Linear Stability of Moist Convecting Atmospheres. Part I: From Linear Response Functions to a Simple Model and Applications to Convectively Coupled Waves

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ABSTRACT

A procedure is presented to systematically construct simple models for the linear stability of moist convecting atmospheres. First, linear response functions of a cumulus ensemble constructed from cloud-system-resolving models are coupled with matrices expressing two-dimensional large-scale linear wave dynamics. For a radiative–convective equilibrium reference state, this model gives two branches of unstable modes: a propagating convectively coupled wave branch and a stationary branch related to storage of column-integrated moist static energy (MSE). The stationary branch is unstable only when radiative feedback is included, while the convectively coupled wave branch is less affected by radiative feedback. With a modular order-reduction procedure from control theory, the linear-response-function-based model is reduced to a system of six ordinary differential equations while still capturing the essential features of the unstable modes (eigenvalues and structures). The six-dimensional system is then split into a slow and a fast manifold. The slow manifold (again, reflecting column MSE storage) is essential for the stationary mode but not for the convectively coupled waves. The fast manifold is then transformed into a form similar to that of prior simple models of convectively coupled waves, thus placing those models and the insights derived from them on a firmer footing. The procedure also better quantifies the parameters of such simple models and allows the stability difference between different reference states to be better understood.

1. Introduction

Cloud-system-resolving model (CSRM) simulations subject to horizontally or zonally homogeneous boundary and forcing conditions can spontaneously develop large-scale circulations in the homogeneous direction. One example is the convectively coupled waves, which can develop without feedbacks from radiation or surface fluxes (e.g., Grabowski and Moncrieff 2001; Tulich et al. 2007; Kuang 2008a; Nasuno et al. 2008; Blanco et al. 2016), and appear comparable in structure and propagation characteristics to such waves observed in nature (e.g., Chang 1970; Hendon and Liebmann 1991; Takayabu and Nitta 1993; Takayabu 1994; Wheeler and Kiladis 1999; Wheeler et al. 2000; Straub and Kiladis 2002; Haertel and Kiladis 2004). Readers are referred to Kiladis et al. (2009) for a review and a historical account

of the observational studies. With additional feedbacks, particularly the radiative feedback, another type of spontaneous development of large-scale circulations appears possible, and the resulting slow-moving or stationary disturbances do not align with the dispersion curves of gravity waves or equatorial waves. This process in closed or periodic numerical domains is known as convective self-aggregation (e.g., Bretherton et al. 2005; Muller and Held 2012; Wing and Emanuel 2014; Holloway and Woolnough 2016; Wing and Cronin 2016; Holloway 2017; Wing et al. 2017). Readers are referred to Mapes (2016) and Wing et al. (2017) for recent reviews. Its connection to the observed Madden–Julian oscillation (MJO) has been suggested (Arnold and Randall 2015; Holloway et al. 2017).

A number of simple models have been proposed for the convectively coupled waves (e.g., Lindzen 1974; Emanuel 1987; Neelin et al. 1987; Wang 1988; Mapes 2000; Khouider and Majda 2006; Fuchs and Raymond 2007; Kuang 2008b, hereafter K08) and convective self-aggregation (Bretherton et al. 2005; Emanuel et al. 2014). These simple models were constructed to facilitate

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understanding, not for quantitative accuracy, and have provided valuable insights into the potential mechanisms of these phenomena. However, treatment of convection in these models largely relied on the physical intuitions of their creators instead of systematic simplifications or approximations of more complete models.

In this study, I shall consider a CSRM to provide an adequate, or at least relevant, representation of moist convecting atmospheres. For the convectively coupled waves, the resemblance between simulated and observed wave structures and dispersion relationships bolsters confidence in such an assumption. For the convective self-aggregation problem, where the dependence on numerical resolution and model configurations is stronger (e.g., Muller and Held 2012), confidence in this assumption is lower. Still, there is value in understanding such behaviors of the CSRM, notwithstanding the potential biases.

For understanding the linear stability of moist convecting atmospheres (as represented by the CSRMs) or their responses to weak forcing, linear response functions constructed from horizontally cyclic limited-domain CSRMs usefully encapsulate the macroscopic behavior of moist convection. When coupled with linearized large-scale dynamics, this approach can provide the needed linear models for such problems. This was used to study the convectively coupled waves (Kuang 2010) and weakly forced mock Walker cells (Kuang 2012). The latter study also noted the need to match the form of convection in the cyclic limited-domain CSRM (e.g., the extent of its organization) with the form of convection in the large-scale moist convecting atmospheres that one seeks to understand.

Models based on the linear response functions, however, are of too-high dimensions to analyze like the simple models referred to earlier. The purpose of this study is to bridge this gap, by developing simple models and constraining their parameters more systematically from the linear-response-functions-based models, with the simplifications and idealizations taken made clearer. This will place the simple models, and the insights derived from them, on a firmer footing.

This paper is Part I of a two-part study, presenting this procedure and its application to convectively coupled waves. The application to convective self-aggregation is described in Part II.

The rest of the paper is organized as follows. Section 2 describes the CSRM, the simulation setups, and the procedure used to construct the linear response functions. Section 3 presents the linear stability results from the linear-response-functions-based models. Section 4 presents the results from a model order-reduction procedure that reduces the problem into a set of six coupled

ordinary differential equations (ODEs). In section 5, I split the system into a fast and a slow manifold and show that the slow manifold is essential for modes that resemble convective self-aggregation but not for the convectively coupled waves. Section 6 describes further simplifications that transform the fast manifold dynamics to a form similar to that of previous simple models of convectively coupled waves and discusses the implications to convectively coupled wave dynamics. The model is then applied to understand the difference in the stability of convectively coupled waves between the two mean states presented in Kuang (2010). Concluding remarks are presented in section 7.

2. Construction of the linear response functions from CSRMs

a. Description of the CSRM and simulation setups

All CSRM experiments were performed with the System for Atmospheric Modeling (SAM), version 6.7.5. A description of an earlier version of this model is given in Khairoutdinov and Randall (2003). The model solves the anelastic equations of motion. The prognostic thermodynamic variables are the liquid water static energy, total nonprecipitating water, and total precipitating water. I use a bulk microphysics scheme and a simple Smagorinsky-type scheme to parameterize the effect of subgrid-scale turbulence. Surface latent and sensible heat fluxes are computed using a bulk aerodynamic formula with a constant 10-m exchange coefficient of 1×10^{-3} and a constant surface wind speed of 5 m s^{-1} to eliminate wind-induced surface heat exchange, the effects of which will be explored in the future. Therefore, for convective self-aggregation, I will be focusing on the problem of linear radiative-convective instability, as emphasized in, for example, Emanuel et al. (2014). Surface momentum fluxes are computed with the Monin-Obukhov similarity theory. Radiation is computed using the National Center for Atmospheric Research Community Atmosphere Model (CAM) radiation package (Collins et al. 2006). Shorter integrations using RRTM radiation (Iacono et al. 2008) gave similar results. For simplicity, I have removed the diurnal cycle by setting the solar zenith angle constant at 51.7° and the solar constant at 685 W m^{-2} .

All experiments are over an ocean surface with fixed temperature and doubly periodic lateral boundary conditions. The domain size is $128 \text{ km} \times 128 \text{ km}$ in the horizontal with a 4-km horizontal resolution. There are 28 vertical layers that extend from the surface to 32 km, the top third of the domain being a wave-absorbing layer, similar to that used in the superparameterized

CAM (SPCAM), which have been shown to produce convectively coupled waves, convective self-aggregation, and the MJO (e.g., [Khairoutdinov et al. 2008](#); [Arnold and Randall 2015](#)). The relatively coarse horizontal and vertical resolutions were chosen both to reduce the cost associated with the calculations of the linear response functions and to make the results directly applicable to the interpretation of SPCAM results in the future.

b. Reference mean states

Two mean states are used in this paper (Part I), and more cases will be discussed in Part II.

The first mean state is that of a radiative–convective equilibrium (RCE) with zero horizontal-mean vertical velocity over the CSRM domain. The sea surface temperature (SST) is set to 30°C to facilitate direct comparisons with [Emanuel et al. \(2014\)](#). Fully interactive radiative feedback is included in this case.

In the other reference state, the domain-mean vertical velocity profile is set to be the mean vertical velocity profile over the large-scale array during the intensive operating period of the Tropical Ocean and Global Atmosphere Coupled Ocean–Atmosphere Response Experiment (TOGA COARE; [Webster and Lukas 1992](#)), as shown in Fig. 1 of [Kuang \(2008a\)](#). The SST is 29.5°C. Based on results for RCE over a range of SSTs (not shown), the SST difference between the RCE and TOGA cases is inconsequential for their linear stability differences discussed in this paper. Radiative cooling is prescribed to be the long-term averages of a control run with interactive radiation. This case will be referred to as the TOGA case. The main purpose of the TOGA case is to elucidate the dependence of convectively coupled waves on the mean state.

c. Construction of the linear response functions

The construction of the linear response functions from cyclic limited-domain CSRMs follows [Kuang \(2010, 2012\)](#) and is briefly described below.

Define the departures of horizontally averaged profiles of temperature T and moisture q of the limited-domain CSRM from a given reference state to constitute the mean-field state vector, denoted as $\mathbf{x}$. If I assume the small-scale convective motions to be in statistical equilibrium with the (mean field) state vector, the expected values of the horizontally averaged convective tendencies $(d\mathbf{x}/dt)_{\text{conv}}$ are unique functions of the state vector. The linear response functions are a linear approximation of this function around the given reference state.

To construct the linear response functions, I added a set of sufficiently complete, time-invariant, horizontally homogeneous, anomalous temperature or moisture tendencies, one at a time, to the forcing of the CSRM.

The full set of the anomalous forcing is denoted as $\mathbf{F}$, the columns of which are the forcing tendencies added in each of the experiments. The specific form and amplitude of the forcings follow those of [Kuang \(2012\)](#). We then ran the CSRM to statistical steady state and averaged the departures of the horizontal-mean temperature and moisture profiles from those of a control experiment without the anomalous tendencies over a long period (10 000 days in the present case). As stratosphere water vapor does not actively participate in the convective process, specific humidity above 150 hPa is considered slaved to the state below and excluded from the state vector. This set of the anomalous (mean field) state vectors, one for each forcing, is denoted as $\mathbf{X}$. Averaged over a long period in statistical steady state, convective tendencies are precisely balanced by the imposed anomalous forcing. Therefore, the linear response functions $\mathbf{M}$ can be constructed through a matrix inversion, $\mathbf{M} = -\mathbf{F}\mathbf{X}^{-1}$ so that

$$\left(\frac{d\mathbf{x}}{dt}\right)_{\text{conv}} = \mathbf{M}\mathbf{x}. \quad (1)$$

The construction is most (least) accurate for eigenmodes of $\mathbf{M}$ associated with the smallest (largest) (in modulus) eigenvalues. There are sometimes large positive eigenvalues. Since the limited-domain CSRM evidently has a stable steady state, we set those positive eigenvalues to -48 day^{-1} . The precise value of this assigned large negative eigenvalue is inconsequential when the tendencies are integrated over an hour or more.

For the RCE case, I further record the anomalous radiative heating for each of the experiments, which form the columns of matrix $\mathbf{Z}$. Contributions from the radiative effects to the linear response functions are computed as $\mathbf{R} = -\mathbf{Z}\mathbf{F}^{-1}\mathbf{M}$, where any positive eigenvalues of $\mathbf{M}$ have been adjusted.

Linear response functions without radiative contributions are computed by subtracting $\mathbf{R}$ from the temperature tendency portion of $\mathbf{M}$. I have also constructed the linear response functions for the RCE case with prescribed radiative cooling, set to the long-term-averaged radiative cooling of the control run with interactive radiation. Results from the two constructions are generally similar.

I have also computed the linear response functions due to the clear-sky radiative effects only or due to cloud radiative effects only. To do so, I first compute the clear-sky radiative heating for the RCE mean state. I then perturb the temperature and humidity in each of the model layers, one variable and one layer at a time, and recompute the clear-sky radiative heating.

FIG. 1. The four quadrants of the 2-h-average linear response function for the RCE case. The horizontal axis is the pressure of the perturbed layer, and the vertical axis is the pressure of the responding layer. Each column of the matrices is normalized by the mass of the perturbed layer.

The difference between the resulting clear-sky radiative heating and that of the RCE mean state gives the clear-sky radiative effect of the temperature and moisture perturbations $\mathbf{R}_{\text{clr}}$. The clear-sky radiative effects defined here are only a function of the *horizontally averaged* temperature and humidity, in line with Emanuel et al. (2014). While, in principle, horizontal temperature and moisture variations could affect the clear-sky radiation because of nonlinearities in radiative transfer, in practice, such effects are small. Difference between the clear-sky radiative effects $\mathbf{R}_{\text{clr}}$ and the full radiative effect $\mathbf{R}$ is attributed to the cloud radiative effect $\mathbf{R}_{\text{cld}}$.

The linear response functions here are constructed from limited-domain CSRM simulations that contain mostly unorganized convection. This is deemed suitable for the initial growth stage of linearly unstable modes. In contrast, for the steady-state problem of weakly forced mock Walker cells of Kuang (2012), propagating convectively coupled waves organized convection into strong squall lines, and this difference in the form of

convection led to different linear responses that were found to be important in that problem.

Figure 1 shows the linear response functions for the RCE case, with the radiative contributions removed, and Fig. 2 shows the all-sky and clear-sky radiative responses $\mathbf{R}$ and $\mathbf{R}_{\text{clr}}$. All tendencies are averaged over 2 h. Moisture perturbations are presented in terms of a 20% reduction in the relative humidity for ease of comparison with Fig. 4 of Emanuel et al. (2014). The color scale in Fig. 1c is saturated for perturbations in the boundary layer moisture to highlight the response to moisture perturbations in the free troposphere.

General features of the linear response functions were discussed in more detail in Kuang (2012). Briefly, a warm or moist anomaly in the subcloud layer (below ~ 900 hPa) leads to cooling and drying locally and warming above; a warm anomaly in the free troposphere leads to cooling at and above the perturbed layer, while a moist anomaly in the free troposphere leads to warming at and above the perturbed layer.

FIG. 2. (a),(c) All-sky and (b),(d) clear-sky radiative responses (2-h averages) to perturbations to the (a),(b) temperature and (c),(d) relative humidity fields. Axes and normalization follow that of Fig. 1.

Clear-sky radiative responses to temperature perturbations resemble Newtonian cooling (negative on the diagonal). Clear-sky radiative response to moisture perturbation shows the expected behavior that a reduction in relative humidity of a layer reduces radiative cooling in this layer while enhancing radiative cooling in the layers below, similar to that shown in Fig. 4 of Emanuel et al. (2014). The 2-h-averaged radiative heating anomalies shown in Fig. 2 include contributions from temperature and moisture anomalies produced over the 2-h period in response to the initial perturbations. This is particularly evident in the upper-tropospheric responses to boundary layer perturbations.

Cloud radiative response, taken as $\mathbf{R}$ minus $\mathbf{R}_{\text{clr}}$, tends to peak in the upper troposphere. It shows strong association with anomalous convective heating: when there is stronger convective heating in the bulk of the troposphere, there is anomalous radiative heating in the upper troposphere and weaker anomalous radiative cooling over a thinner layer near the tropopause. This radiative response is largely associated with the

anvil clouds and is dominated by longwave effects partially compensated by shortwave effects.

The linear response functions for the TOGA case are presented in Fig. 3. There are no radiative contributions. Note the scale difference between Figs. 1 and 3. This was done because the mean rainfall in the TOGA case (8.9 mm day^{-1}) is about 2.5 times that in the RCE case (3.6 mm day^{-1}). See Fig. 2 of Kuang (2010) for a comparison of the vertical structures of mean convective heating, as well as the moister mean relative humidity profile of the TOGA case.

A comparison of Figs. 1 and 3 shows that responses to lower-tropospheric temperature anomalies tend to have a stronger upper-troposphere extension in the RCE case compared to the TOGA case. There is also a hint of a stronger heating response and a weaker moisture response to midtroposphere moisture perturbations in the TOGA case (after adjusting for the factor-of-2.5 difference in mean convective heating). Note however that the fast-decaying eigenvectors tend to dominate these figures and mask the more slowly decaying modes,

FIG. 3. As in Fig. 1, but for the TOGA case with the color scales 2.5 times larger.

which can be more important for coupling with large-scale dynamics. The model order-reduction procedure described later will allow a more informative comparison between the two in the context of a simple model.

3. Linear stability results based on the linear response functions

We shall now consider the linear stability problem when moist convection is coupled to 2D linear gravity waves. This prototype problem captures the basic aspects of the coupling between convection and large-scale dynamics, upon which effects such as those from an equatorial β plane can be added (e.g., Andersen and Kuang 2008).

For each horizontal wavenumber k , I can write the system as

$$\frac{d}{dt} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{M} & \mathbf{A} \\ k^2 \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}, \quad (2)$$

where $\mathbf{x}$, as stated earlier, is a vector that contains the vertical profiles of temperature T and specific humidity

q , $\mathbf{M}$ is the linear response function derived from the CSRM, $\mathbf{A}$ represents the effect of vertical temperature and moisture advection on $\mathbf{x}$, $k^2 \mathbf{C}$ is the effect of T and q on the vertical velocity profile $\mathbf{w}$ (through hydrostatic balance, horizontal momentum equation, and continuity), and $\mathbf{D}$ represents momentum damping, treated here as Rayleigh damping.

For simplicity, I neglect the virtual effect in the large-scale wave dynamics, which was found to have only a minor contribution, and assume a rigid lid ($w = 0$) at 175 hPa, as a radiating upper boundary condition is not essential for the instabilities that I shall consider.

The growth rates and phase speeds of the unstable modes for the RCE case with a Rayleigh damping time of 1 day are shown in the top panels of Fig. 4. With full radiative feedback, there are two branches of unstable modes: a propagating convectively coupled wave branch with wavelengths of thousands of kilometers and speeds near 20 ms^{-1} (only the positive phase speeds are shown) and a stationary branch, possibly identified as convective self-aggregation, with growth rates peaking at a wavelength of $\sim 15\,000 \text{ km}$. Representative structures of the

FIG. 4. (right) Linear growth rates and (left) the phase speed of the unstable modes for the RCE case for (top) the full model and (bottom) the reduced sixth-order model. The different choices for radiative feedbacks are represented by different colors: full all-sky radiative feedback (black), no radiative feedback (red), clear-sky radiative feedback only (blue), and cloud radiative feedback only (green).

unstable convectively coupled wave mode and the stationary mode are shown in Figs. 5 and 6.

One should not extrapolate the growth rates for the convectively coupled waves to higher horizontal wavenumbers to infer ultraviolet catastrophe (unbound growth rates as horizontal wavenumber increases). First, the scale separation between convection and the large-scale flow inherent in the framework adopted here will not apply when the horizontal wavelength becomes shorter than a few hundred kilometers. Moreover, the linear response functions, and thus the growth-rate calculations, assume that convection is always in statistical equilibrium with the large-scale flow. When the finite time that convection takes to respond to the large-scale flow is taken into account, growth rates for the convectively coupled wave branch at high wavenumbers are preferentially reduced (e.g., Emanuel et al. 1994).

The stationary mode decays without radiative feedbacks, while clear-sky and cloud radiative effects appear to destabilize the stationary modes roughly equally for this reference state. The convectively coupled waves are less affected by the different choices of radiative feedbacks.

The growth rates and the phase speeds of the unstable modes for the TOGA case are shown in Fig. 7. The strength of Rayleigh damping is reduced to 0.1 day^{-1} so that convectively coupled waves can be unstable for this case. When the same (0.1 day^{-1}) Rayleigh damping is used in the RCE case, growth rates for convectively coupled waves are much greater, while the maximum growth rates for the stationary mode were little changed except with the peak shifting to lower wavenumbers (not shown). That the TOGA case has weaker convectively coupled wave instability was noted in Kuang (2010), and a more concrete

FIG. 5. (top) Temperature, (middle) specific humidity, and (bottom) pressure velocity structures of the unstable propagating mode in Fig. 4 with full radiation and a wavelength of 2500 km with (left) the full model and (right) the reduced sixth-order model.

discussion of the difference between the two is presented in section 6.

I used Rayleigh damping in this study because of its simplicity so I can focus on the thermodynamic aspects. Preliminary results using CSRM-derived linear response function for horizontal momentum, as done in Kuang (2012), show that there are also an unstable convectively coupled wave branch and an unstable stationary mode branch. Analyses of such models are more complicated and are left to future work.

4. Model order reduction

I now reduce the dimension or order of the model described in section 3 so that its dynamics can be understood more easily. Starting with Eq. (2), I first reduce the order of vertical velocity. To do so, I consider $\mathbf{x}$ as

the input, $d\mathbf{x}/dt$ as the output, and $\mathbf{w}$ as the internal state variable. This puts Eq. (2) in a form commonly used in control theory for linear, time-invariant systems, and stable algorithms exist to optimally reduce the order. I used the algorithm of Safonov and Chiang (1989). The order-reduction procedure linearly transforms then truncates the internal-state space to retain states with the highest Hankel singular values, such that for a given order that is retained, the error bound provided by Glover (1984) on the outputs for arbitrary inputs at all frequencies is minimized. The order reduction is independent of the horizontal wavenumber k , as multiplicative factors of $\mathbf{C}$ (and $\mathbf{B}$) do not enter the order reduction. We reduce the order of vertical velocity to 2. This reduction has minimal effects on the growth rates and phase speeds of the unstable modes (not shown). On the other hand, when only one vertical

FIG. 6. (left) Temperature, (middle) specific humidity, and (right) pressure velocity structures of the unstable stationary mode in Fig. 4 with full radiation and a wavelength of 15 000 km with the full model (solid) and with the reduced sixth-order model (dashed).

velocity mode is retained, the behavior of the system can no longer be captured. These results indicate that two vertical velocity modes are sufficient and also necessary to capture the instability mechanisms.

After reducing the order of vertical velocity, I have a system similar to Eq. (2) except with two modes for the vertical velocity. I then consider the two vertical velocity modes as the input, and their time derivative as the output, and $\mathbf{x}$ (profiles of temperature and humidity) as the internal state variables. Purely from the perspective of reproducing the full system's behavior with a model of the lowest order, it is optimal to reduce temperature and humidity at all heights together. However, each of the resulting modes from such a procedure will involve temperature and humidity at all heights. For better interpretability, I shall perform a modular model reduction with structural constraints so that order reductions are done within predetermined subspaces. Specifically, based on our physical understanding (or preconception) of the system and the structures of the unstable modes of the full model shown in Figs. 5 and 6, I define three separate subspaces: the

first consists of temperature and moisture in the boundary layer (set to be below 800 hPa), and the other two are the free-troposphere temperature and moisture fields above the boundary layer, respectively. State transformation is allowed only within each subspace but not across them. This is accomplished by zeroing out the cross-subspace terms in the observability and controllability Gramians in the order-reduction procedure (Keil and Gouze 2003). Again, the order reduction is independent of the horizontal wavenumber k .

In the spirit of a minimal complexity model, I set the number of thermodynamic modes retained to 4, the lowest order to reproduce the qualitative behavior of the full model. In all cases that I have examined, the four thermodynamic modes retained include two free-troposphere temperature modes, one free-troposphere moisture mode, and one boundary layer mode (examples of their vertical structures are shown later in Fig. 8). This partition is not imposed a priori and is the result of the order reduction; the structural constraint itself does not specify the number of modes retained in each

FIG. 7. (right) Linear growth rates and (left) the phase speed of the unstable modes for the TOGA case when the full model is used (black) and when the reduced sixth-order model is used (blue). There is no radiative feedback.

of the three subspaces, which is free to vary between 0 and 4.

Together with the two vertical velocity modes retained, I arrive at a sixth-order model. Growth rates and phase speeds of this reduced model for cases without radiation, with clear-sky radiation only, with cloud radiation only, and with full radiation are shown in the bottom panels of Fig. 4. Order reduction was done for the different radiation cases separately.

Vertical structures of the unstable modes for this reduced model are compared with those of the full model in Figs. 5 and 6. Having only one free-troposphere moisture mode, the reduced system cannot capture the evolving vertical structures of the free-troposphere

humidity as seen in the full model and in observations (e.g., Kiladis et al. 2009). Allowing five modes for temperature and moisture adds another free-troposphere moisture mode and substantially improves this aspect but at the cost of interpretability and is not pursued in this paper. Aside from this difference, the vertical structures of the unstable modes in the reduced and full models show broad consistency.

One nonuniqueness in the above procedure is whether to reduce the order of $\mathbf{w}$ or $\mathbf{x}$ first. Reducing $\mathbf{x}$ first to four modes then reducing $\mathbf{w}$ to two modes produces mostly similar but slightly less accurate results. This nonuniqueness represents a subjective aspect of the procedure advocated here.

FIG. 8. Basis functions used in the reduced sixth-order model for T_1 , T_2 , q , h_b , w_1 , and w_2 (see legends) in terms of the (left) temperature, (middle) specific humidity, and (right) negative pressure velocity.

FIG. 9. (left) Temperature and (right) specific humidity structures of the slowest-decaying eigenvector for the full linear response function $\mathbf{M}$ (red) and the reduced matrix $\mathbf{M}^{4 \times 4}$ (blue).

The free-troposphere temperature modes and vertical velocity modes are further rotated and scaled so that the equations can be written as

$$\frac{d}{dt} \begin{pmatrix} T_1 \\ T_2 \\ q \\ h_b \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \mathbf{M}^{4 \times 4} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ a_1 & a_2 \\ 0 & 0 \end{pmatrix} \\ \hline \begin{pmatrix} k^2 c_1^2 & 0 & 0 & 0 \\ 0 & k^2 c_2^2 & 0 & 0 \end{pmatrix} & \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix} \end{pmatrix} \begin{pmatrix} T_1 \\ T_2 \\ q \\ h_b \\ w_1 \\ w_2 \end{pmatrix}, \quad (3)$$

where the variables T_1 , T_2 , q , h_b , w_1 , and w_2 denote the amplitudes of the four retained temperature and moisture modes and the two vertical velocity modes, whose vertical structures (i.e., the basis functions) are shown in Fig. 8 for the RCE case without radiative feedbacks. The basis functions for the other cases (including the TOGA case) are sufficiently similar that they are not shown here for brevity. These basis functions are remarkably similar to those used in the previous simple models (e.g., Mapes 2000; Khouider and Majda 2006; K08) and demonstrate that they are indeed

optimal basis functions for such simple models. The other symbols in Eq. (3) are listed below: $\mathbf{M}^{4 \times 4}$ is the reduced linear response function, ε is the Rayleigh damping coefficient, c_1 and c_2 are the speeds of the first- and second-mode dry gravity waves, and a_1 and a_2 represent vertical advection of free-troposphere moisture q by w_1 and w_2 . The rotation and scaling were done to make the upper-right 2×2 elements of the matrix in Eq. (3) an identity matrix and the lower-left 2×2 elements a diagonal matrix. Furthermore, the basis functions of the temperature modes are scaled so that they are positive in the lower troposphere and their peak positive value is $\pi/2$. This, together with Eq. (3), fixes the scales of the vertical velocity modes. How I set the scales of the boundary layer and the free-troposphere moisture basis functions is discussed in section 5. The rotation and scaling have no effects on the eigenvalue calculation. In Eq. (3), I have further set the effects of vertical advection on the boundary layer mode h_b as well as the effect of the boundary layer mode on the vertical velocity modes to zero. These two simplifications may be justified respectively by the fact that vertical velocity tends to be small near the surface and that the boundary layer is relatively thin. In practice, effects of these simplifications on the stability calculation are found to be secondary.

5. Separation of fast and slow manifolds

Let us first examine the role of the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$, whose structure is shown in Fig. 9, along with that of the slowest-decaying eigenvector of the full linear response functions $\mathbf{M}$. As noted in Kuang (2010), the slowest-decaying eigenvector of $\mathbf{M}$ resembles the equilibrium profiles of the Betts–Miller scheme. Reducing $\mathbf{M}$ to $\mathbf{M}^{4 \times 4}$ clearly distorts this structure. This distortion is due to the structural constraints imposed during the modular order reduction. If I remove the structural constraints and reduce temperature and moisture at all levels together, I can recover the full eigenvector structure almost perfectly.

The decay rate of the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$ is 0.15 day^{-1} for the RCE case. When it is set to zero, the growth rate of the unstable stationary mode increases considerably, while when it is set to be very large (e.g., 10^4 day^{-1}), the stationary mode is stable in all cases that I have examined (Fig. 10). However, the unstable convectively coupled wave modes are not strongly affected by such changes in

the slowest-decaying eigenvector. This suggests the following framework to simplify the system to understand convectively coupled waves and the stationary mode separately.

Let us now split the phase space of the sixth-order model expressed in Eq. (3) into two subspaces or manifolds: a slow manifold spanned by the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$ and the vertical velocity that it drives (through pressure gradient force and continuity) and a fast manifold spanned by the remaining three eigenvectors and the vertical velocity that they drive. The amplitude along the direction of the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$ (or the slow mode) is denoted as e^s . The amplitudes of two vertical velocity modes in the slow manifold are denoted as w_1^s and w_2^s , and those in the fast manifold are denoted as w_1^f and w_2^f . The total vertical velocities w_1 and w_2 are the sum of the respective fast and slow contributions. By construction, the two manifolds interact only through effects of vertical advection on temperature and humidity and not through convection. This gives the following equations:

$$\frac{d}{dt} \begin{pmatrix} e^s \\ w_1^s \\ w_2^s \\ T_1^f \\ T_2^f \\ q^f \\ w_1^f \\ w_2^f \end{pmatrix} = \begin{bmatrix} -\lambda^s & a_1^s & a_2^s & 0 & 0 & 0 & a_1^s & a_2^s \\ \xi_1^s k^2 & -\varepsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_2^s k^2 & 0 & -\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & * & * & * & -1 & 0 \\ 0 & 0 & -1 & * & * & * & 0 & -1 \\ 0 & a_1^f & a_2^f & * & * & * & a_1^f & a_2^f \\ 0 & 0 & 0 & (c_1^f k)^2 & 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & 0 & (c_2^f k)^2 & 0 & 0 & -\varepsilon \end{bmatrix} \begin{pmatrix} e^s \\ w_1^s \\ w_2^s \\ T_1^f \\ T_2^f \\ q^f \\ w_1^f \\ w_2^f \end{pmatrix}, \quad (4)$$

where λ^s is the decay rate of the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$; a_1^s and a_2^s are the effect of vertical advection by w_1 and w_2 on the slow mode amplitude e^s ; $\xi_1^s k^2$ and $\xi_2^s k^2$ are effects of the slow mode on w_1 and w_2 ; T_1^f , T_2^f , and q^f are the amplitudes of three linearly independent modes of the fast subspace, the choice of which will be described shortly; a_1^f and a_2^f are the effect of vertical advection by w_1 and w_2 on q^f ; and c_1^f and c_2^f are the dry wave speeds for the adjusted modes (see end of section 5) but have nearly the same values as c_1 and c_2 .

To simplify interpretation, the modes represented by T_1^f , T_2^f , and q^f have been chosen as the following:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad (5)$$

where the four elements of the columns are values for T_1 , T_2 , q , and h_b , respectively. The reason for this choice is as follows.

Let us write the slowest-decaying left eigenvector of $\mathbf{M}^{4 \times 4}$ as

$$\begin{pmatrix} h_{T_1} \\ h_{T_2} \\ h_q \\ 1 \end{pmatrix}. \quad (6)$$

I have chosen the scales of the basis functions of q and h_b (shown in Fig. 8) in such a way that $h_{T_1} = 1$ and $h_q = 1$. Since h_{T_2} is small in practice (-0.09 for the RCE case and -0.02 for the TOGA case), I set it to zero for simplicity. From the principle of biorthogonality, the slowest-decaying left eigenvector of $\mathbf{M}^{4 \times 4}$ is orthogonal

to the fast thermodynamic subspace so that, within the fast manifold, I have

$$h_b = -(T_1^f + q^f). \quad (7)$$

The reader can verify that the vectors in Eq. (5) reside in the fast subspace.

Within the framework of the reduced-order model, the sum of T_1 , q , and h_b can be interpreted as the column-integrated moist static energy (MSE), while T_2 (a vertical dipole) does not contribute to the column-integrated MSE. Therefore T_1^f , T_2^f , and q^f have the simple interpretation as modes that vertically redistribute MSE within the column with no contributions to the column-integrated MSE, and Eq. (7) is simply a statement of column MSE conservation.

Within the fast subspace, an implied value for the boundary layer mode h_b can be computed from T_1^f , q^f , and Eq. (7). The choice of using h_b as the implied variable was made because effects of vertical advection on h_b are small. If I were to use q as the implied variable, then vertical advection of free-tropospheric humidity would have projections on all modes, complicating the interpretation.

To keep the upper-right 2×2 elements the identity matrix and the lower-left 2×2 elements a diagonal matrix in the fast subspace portion of Eq. (4), I have slightly readjusted the vertical profiles of T_1 , T_2 , w_1 , and w_2 (shown as dashed line in Fig. 8). The ability to split the subspaces cleanly is worth the need for this small readjustment.

With this separation, I can focus on the fast manifold when studying the convectively coupled waves, where the slow manifold is not essential, as described in the next section. The fast and slow manifold separation can also be used to better understand the stationary mode, where the fast manifold may be considered as steady state. This will be described in Part II.

6. Convectively coupled wave dynamics

To focus on the convectively coupled waves, I eliminate the slow manifold and retain only the fast manifold, which gives

$$\frac{d}{dt} \begin{pmatrix} T_1^f \\ T_2^f \\ q^f \\ w_1^f \\ w_2^f \end{pmatrix} = \left[\begin{array}{ccc|cc} \mathbf{M}^{3 \times 3} & -1 & 0 \\ & 0 & -1 \\ & a_1^f & a_2^f \\ \hline (c_1^f k)^2 & 0 & 0 & -\varepsilon & 0 \\ 0 & (c_2^f k)^2 & 0 & 0 & -\varepsilon \end{array} \right] \begin{pmatrix} T_1^f \\ T_2^f \\ q^f \\ w_1^f \\ w_2^f \end{pmatrix}, \quad (8)$$

where $\mathbf{M}^{3 \times 3}$ is the 3×3 submatrix shown as asterisks in Eq. (4). We shall focus on the case without radiative feedback because radiative feedback is not essential to the convectively coupled waves (Fig. 4).

a. Recast into the form of previous simple models

A number of additional simplifications and manipulations are possible so that the system can be cast in a form similar to that of previous studies such as Mapes (2000), Khouider and Majda (2006), and K08.

First, because the fastest-decaying eigenvector of $\mathbf{M}^{3 \times 3}$ has a large negative real eigenvalue (with values of -17 day^{-1} for RCE and -25 day^{-1} for TOGA), the thermodynamic state of the fast manifold, to a good approximation, resides in a subspace spanned by the remaining two eigenvectors so that the projection of the thermodynamic state of the fast manifold to the direction orthogonal to this subspace is small. This gives an equation relating q^f to T_1^f and T_2^f :

$$q^f + f_1 T_1^f + f_2 T_2^f = 0. \quad (9)$$

Both f_1 and f_2 are positive and shown in Table 1. Note that the direction orthogonal to this subspace is not the direction of the fastest-decaying eigenvector, which is not orthogonal to the other eigenvectors.

Since I am operating in the fast manifold, an equation for the boundary layer MSE anomaly h_b is implied given Eqs. (7) and (9), leading to the quasi-equilibrium (QE) approximation used in K08 [Eq. (14) therein], with T_1^f and T_2^f in the place of T_1 and T_2 in that paper:

$$\frac{dh_b}{dt} = F^{K08} \left[f^{K08} \frac{dT_1^f}{dt} + (1 - f^{K08}) \frac{dT_2^f}{dt} \right], \quad (10)$$

where $F^{K08} = f_1 + f_2 - 1$ and $f^{K08} = (f_1 - 1)/(f_1 + f_2 - 1)$. Time derivative is applied on both sides of Eq. (10), as this is the form that is used to infer convective heating needed to maintain QE in the presence of large-scale forcings of T_1^f and T_2^f .

While the QE relationship can be equivalently expressed in q^f or in h_b , the h_b form [Eq. (10)] ties more directly to the more familiar idea of a shallow QE and provides a better connection to K08, where it is used to provide a closure on convective heating. A free-troposphere humidity anomaly q^f is implied by Eq. (7) given T_1^f and h_b . Deviations of the actual q from this implied value would indicate the presence of the slowest-decaying eigenvector, which, however, makes only a small contribution to the convective heating. The discussions above therefore indicate that the QE relation used in K08 had assumed (implicitly)

FIG. 10. (right) Linear growth rates and (left) the phase speed of the unstable modes for the reduced sixth-order model for the RCE case with full radiative feedback, with unaltered eigenvalues (black), with the eigenvalue of the slowest-decaying eigenvector set to zero (blue), and with the eigenvalue of the slowest-decaying eigenvector set to -10^4 day^{-1} (red).

that the slowest-decaying eigenvector has been filtered out or, equivalently, that contributions of the different modes there to the column-integrated MSE had been removed.

The convective tendencies of T_1^f , T_2^f , and q^f , denoted as J_1 , J_2 , and J_q , are given by

$$\begin{pmatrix} J_1 \\ J_2 \\ J_q \end{pmatrix} = \mathbf{M}^{3 \times 3} \begin{pmatrix} T_1^f \\ T_2^f \\ q^f \end{pmatrix}, \quad (11)$$

and convective heating for h_b (denoted as J_{h_b}) is implied, again from Eq. (7) (i.e., column MSE conservation within the fast subspace). Note that the QE relationship [Eq. (9)] cannot be used to replace q^f with T_1^f and T_2^f in Eq. (11) because $\mathbf{M}^{3 \times 3}$ contains the fastest-decaying eigenvector.

I now eliminate T_1 from the first two rows of Eq. (11). This gives us the following relationship:

$$J_2 = -\gamma_0 J_1 - \gamma_q q^f - \gamma_T T_2^f, \quad (12)$$

which is a combination of Eqs. (20) and (34) of K08 and is a parameterization of the shape of convective heating.

A positive γ_0 means that without T_2^f and q^f anomalies, convective heating anomalies are top-heavy (note the sign change compared to K08); a positive γ_q means that an anomalously humid free troposphere (with a corresponding drier and colder boundary layer so that the net contribution to column-integrated MSE is zero) corresponds to a more top-heavy convective heating profile, and a positive γ_T simply represents the tendency for convection to remove a T_2^f anomaly, which is the congestus damping in Mapes (2000). Note that q^f instead of q is used here; an anomalously humid free troposphere that is part of the slowest-decaying eigenvector does not control the shape of convective heating effectively. Also note that for a given q^f or T_2^f , if J_1 is zero, a T_1^f anomaly is implied given the QE condition [Eq. (9)]. In other words, the effects of q and T_2 on the shape of convection (the $\gamma_q q^f$ and $\gamma_T T_2^f$ terms) are the residuals after the fastest-decaying eigenvector has been removed

TABLE 1. Calculated parameters for the RCE and TOGA cases. Parameters found to be key to the stability differences are in bold. Rayleigh damping coefficient ε is prescribed not calculated but is included here for completeness.

Symbol	RCE	TOGA	Unit	Description
a_1^f, a_2^f	1.0, 0.019	0.98, 0.0024	—	Increase in q^f per unit w_1 and w_2 , Eq. (14)
c_1^f, c_2^f	0.99, 0.42	1.0, 0.43	50 m s^{-1}	Dry speeds of the first and second modes
d_1, d_2	0.40, 0.98	0.68, 0.067	—	Decrease in q^f per unit J_1 and J_2 , Eq. (13)
d_q	1.1	0.95	day^{-1}	Decrease in q^f per unit q^f , Eq. (13)
f_1, f_2	1.9, 1.3	2.3, 1.4	—	Coefficients relating q^f to T_1^f and T_2^f in QE, Eq. (9)
γ_0	0.18	0.10	—	Top-heaviness of convective heating anomalies when $q^f = T_2^f = 0$, Eq. (12)
γ_q, γ_T	0.47, 0.78	1.3, 3.4	day^{-1}	Dependence of the shape of convective heating on q^f and T_2^f , Eq. (12)
ε	1	0.1	day^{-1}	Rayleigh damping coefficient

by the QE condition (recalling, again, that the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$, which induces little convective heating, was also eliminated).

One could choose to eliminate T_2^f (or q^f) from Eq. (11) instead. The choice to eliminate T_1^f is based on the fact that this choice yields a form that is more consistent with prior simple models and has a clearer physical interpretation.

Using the first two rows of Eq. (11), I can write T_1^f and T_2^f in terms of J_1 , J_2 , and q^f so that the equation for the convective tendency of q^f , J_q , becomes

$$J_q = -d_1 J_1 - d_2 J_2 - d_q q^f. \quad (13)$$

Using Eq. (13) to rewrite Eq. (8) in terms of J_1 and J_2 , I have

$$\begin{aligned} \frac{dT_{1,2}^f}{dt} &= J_{1,2} - w_{1,2}^f, \\ \frac{dw_{1,2}^f}{dt} &= (c_{1,2}^f k)^2 T_{1,2}^f - \varepsilon w_{1,2}^f, \\ \frac{dq^f}{dt} &= a_1^f w_1^f + a_2^f w_2^f - d_1 J_1 - d_2 J_2 - d_q q^f. \end{aligned} \quad (14)$$

The implied equation for h_b , as we are in the fast subspace, is

$$\begin{aligned} \frac{dh_b}{dt} &= (1 - a_1^f) w_1^f - a_2^f w_2^f + (d_1 - 1) J_1 \\ &\quad + d_2 J_2 + d_q q^f. \end{aligned} \quad (15)$$

Since a_1^f is close to 1 and a_2^f is close to zero [a consequence of the slowest-decaying eigenvector of $\mathbf{M}^{4 \times 4}$ being dominated by free-troposphere humidity variations (Fig. 9)], Eq. (15) may be simplified to

$$\frac{dh_b}{dt} = -b_1 J_1 - b_2 J_2 - b_q q^f, \quad (16)$$

with $b_1 = 1 - d_1$, $b_2 = -d_2$, and $b_q = -d_q$. The relationships between the b parameters and the d parameters are simply a statement of column MSE conservation. The parameter b_1 is greater than zero (i.e., d_1 is less than 1) because it represents the efficiency with which J_1 moves MSE from the boundary layer to the free troposphere.

Equations (14) and (16), together with the QE relationship [Eq. (10)] and the parameterization of the shape of convective heating [Eq. (12)], form a complete system, which will be referred to as the simple convectively coupled wave model. Note that only two equations from Eq. (11) are explicitly used, and the third is replaced by the QE relationship, which provides the

closure on J_1 . Also, Eq. (10) is a weaker version of the QE relationship than the version without the time derivative, and the latter can be used to eliminate a spurious mode with zero growth rate and phase speed that arises as a result.

The growth rates and the phase speeds of the unstable modes from this system for the RCE and TOGA cases, with parameters shown in Table 1, are given in Fig. 11. The wave structures are similar to those in the right column of Fig. 5 and are omitted.

This system is essentially that of K08, which builds upon Mapes (2000) and Khouider and Majda (2006), with additions from Kuang (2011). However, the parameter estimates and the model formulations here are better justified. In particular, Eqs. (13) and (16) and the parameter values listed in Table 1 make it clear that, all else being equal, negative second-mode convective heating (or stratiform heating) moistens the free troposphere and reduces the boundary layer MSE (and the opposite is true for congestus heating), consistent with our expectations. The opposite behavior in K08 can now be seen as caused by lumping the effect of free-troposphere humidity into the effect of the second-mode heating: in the default case of K08, $\gamma_0 = \gamma_T = 0$, so the q^f terms in Eqs. (13) and (16) can be lumped into the J_2 terms, giving an effective d_2 and b_2 , denoted as d_2^{K08} and b_2^{K08} . This lumping was noted in Kuang (2011), where the effect of free-troposphere humidity was separated out, but the parameter estimates there were rather ad hoc.

b. Comparison with K08

K08 provided rough estimates of the parameters for the TOGA case through regressions. Given the strong correlations among different variables, this was acknowledged as “educated guesses of plausible values.” We are now in a position to evaluate those estimates. A comparison of parameters used in K08 and those computed here is shown in Table 2. For this comparison, the q^f and h_b variables are scaled so that the b_1 and d_1 parameters computed here are the same as those in the K08 study. Table 2 shows that the parameter estimates in K08 are largely consistent with those computed here. The main differences are 1) in γ_T , which was neglected in the default case of K08 and represents the congestus damping effect identified by Mapes (2000); 2) in F^{K08} , which was underestimated by a factor of 2 in K08 but was found to not fundamentally affect the stability (see Fig. 4 of K08); and 3) in γ_q , which controls the strength of the moist stratiform instability and was underestimated by $\sim 20\%$ in K08.

The above discussions thus place the simple model analyzed in detail in K08 on a firmer footing. Readers

FIG. 11. (right) Linear growth rates and (left) the phase speed of the unstable modes for the simple model of convectively coupled waves described by Eqs. (10), (12), (14), and (16) for the RCE case with Rayleigh damping of 0.1 (black) and 1 day⁻¹ (blue) and for the TOGA case with Rayleigh damping of 0.1 day⁻¹ (red).

are referred to [Mapes \(2000\)](#) and [K08](#) for detailed analysis of the model and discussions of the dynamics of convectively coupled waves, where a direct stratiform instability, which relies on a top-heavy shape of convective heating anomalies in the absence of anomalies in free-troposphere humidity and second-mode temperature, and a moisture-stratiform instability, which relies on the effect of free-troposphere humidity on the shape of convection, were identified.

c. Differences between RCE and TOGA

The more systematic estimates of the parameters now also allow us to better understand the reason for the stability differences between different cases.

Experimentations with the simple convectively coupled wave model [Eqs. (10), (12), (14), and (16)] show that the parameters highlighted in bold in [Table 1](#) are responsible for the differing stability between the RCE and TOGA cases. These are parameters that control the shape of anomalous convective heating. The

parameter γ_0 is considerably larger in the RCE case. This indicates that without q^f and T_2^f anomalies, convective heating anomalies are more top-heavy in the RCE case, which allows for stronger direct-stratiform instability. Some indications of the top-heaviness of the RCE case were seen in comparisons of [Figs. 1](#) and [3](#) in terms of a stronger upper-troposphere extension of the convective responses ([section 2](#)), but the difference can now be better quantified. The sensitivity of the shape of convection to free-troposphere moisture [γ_q in Eq. (12)] and the second-mode temperature [γ_T in Eq. (12)] is also much stronger in the TOGA case than in the RCE case. The stronger sensitivity to free-troposphere moisture allows for a stronger moisture-stratiform instability ([K08](#)), while the stronger sensitivity to the second-mode temperature indicates a stronger congestus damping ([Mapes 2000](#)). These two effects oppose each other, with the net effect of reducing the growth rates for the TOGA case, compared to RCE. The stronger sensitivities are partially explained by the

TABLE 2. Parameter values for the TOGA case used in [K08](#) and calculated here (the values are scaled to facilitate comparison; see text for more explanations).

Symbol	K08	This study	Unit	Description
a_1^f, a_2^f	1.4, 0	1.4, 0.0	—	Increase in q^f per unit w_1 and w_2 , Eq. (14)
b_1, b_2^{K08}	1, 2	1, 1.9	—	Decrease in h_b per unit J_1 and J_2 , with the effect of q^f lumped in (see end of section 6a)
c_1^f, c_2^f	1, 0.5	1.0, 0.43	50 m s ⁻¹	Dry speeds of the first and second modes
d_1, d_2^{K08}	1.1, -1	0.97, -0.93	—	Decrease in q^f per unit J_1 and J_2 , with the effect of q^f lumped in (see end of section 6a)
γ_0	0	0.11	—	Top-heaviness of convective heating anomalies when $q^f = T_2^f = 0$, Eq. (12)
γ_q, γ_T	0.7, 0	0.91, 3.5	day ⁻¹	Dependence of the shape of convective heating on q^f and T_2^f , Eq. (12)
$f^{\text{K08}}, F^{\text{K08}}$	0.5, 4	0.48, 8.4	—	Coefficients relating h_b to T_1^f and T_2^f in QE, Eq. (10)
ε	0.1	0.1	day ⁻¹	Rayleigh damping coefficient

stronger mean heating in the TOGA case, which is about 2.5 times stronger: if free-troposphere moisture or the second-mode temperature affects a single updraft in the same way, the TOGA case will have a stronger sensitivity as it has more updrafts. However, the difference in mean heating does not entirely explain the differing sensitivities, indicating that convective updrafts in the TOGA case are more sensitive to both free-troposphere moisture and second-mode temperature perturbations (with corresponding changes in the boundary layer MSE to keep column-integrated MSE unchanged). The reason for this is unclear and warrants further study.

Further experiments with the simple convectively coupled wave model confirm the findings of Kuang (2010) that in the TOGA case, zeroing γ_q almost entirely removes the unstable convectively coupled waves, implying that the moisture-stratiform instability is the dominant mechanism, while in the RCE case, zeroing γ_0 or γ_q significantly weakens but does not remove the convectively coupled wave instability, which is only removed when both γ_0 and γ_q are set to zero, implying that both the direct-stratiform instability and the moisture-stratiform instability contribute to the convectively coupled wave instability in RCE.

There are additional notable parameter differences between the two cases: in the TOGA case, the moistening of the free troposphere by anomalous second-mode heating J_2 , d_2 in Eq. (13), is much weaker compared to that in the RCE case, presumably because the TOGA case has a moister free troposphere (e.g., Fig. 2 of Kuang 2010) and there is less evaporation of rain. While the second-mode heating J_2 , when positive, has been referred to as stratiform heating, it should be recognized that the term “stratiform heating” is used here in a loose sense and can be produced by greater mass flux in the upper troposphere than in the lower troposphere and does not necessarily require evaporation of rain. Similarly, the reduction of boundary layer MSE by anomalous second-mode heating [$b_2 = -d_2$; Eq. (16)] is also weaker in the TOGA case, as less rain evaporation leads to weaker precipitation-driven downdrafts. The large difference in the d_2 (and b_2) values between the TOGA and RCE cases however turns out to have only a secondary effect on the stability of convectively coupled waves. This is because J_2 can also affect q^f through the QE relationship [most clearly seen through Eq. (9)], reducing the sensitivity of the system to the d_2 parameter.

7. Conclusions

In studies of moist convecting atmospheres, simple models have long played a vital role in providing

insights. However, formulations and parameters in these simple models’ treatment of convection are often based on intuition and not systematically justified.

In this paper, I have presented a procedure to systematically construct simple models for the linear stability of moist convecting atmospheres, along with their parameters, from a more comprehensive model, such as a CSRM.

In this study, I have focused on the prototype problem of convection coupling with large-scale 2D linear gravity waves. For a given reference state, I start by constructing linear response functions of the CSRM. These linear response functions, when coupled with large-scale linear wave dynamics, provide a suitable model for this problem. For an RCE reference state with full radiative feedback, this linear-response-function-based model gives two branches of unstable modes: a propagating convectively coupled wave branch and a stationary branch. The stationary branch is unstable only when radiative feedback is included, while the convectively coupled wave branch is less affected by radiative feedback.

I then used a modular order-reduction procedure from control theory to reduce the linear-response-function-based model to a system of six ODEs, which are found to capture the essential features of the full model. The basis functions obtained from the order-reduction procedure match those used in previous simple models remarkably well, thus more formally justifying their use.

The reduced-order system is then split into a slow and a fast manifold, and the former is found essential for the stationary mode but not for the convectively coupled waves. This finding indicates that the essential convectively coupled wave dynamics are contained within the fast manifold.

Through a further QE assumption based on the fastest-decaying eigenvector, the fast manifold of the reduced-order model is transformed into a form similar to that of prior simple models of convectively coupled waves, whose parameters are thereby computed from the CSRM. The results show that the formulation used in K08 lumped the effect of free-troposphere moisture on convective moistening and boundary layer MSE into the effect of second-mode convective heating. The procedure here also made clear that, in previous simple models of convectively coupled waves, variations in the column-integrated MSE have been implicitly filtered out. Comparisons of the formulations and parameters place the previous simple models and the insights derived from them on a firmer footing.

With better quantifications of the parameters, we can also better understand the stability difference between different reference states. In particular, the different

stability of convectively coupled waves for the RCE and TOGA cases found in Kuang (2010) can now be attributed to their differences in the control of the shape of convective heating.

In this paper, I have focused on convectively coupled waves. In Part II, I shall examine the stationary mode and the effect of radiative feedback by considering the interaction between the slow and the fast manifolds, with the fast manifold set to a steady state, which represents an extension of the moisture mode and weak temperature gradient approach currently used in such problems.

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