = \tilde{B}_{11} \left(\frac{\tilde{L}}{L} \right) = \tilde{B}_{11} \tilde{L} |\mathbf{U}| / F. \quad (36)$$

Solving for F , we obtain $F = \tilde{B}_{11} (\tilde{L}/C) \approx 535 \times 10^3 \text{ m}^2 \text{s}^{-1}$, so (35) becomes

$$L(\text{km}) \approx 535/|\mathbf{U}|. \quad (37)$$

If one accepts the premises of this section, (37) gives the smoothing length scale for each wind speed. Wind speeds of $|\mathbf{U}| = 10, 20$, and 30 m s^{-1} give scales of $L = 54, 27$, and 18 km , respectively. This empirical description represents the physics of stratified atmospheric boundary layer over complex terrain. Slower winds give effectively smoother terrain and smaller drag coefficients. Functionally, the drag matrix is $\mathbf{B} = \mathbf{B}(L)$, where $L = L(|\mathbf{U}|)$.

7. Comparing wave drag laws with WRF drag time series

The goal of this section is to compare the matrix drag law [(25), (26), (34), and (37)] against the WRF wave drag time series. To review, our drag law uses the instantaneous regional wind speed to compute the smoothing length L and then uses the corresponding drag matrix $\mathbf{B}(L)$ to compute

FIG. 7. Drag component (D_x, D_y) scatterplot derived from multiplying the 3-hourly wind vector $\mathbf{U} = (U, V)$ times the drag matrix $\mathbf{B}(L)$ for the full anisotropic nonlinear model.

the drag vector. In Fig. 7, we show a scatterplot of the 735 three-hourly drag vectors predicted from the nonlinear anisotropic theory. The matrix theory rotates the drag nicely into the southeastward direction similar to the WRF drag in Fig. 3.

In Figs. 8 and 9, we show the time series of zonal and meridional WRF drag components and the predicted drag components from the full matrix model. The agreement for both components appears good, and the Pearson correlation coefficients are $CC = 0.91$ and $CC = 0.87$, respectively, for D_x and D_y . Note the episodic nature of the drag time series caused by the sensitivity of drag to wind speed and direction. A typical high-drag event only lasts for 24–48 h. The sharp peak at day 21 corresponds to the large aircraft-observed momentum flux on 21 June 2014 discussed in section 2.

To see whether our treatments of anisotropy and nonlinearity significantly improve drag predictions, we degrade the full model (say, drag law 4) to three simpler drag laws (drag laws 1–3) in Table 3. Drag law 1 is linear and isotropic. Drag laws 2 and 3 include either anisotropy or nonlinearity. Drag law 4 has both. To linearize the model, we use the RMS wind speed for the full test period $|\mathbf{U}| = 14.4 \text{ m s}^{-1}$ to determine a fixed value for smoothing length scale. From (37), this gives $L = 33 \text{ km}$. To make the model isotropic, we imagine an isotropic South Island, with a “scalar” drag matrix B_{ISO} given in Table 2 and (33).

The Pearson CCs rise from $CC = 0.75$ and 0.71 for the degraded linear isotropic model (drag law 1) to $CC = 0.91$ and 0.87 for the full model (drag law 4). The mean absolute error (MAE) decreases from $\text{MAE} = 11.9$ and 10.0 GN to $\text{MAE} = 7.8$ and 7.3 GN . Likewise, the root-mean-square

FIG. 8. Time series of WRF zonal wave drag D_x (blue) compared with the prediction (red) from the drag matrix $\mathbf{B}(L)$. The time axis is days after 1 Jun 2014. Most zonal drag values are positive (i.e., eastward).

FIG. 9. Time series of WRF meridional wave drag D_y (blue) compared with the prediction (red) from the drag matrix $\mathbf{B}(L)$. The time axis is days after 1 Jun 2014. Most meridional drag values are negative (i.e., southward).

error (RMSE) decreases from model 1 to model 4. By all of these measures, the full nonlinear anisotropic model makes the best prediction.

8. Summary and conclusions

We have shown that a 2×2 drag matrix with variable elements $\mathbf{B}(L)$ can accurately predict the time series of mountain wave drag on the South Island of New Zealand. To obtain good agreement, it is necessary to include both terrain anisotropy and dynamic nonlinearity. The anisotropy is handled using a linear hydrostatic positive-definite wave drag matrix. The nonlinearity is handled using variable terrain smoothing as a function of wind speed. This approach is consistent with recent observational results (Smith and Kruse 2017) that large drag events are caused by a dominance of short gravity waves that carry more momentum flux. The representation using a drag matrix is a significant simplification over the previous methods.

The proposed drag matrix formulation requires that we neglect the Coriolis force and nonhydrostatic effects. These assumptions are well supported by previous work and recent aircraft wave observations (Smith et al. 2016; Smith and Kruse 2017). The hydrostatic assumption can lead to large errors if applied to small-scale rough terrain. We avoid this problem by always smoothing the terrain into the hydrostatic range. For simplicity, we have also taken the stability frequency N and the air density ρ as constant. Perhaps it is surprising that the drag vector can be well forecast knowing only the regional wind vector. No meteorological information

regarding fronts, vertical wind shear, static stability, moisture, or diurnal/seasonal solar insolation is needed.

Two mathematical properties of the wave drag matrix $\mathbf{B}$ are important. Matrix symmetry $\mathbf{B} = \mathbf{B}^T$ leads to real eigenvalues and orthogonal eigenvectors. This property guarantees that there will be two perpendicular principal wind directions that will generate no transverse drag. A measure of the drag anisotropy is the ratio of the two real eigenvalues. The wave drag matrix is also “positive definite,” guaranteeing that vertical wave energy flux will be positive for any wind direction and that both eigenvalues will be positive. The drag matrix $\mathbf{B}$ is compared with two other symmetric matrices $\mathbf{A}$ and $\mathbf{S}$ representing different aspects of terrain anisotropy.

The nonlinearity in wave generation is treated by smoothing the terrain to qualitatively take into account upstream blocking, flow around steep isolated peaks, and cold air trapped in valleys. Wind speeds of $|\mathbf{U}| = 10, 20$, and 30 m s^{-1} give smoothing scales of $L = 54, 27$, and 18 km , respectively. At slow wind speeds, we find that a smoothing scale as long as $L = 60 \text{ km}$ is suitable. This eliminates wave drag contributions from short-scale waves. As wind speed increases, we reduce the smoothing scale to almost $L = 10 \text{ km}$ to retain rougher terrain and shorter waves. The elements of the drag coefficient matrix $\mathbf{B}(L)$ increase accordingly. We retain the hydrostatic assumption throughout.

The increased terrain smoothing at low wind speeds works to decrease the typical mountain height h_T and maintain some validity of the linear theory. For example, with a slow wind of $U = 10 \text{ m s}^{-1}$, the smoothing scale climbs to 54 km , and the $h_T \approx 450 \text{ m}$ (Table 1).

TABLE 3. Error characteristics of four wave drag formulations for zonal (D_X) and meridional (D_Y) components.

Model	Anisotropic	Nonlinear	CC D_X	CC D_Y	MAE D_X	MAE D_Y	RMSE D_X	RMSE D_Y
1	No	No	0.75	0.71	11.9	10.0	20.5	15.5
2	Yes	No	0.84	0.81	10.8	10.3	15.8	13.2
3	No	Yes	0.81	0.75	10.6	8.5	18.9	14.7
4	Yes	Yes	0.91	0.87	7.8	7.3	12.5	11.2

With $N = 0.01 \text{ s}^{-1}$, the nonlinearity parameter $Nh_T/U \approx 0.45$, within a reasonable range.

For South Island, the drag matrix has principal axes oriented at about -42° to the cardinal eastward direction. This direction agrees with the NZ orientation determined from the second moments (**A**) and the slope variance (**S**) of the terrain. The drag anisotropy is large: approximately 4. Strong winds across the terrain from the northwest can generate drags in excess of 100 GN. The combination of the anisotropy and nonlinearity gives the drag time series an irregular “episodic” characteristic. The drags are usually small, but they can rise quickly to large values when the wind blows strongly directly across the terrain. A linear isotropic model (e.g., drag law 1) would give a smoother and less variable drag time series.

The quantitative success of the theory may be caused in part by the large area of South Island, over which we average. Wave drag on such a large area seems to be very predictable if U and V are known. Drag in a smaller area (e.g., a $20 \text{ km} \times 20 \text{ km}$ cell) may be more difficult to predict as special types of terrain or wave dynamics may apply locally (Bacmeister and Pierrehumbert 1998; Rottman and Smith 1989). On the other hand, our use of a single averaged environmental wind vector (U , V) and drag vector (D_X , D_Y) may degrade the accuracy because of heterogeneities within such a large region. We have not investigated the sensitivity to averaging area.

The current method could be applied to other mountainous regions. The first step is to perform two linear theory FFT hydrostatic drag calculations along cardinal directions to determine the three elements of the drag matrix (B_{XX} , B_{XY} , and B_{YY}). Repeat for three or four different smoothing lengths L . This procedure will account for the roughness of the target terrain. Finally, select a $L(|U|)$ function, either using ours [(37)] or using WRF simulations to derive your own. Operationally, drag is determined quickly as the product of a 2×2 drag matrix and the regional wind vector.

The application of these results to wave drag parameterization design will vary depending on the grid size of the global model (Vosper et al. 2016). In general, the subgrid parameterized terrain [i.e., $h(x, y)$] will be the residual after the original terrain is smoothed to fit the GCM grid. This residual terrain may have zero volume and a reduced variance. The residual terrain will

be used to derive the drag matrix **B**(L). As the model grid cell shrinks, the shorter terrain scales will play a larger role in the subgrid wave drag. The larger-scale waves will be captured in the GCM grid.

Finally, the wave drag predicted here is not the full representation of momentum transfer between Earth and the atmosphere. Pressure gradients from cold-air damming or from geostrophic flow can act on mountains but are not included in wave drag. Turbulent wind stress on small hills and vegetation is also not included.

Acknowledgments. This work was supported by Grant NSF-AGS-1338655 from the Physical and Dynamical Meteorology branch of the Geoscience division of the National Science Foundation. Terrain datasets were prepared by Laurent Bonneau. A discussion with Julio Bacmeister was helpful. Thanks also to David Fritts, James Doyle, Steve Eckermann, and Michael Taylor for their work in organizing the DEEPWAVE field project in New Zealand, which motivated this work. We acknowledge the high-performance computing from Cheyenne provided by the NCAR’s Computational and Information Systems laboratory sponsored by the National Science Foundation. Three anonymous reviewers offered extensive useful comments.

APPENDIX

Further Properties of the Drag Matrix

a. Axisymmetric terrain

If the terrain is axisymmetric so that $h(x, y)$ can be written as $h(r)$, where $r = (x^2 + y^2)^{1/2}$, then $P(k, l)$ is isotropic and $B_{XY} = 0$ because of integrand cancellation in (26) between the first and third quadrants where $kl > 0$ and the second and fourth quadrants where $kl < 0$. The diagonal terms are equal, so $B_{XX} = B_{YY}$. The drag vector is parallel to the wind vector, and the drag magnitude is independent of wind direction.

b. Proof of positive definiteness

To see why $\det(\mathbf{B}) > 0$ from (26), we consider how we might approach the limiting case of $\det(\mathbf{B}) = 0$. For this, we must have the terrain power spectrum $P(k, l) = \hat{h}\hat{h}^* > 0$ focused along radial lines in (k, l) space. For example, if

$P > 0$ for $k = l$ and $P = 0$ otherwise, then $k^2 = kl$ and $l^2 = kl$ in the integrand, so $B_{XX} = B_{YY} = B_{XY}$. It follows that $\det(\mathbf{B}) = B_{XX}B_{YY} - B_{XY}^2 = 0$. Likewise, if $P > 0$ for $k = -l$ only, then $k^2 = -kl$ and $l^2 = -kl$ in the integrand so $B_{XX} = B_{YY} = -B_{XY}$, so again $\det(\mathbf{B}) = B_{XX}B_{YY} - B_{XY}^2 = 0$. These conditions correspond to an infinite ridge with orientation of $\pm 45^\circ$. The same principle applies if the infinite ridge is oriented along the cardinal directions. A north–south terrain with $P > 0$ only for $l = 0$ has $B_{XX} > 0$ while $B_{XY} = B_{YY} = 0$ so $\det(\mathbf{B}) = 0$. For compact terrains with $P > 0$ off the radial line, $\det(\mathbf{B}) > 0$.

c. Unphysical drag matrix

Consider an arbitrary symmetric matrix with dominant off-diagonal elements such as

$$\mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},$$

with $\det(\mathbf{B}) < 0$. This drag matrix is not positive definite and therefore is unphysical in the present context. A northwesterly $\mathbf{U} = (1, -1)$ or southeasterly $\mathbf{U} = (-1, 1)$ wind would give $\mathbf{U} \cdot \mathbf{D} = -2$, violating (30). The angle between the wind and drag vectors exceeds 90° .

d. Rotated terrain

If a terrain is rotated counterclockwise by angle θ , the drag matrix $\mathbf{B}_1$ is altered to $\mathbf{B}_2$. There is no need to recompute the drag matrix elements from (26). Instead, using the orthogonal rotation matrix

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix},$$

we have $\mathbf{B}_2 = \mathbf{Q}^T \mathbf{B}_1 \mathbf{Q}$.

e. Adding terrains

Consider two terrains $h_1(x, y)$ and $h_2(x, y)$ adding to give $h_3(x, y) = h_1(x, y) + h_2(x, y)$. If both terrains are positive or zero everywhere but not overlapping so that

$$\iint h_1(x, y)h_2(x, y) dx dy = 0,$$

their variances [see (3)] are additive, but their drag matrices are only approximately additive;

$$\mathbf{B}_3 \approx \mathbf{B}_1 + \mathbf{B}_2.$$

The inequality arises from the influence of one terrain on the pressure field over the other terrain.

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