Cellular Statistical Models of Broken Cloud Fields. Part IV: Effects of Pixel Size on Idealized Satellite Observations

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ABSTRACT

In the fourth part of our “Cellular Statistical Models of Broken Cloud Fields” series we use the binary Markov processes framework for quantitative investigation of the effects of low resolution of idealized satellite observations on the statistics of the retrieved cloud masks. We assume that the cloud fields are Markovian and are characterized by the “actual” cloud fraction (CF) and scale length. We use two different models of observations: a simple discrete-point sampling and a more realistic “pixel” protocol. The latter is characterized by a state attribution function (SAF), which has the meaning of the probability that the pixel with a certain CF is declared cloudy in the observed cloud mask. The stochasticity of the SAF means that the cloud-clear attribution is not ideal and can be affected by external or unknown factors. We show that the observed cloud masks can be accurately described as Markov chains of pixels and use the master matrix formalism (introduced in Part III of the series) for analytical computation of their parameters: the “observed” CF and scale length. This procedure allows us to establish a quantitative relationship (which is pixel-size dependent) between the actual and the observed cloud-field statistics. The feasibility of restoring the former from the latter is considered. The adequacy of our analytical approach to idealized observations is evaluated using numerical simulations. Comparison of the observed parameters of the simulated datasets with their theoretical expectations showed an agreement within 0.005 for the CF, while for the scale length it is within 1% in the sampling case and within 4% in the pixel case.

1. Introduction

Clouds play an important role in Earth’s climate system, both contributing and responding to climate change (Stephens 2005). The complexity of cloud processes and their interactions with aerosols and radiation make clouds a major source of the uncertainty for estimates of Earth’s energy budget (e.g., Boucher et al. 2013; Flato et al. 2013). In particular, change in cloud-field structure such as the transition from stratocumulus to cumulus clouds over the subtropical oceans attracts wide interest in climate studies (e.g., Yamaguchi et al. 2015; de Roode et al. 2016). This transition driven by advection of the cloud system over a warming sea surface represents fundamental cloudy boundary layer

processes as well as cloud feedbacks in a warmer-ocean future. Current climate models’ disagreement on the change of the subtropical low-cloud amount under a global warming scenario results in considerable uncertainties in global-mean temperature predictions (Bony and Dufresne 2005; Webb et al. 2013; Tsushima et al. 2016).

The combination of observations, theory, and models is essential for the understanding of how clouds contribute and respond to climate change. While physically based dynamical models such as large-eddy simulations (LESs) remain the essential tool for study of the cloud structure and development, some important features of cloud fields can be captured using much simpler stochastic models (e.g., Titov 1990; Evans and Wiscombe 2004; Hogan and Kew 2005; Venema et al. 2006; Schmidt et al. 2007; Prigarin and Marshak 2009). In particular, the most basic cloud-field characteristics, such as cloud

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fraction (CF) and scale (characterized by sizes of clouds and gaps between them), can be adequately described by models representing cloud fields as binary mixtures of cloudy and clear areas (e.g., [Su and Pomraning 1994](#); [Zuev and Titov 1995](#); [Astin and Latter 1998](#); [Astin et al. 2001](#); [Prigarin et al. 2002](#); [van de Poll et al. 2006](#); [Alexandrov et al. 2010a,b](#), hereinafter [Part I](#) and [Part II](#), respectively). Beyond the computational simplicity, another advantage of such models is that they can be used for statistical parameterization of cloud masks (CMs), which are routinely derived from satellite observations on a global scale (e.g., [Stubenrauch et al. 2013](#)).

This study continues the series of [Part I](#), [Part II](#), and [Alexandrov and Marshak \(2017, hereinafter Part III\)](#) developing statistical parameterization and modeling of the cloud structure in binary-mixture framework. The approach adopted in this series is based on cloud-mask statistics of 2D broken cloud fields derived from observations made along linear transects (chords). Such observations consist of the lengths of cloudy and clear intervals in each transect. The cloud statistics in this approach are always finite, so it works equally well for cumulus and stratocumulus cloud fields with a smooth transition between these two types (which are symmetrical to each other under cloud–clear interchange). This is an advantage compared to, for example, area-based characterization which yields infinite cloud areas in stratocumulus (Sc) cases and infinite clear areas in cumulus (Cu) cases. In [Part II](#) the analytical expressions derived in [Part I](#) were demonstrated to adequately describe the statistics of shallow, broken cloud fields generated using a realistic LES model. In [Part III](#) the results of [Part I](#) were interpreted in terms of the theory of Markov processes (e.g., [Kulkarni 2011](#); [Ibe 2013](#)) assuming that each transect consisting of subsequent cloudy and clear segments is a realization of a binary Markov process, which can take only two values: occupied (cloudy: ●) or empty (clear: ○). The main results of our previous studies are outlined in [section 2](#).

A number of statistical studies have been devoted to binary Markovian mixtures both in general ([Sanchez et al. 1994](#); [Astin and Di Girolamo 1999](#)) and in application to cloud-field properties ([Su and Pomraning 1994](#); [Astin and Latter 1998](#); [Astin et al. 2001](#); [van de Poll et al. 2006](#)). At the same time Markovian cloud models are extensively used in the stochastic radiative transfer theory and simulations ([Levermore et al. 1988](#); [Titov 1990](#); [Zuev and Titov 1995](#); [Su and Pomraning 1995](#); [Pomraning 1989, 1996, 1998](#); [Malvagi et al. 1993](#); [Lane et al. 2002](#); [Kassianov 2003](#); [Byrne 2005](#); [Kassianov and Veron 2011](#); [Doicu et al. 2013, 2014a,b](#); [Efremenko et al.](#)

[2016](#)). The Markovian approach to cloud fields has been also used for an analysis of ground-based measurements of sunshine duration and vertical visibility (e.g., [Morf 2011](#), and references therein).

In this study we focus on the effects of the finite resolution of idealized satellite measurements on the observed cloud statistics. The effects of instrument resolution (footprint or pixel size) on the estimate of cloud fraction has been studied in the past using both idealized framework (e.g., [Shenk and Salomonson 1972](#); [Di Girolamo and Davies 1997](#); [Astin 1997](#)) and actual satellite products (e.g., [Wielicki and Parker 1992](#)).

Cloud detection (masking) techniques widely vary depending on the type of satellite instrument ([Wielicki and Parker 1992](#); [Stubenrauch et al. 2013](#)). In addition to this, cloudy–clear pixel-state attribution in existing satellite data analysis algorithms depends on whether the analysis is focused on cloud or aerosol properties. For aerosol retrievals, the cloud mask essentially assesses the likelihood of a pixel being obstructed by clouds, and pixels falling into “gray area” are rejected as cloudy. Conversely, in cloud-focused algorithms pixels with thin clouds not suitable for cloud-based analysis are rejected as clear. The currently used cloud-masking techniques are based on the differences between cloud and clear-sky signals either spectrally or in variability (which is much higher in cloudy cases). Here are some examples of CM algorithms used in Moderate Resolution Imaging Spectroradiometer (MODIS) data products. The standard MODIS algorithm (e.g., [Platnick et al. 2017](#); [Kokhanovsky and de Leeuw 2009](#)) uses a variety of spectral cloud detection tests to indicate a level of confidence that a clear-sky scene is observed. In the MODIS dark target aerosol retrieval algorithm ([Levy et al. 2013](#)), the CM procedure examines the standard deviation in every group of 3×3 pixels. Any such group with standard deviation exceeding the set threshold is labeled as cloudy, and all nine pixels in the group are discarded ([Martins et al. 2002](#)). The Multi-Angle Implementation of Atmospheric Correction (MAIAC) cloud-masking algorithm ([Lyapustin et al. 2011](#)) uses spatial and temporal covariance analysis to distinguish the cloud-free surface spatial pattern (which is reproducible in the short time frame) from cloudy scenes randomly disturbing this pattern. For the retrieval of aerosol properties over land from MODIS data four CM techniques used together are based on magnitude and spectral dependence of both brightness temperature and reflectance in the visible range ([Kokhanovsky et al. 2009](#)). The MODIS CM procedure for retrieving cloud properties consists of a series of threshold tests applied to 22 out of the 36 MODIS spectral bands and

depending on the surface type and the time (day or night) of observations (Ackerman et al. 1998; Frey et al. 2008). The MODIS cloud products and their spatial and temporal distributions are described in detail by King et al. (2013).

An ultimate goal of investigation of the influence of observation resolution on satellite-based retrievals is to accurately derive the underlying macroscopic cloud statistics from coarse-resolution cloud masks. To achieve this goal, some approaches use box-counting technique based on assumed scale invariance (fractality) of cloud fields. Lovejoy et al. (1987) initially designed this method for rain data, while later it was applied to cloud observations by Wielicki and Parker (1992). However, many studies suggest that fields of, for example, small cumuli clouds do not necessarily exhibit scale invariance over all scales (Gabriel et al. 1988; Wielicki and Parker 1992; Feijt and Jonker 2000). Another approach (Di Girolamo and Davies 1997; Jones et al. 2012) is based on machine learning with real high-resolution data, and thus does not need to make wide-ranging idealized assumptions to get the predictions to work. Recent availability of extremely high-resolution (15 m) data from Advanced Spaceborne Thermal Emission and Reflection Radiometer (ASTER) onboard the EOS *Terra* spacecraft (Abrams 2000) gave a boost to the research of pixel size effects on cloud retrievals (e.g., Zhao and Di Girolamo 2007; Dey et al. 2008; Jones et al. 2012).

While the abovementioned computer algorithms are operationally used in real satellite data analysis, it is still important to advance the mathematical methodology, as it is done in this study, in order to better understand the nuances of the problem. We will start our presentation with simplified discrete-sampling approach, in which the observed cloudy–clear state of the sampled point coincides with its actual state. Then we will introduce a more realistic “pixel” observation protocol characterized by a state attribution function (SAF), which depends on the actual CF within the pixel’s field of view. This function has the meaning of the probability that the pixel with certain CF will be declared “cloudy” in the cloud mask. SAF can be chosen to be deterministic (e.g., equal to 1 for CF larger than 50% and to 0 otherwise) or stochastic (e.g., proportional to the CF). The stochasticity of SAF reflects the influence of external factors (solar-viewing geometry, detector sensitivity to low light, etc.) affecting the cloud–clear attribution in real satellite datasets. We will demonstrate that both sampling and pixel observations can be represented by Markov chains, whose statistics depend on both actual cloud-field parameters and the observations’ resolution. These results are obtained using the notion of the master matrix (introduced in Part III),

which unifies the statistics of the cloud fraction and the Markovian properties of the cloud field.

2. Binary Markov model of broken cloud field

The theoretical Markovian framework for statistical parameterization of broken cloud fields was outlined in Part III of this series of papers, generalizing the results of Part I. Here we briefly recall the main points made in Part III.

a. Markovian properties

The realizations of a binary Markov model are stochastic functions on real line $\mathbb{R}$, which can take only two (generally nonnumeric) values (states): $\bullet$ (occupied, cloudy) or $\circ$ (empty, clear). The Markovian property of the model means that for any chosen initial point $x_0 \in \mathbb{R}$ the probabilities of the points with $x > x_0$ to be in one of the two states depend only on the state of x_0 itself (and not on the states of the points with $x < x_0$). These probabilities are combined into the transition probability matrix of the form

$$\mathbf{P} = \begin{pmatrix} P_{\bullet\bullet} & P_{\bullet\circ} \\ P_{\circ\bullet} & P_{\circ\circ} \end{pmatrix}, \quad (1)$$

where P_{ij} is the probability of transition from the state i at x_0 into state j at $x > x_0$ (i and j can be either $\bullet$ or $\circ$). By definition of transition probabilities, each row of $\mathbf{P}$ sums to unity:

$$P_{\bullet\bullet} + P_{\bullet\circ} = 1 \quad \text{and} \quad P_{\circ\bullet} + P_{\circ\circ} = 1. \quad (2)$$

We assume the model to be spatially homogeneous, so the transition matrix depends only on the distance (lag) $L = x - x_0$, rather than on x and x_0 themselves: $\mathbf{P} = \mathbf{P}(L)$. The transition matrices for two consequent intervals with lengths L_1 and L_2 obey the group property

$$\mathbf{P}(L_1 + L_2) = \mathbf{P}(L_1)\mathbf{P}(L_2) \quad \text{and also} \quad \mathbf{P}(L \rightarrow 0) = \mathbf{I}, \quad (3)$$

where $\mathbf{I}$ is the identity matrix. The state of the initial point of the interval can be also considered as a random variable taking value $\bullet$ with the probability u and $\circ$ with the probability $v = 1 - u$. It is convenient to describe such state by a “state matrix” with identical rows:

$$\mathbf{U} = \begin{pmatrix} u & v \\ u & v \end{pmatrix}. \quad (4)$$

The state matrices for definitely known (pure) cloudy and clear states have the form Eq. (4) with $u = 1$ and $u = 0$, respectively:

$$\mathbf{U}^\bullet = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{U}^\circ = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \quad (5)$$

If the state of a point x is specified by $\mathbf{U}(x)$, then the state of the point $x + L$ will be specified by the matrix

$$\mathbf{U}(x + L) = \mathbf{U}(x) \mathbf{P}(L). \quad (6)$$

The state matrices have the following properties:

$$\det \mathbf{U} = 0, \quad \text{Tr} \mathbf{U} = 1, \quad (7)$$

$$\mathbf{U}'\mathbf{U} = \mathbf{U}, \quad \text{thus,} \quad \mathbf{U}^2 = \mathbf{U}, \quad (8)$$

where $\mathbf{U}'$ is another state matrix.

b. Cloud-field statistics

Cloud fields representing a binary Markov model on $\mathbb{R}$ are infinite 1D patterns of interchanging clear and cloudy intervals of finite lengths. The statistical distributions of lengths of these intervals appear to be exponential (Levermore et al. 1988; Pomraning 1989; Part I) with the respective means $L_\bullet$ and $L_\circ$ (note that in Part I and Part II we used the notations L_c for “clouds” and L_g for “gaps”). The pair of numbers $(L_\bullet, L_\circ)$ provides complete parameterization of the model in the infinite space. This parameterization, however, is not unique and not the most convenient. An alternative set of two independent parameters can be chosen as (cf. Levermore et al. 1988; Pomraning 1989; Part III)

$$\bar{c} = \frac{L_\bullet}{L_\bullet + L_\circ} \quad \text{and} \quad L_* = \frac{L_\bullet L_\circ}{L_\bullet + L_\circ}, \quad (9)$$

where $\bar{c}$ is the mean cloud fraction on $\mathbb{R}$ and L_* is the double of the geometric mean of $L_\bullet$ and $L_\circ$:

$$\frac{1}{L_*} = \frac{1}{L_\bullet} + \frac{1}{L_\circ}, \quad (10)$$

which can be considered as a universal scale length of the cloud field. This quantity can be called “autocorrelation length” (cf. Levermore et al. 1988; Pomraning 1989) since it enters the exponent of the corresponding autocorrelation function (Morf 1998, 2011). Note, however, that in order to define autocorrelation function $\bullet$ and $\circ$ should be assigned with numerical values (e.g., 0 and 1). The inverse of the relationship (9) is

$$L_\bullet = \frac{L_*}{\bar{s}} \quad \text{and} \quad L_\circ = \frac{L_*}{\bar{c}}, \quad (11)$$

where $\bar{s} = 1 - \bar{c}$.

In addition to infinite-length realizations of the binary Markov model, the ensemble of finite-length samples

extracted from these realizations was considered in Part I. It was demonstrated in Part II that the same analysis is valid for finite 1D transects extracted from a 2D Markov field (in that case LES-simulated cloud masks). The sampling procedure is specified by the length L of the sample and the probabilities of its initial point to be cloudy or clear. We denote both sample length and transition lag by the same letter L since the formalism described below provides a unified description of the CF statistics in the sample and the probabilities of transition between its ends. If the samples are chosen at random, the initial state is described by the state matrix

$$\mathbf{C} = \begin{pmatrix} \bar{c} & \bar{s} \\ \bar{c} & \bar{s} \end{pmatrix}. \quad (12)$$

We will refer to such state as the “random state.” Besides being a state matrix, $\mathbf{C}$ can be considered as the transition matrix corresponding to $L = \infty$. In this role the matrix $\mathbf{C}$ enters the expressions for some important quantities, such as, for example, the finite-lag transition matrix. While random sample selection is the most natural, we reserve a possibility to consider biased samplings (e.g., where all selected samples start with cloudy point), which have a generic initial state matrix [Eq. (4)].

c. Master matrix

In Part III we derived cloud fraction probability distribution functions (PDFs) separately for the four combinations of the initial and final states of the samples and combined them into the master matrix:

$$\mathbf{F}(c) = \begin{bmatrix} F_{\bullet\bullet}(c) & F_{\bullet\circ}(c) \\ F_{\circ\bullet}(c) & F_{\circ\circ}(c) \end{bmatrix}, \quad (13)$$

which in the $(\bar{c}, L_*)$ parameterization has the form

$$\mathbf{F}(c) = \begin{bmatrix} \frac{Z}{2s} I_1(Z) & \bar{s}r I_0(Z) \\ \bar{c}r I_0(Z) & \frac{Z}{2c} I_1(Z) \end{bmatrix} e^{-(\bar{s}c + \bar{c}s)r} + \begin{bmatrix} \delta(s)e^{-\bar{s}r} & 0 \\ 0 & \delta(c)e^{-\bar{c}r} \end{bmatrix}, \quad (14)$$

where I_0 and I_1 are the modified Bessel functions, $s = 1 - c$,

$$r = \frac{L}{L_*} \quad (15)$$

is the scale ratio, and

$$Z = 2r\sqrt{\bar{c}\bar{s}}cs. \quad (16)$$

In the short sample limit ($r \rightarrow 0$, $L \ll L_*$) the master matrix has the form

$$\mathbf{F}^{(0)}(c) = \begin{bmatrix} \delta(s) & 0 \\ 0 & \delta(c) \end{bmatrix}, \quad (17)$$

while in the long sample limit ($r \rightarrow \infty$, $L \gg L_*$) it becomes

$$\mathbf{F}^{(\infty)}(c) = \mathbf{C}\delta(c - \bar{c}). \quad (18)$$

Each matrix element $F_{ij}(c)$ is the cloud cover probability density conditional by having the first point of the interval in the state i , and the last point in the state j . This means that given an initial state described by a matrix $\mathbf{U}$ of the form Eq. (4) we can compute the cloud fraction PDF as

$$F_{\text{cf}}(c) = \text{Tr}[\mathbf{U}\mathbf{F}(c)]. \quad (19)$$

If the ensemble of samples is randomly chosen, then $\mathbf{U} = \mathbf{C}$ from Eq. (12). In this case Eq. (19) leads to the PDF computed in Part I, which in the $(\bar{c}, L_*)$ parameterization has the following form

$$F_{\text{cf}}(c) = 2\bar{c}\bar{s}re^{-(\bar{s}c + \bar{c}s)r} \left[I_0(Z) + (\bar{c}c + \bar{s}s)r \frac{I_1(Z)}{Z} \right] + \bar{s}e^{-\bar{c}r}\delta(c) + \bar{c}e^{-\bar{s}r}\delta(s). \quad (20)$$

In the short and long sample limits this expression transforms into

$$F_{\text{cf}}^{(0)}(c) = \bar{s}\delta(c) + \bar{c}\delta(s) \quad (21)$$

and

$$F_{\text{cf}}^{(\infty)}(c) = \delta(c - \bar{c}), \quad (22)$$

respectively. Note that in the case of random sampling the mean cloud fraction does not depend on the sample length and is equal to $\bar{c}$ (see Part I and Part III).

The master matrix elements $F_{ij}(c)$ can be also considered as transition probabilities between the sample's initial state i and final state j conditioned by the cloud fraction in the sample. This means that the transition matrix $\mathbf{P}$ of the Markov process can be derived from the master matrix by integrating out the CF dependence:

$$\mathbf{P} = \int_0^1 \mathbf{F}(c) dc. \quad (23)$$

The direct computation performed in Part III yielded the following expression for the transition matrix:

$$\mathbf{P} = e^{-r}\mathbf{I} + (1 - e^{-r})\mathbf{C}, \quad (24)$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{C}$ is defined by Eq. (12). The factor $e^{-r} < 1$ balances the fractions of $\mathbf{I}$ and $\mathbf{C}$ in the transition matrix; thus, in the limits of $r \rightarrow 0$ and $r \rightarrow \infty$ Eq. (24) takes the respective forms

$$\mathbf{P}^{(0)} = \mathbf{I} \quad \text{and} \quad \mathbf{P}^{(\infty)} = \mathbf{C}, \quad (25)$$

which are in agreement with Eqs. (17) and (18).

d. Cloud-clear interchange

The expressions for cloud-field characteristics presented in this section have certain symmetries under cloud-clear interchange (CCI). These symmetries can substantially reduce analytical computations by allowing for use of simple rearrangement of indices and coefficients in existing expressions in order to derive new ones. CCI acts on expressions for cloud-field parameters by interchanging (in both lhs and rhs) u and v , c and s (thus, also $\bar{c}$ and $\bar{s}$) simultaneously with the indices $\bullet$ and $\circ$. For example, the expressions $L_\circ = L_*/\bar{c}$ and $L_\bullet = L_*/\bar{s}$ in Eq. (11) can be obtained one from the other if we notice the invariance of L_* under CCI [see Eq. (10)]. This invariance naturally extends to all functions of $r = L/L_*$. The expressions for the elements of the master matrix equation [Eq. (14)] are also related via CCI: $F_{\circ\circ}$ can be obtained from $F_{\bullet\bullet}$, while $F_{\circ\bullet}$ can be obtained from $F_{\bullet\circ}$ (and vice versa). Below we will denote such relationships by a double-sided arrow: $L_\bullet \leftrightarrow L_\circ$, $F_{\bullet\bullet} \leftrightarrow F_{\circ\circ}$, and $F_{\bullet\circ} \leftrightarrow F_{\circ\bullet}$.

We should also note that since

$$\int_0^1 dc = - \int_1^0 ds = \int_0^1 ds, \quad (26)$$

CCI commutes with integration over c : if $A \leftrightarrow B$, their integrals are also related. For example, this is true (and can be directly verified) for the elements of the transition probability matrix $\mathbf{P}$, which are the integrals of the corresponding matrix elements of $\mathbf{F}$, thus inheriting their symmetries: $P_{\bullet\circ} \leftrightarrow P_{\circ\bullet}$ and $P_{\bullet\bullet} \leftrightarrow P_{\circ\circ}$.

3. Markov chain of observations

In this study we assume that the cloud masks generated using 1D binary Markov model constitute the “real world,” to which we apply idealized “satellite observation” procedures. These procedures will result in the creation of lower-resolution “observed” cloud masks, which are different from the original ones and depend on the observational procedure used. We will study the relationships between the statistical parameters of the original and observed cloud masks and investigate possibilities to restore the former from the latter.

We will consider two types of observation scenarios. The first type simply returns an array of N observation points corresponding to actual points in the observed cloud field sampled on a regular grid with spacing l_p . The observed cloudy-clear state of a point in this case coincides with its actual state. The second, more realistic, type of observations returns an array of N consecutive “pixels” corresponding to intervals (of the length l_p) in the actual cloud field. The cloudy-clear state of each pixel is assigned using a statistical procedure based on cloud fraction in the interval (field of view) corresponding to this pixel. This type of observation, unlike the simple sampling, implies a “subjective” decision by the algorithm since the actual pixels may be partially cloudy, while the observed pixels are assigned with either cloudy or clear state.

We will demonstrate later in the paper that the results of both types of observations can be described as realizations of Markov chains (discrete Markov processes). While in the discrete-sampling approach this representation is exact, the definition of pixel Markov chain involves an approximation. We denote the transition probability matrix of observational Markov chain by

$$\mathbf{Q} = \begin{pmatrix} Q_{\bullet\bullet} & Q_{\bullet\circ} \\ Q_{\circ\bullet} & Q_{\circ\circ} \end{pmatrix}, \quad (27)$$

with the elements satisfying Eq. (2):

$$Q_{\bullet\bullet} + Q_{\bullet\circ} = 1 \quad \text{and} \quad Q_{\circ\bullet} + Q_{\circ\circ} = 1. \quad (28)$$

Let us call a set of consecutive cloudy observations “observed cloud” and a set of consecutive clear observations “observed gap.” The lengths of the observed clouds and gaps obey statistical distributions similar to those in discrete cellular model described in Part I. These distributions for the cloudy or clear interval of the length m have the form

$$n_i(m) = \left[1 + \frac{1 - (1 - q_i)m}{1 + (1 - q_i)(N - 1)} \right] (1 - q_i)q_i^{m-1} \quad (29)$$

for $m < N$, while

$$n_i(N) = \frac{q_i^{N-1}}{1 + (1 - q_i)(N - 1)}. \quad (30)$$

Here $q_i = Q_{ii}$ ($i = \bullet$ or $\circ$) is the probability for cloud or gap to continue to the next point. Note that in the model from Part I that we use here the probability q_i is independent from the state of the preceding point in the sample (so that model is not Markovian). However, as long as we consider subsequent points within the

observed cloud (or gap) in binary Markov model, this does not matter since all points in such discrete sub-sample have the same state. In the observed cloud the probability for the next point to be cloudy (so the cloud continues) is $q_{\bullet} = Q_{\bullet\bullet}$ and to be clear (so the cloud ends) is $1 - q_{\bullet} = Q_{\bullet\circ}$. Similarly, in the observed gap the probability for the next point to be clear (so the gap continues) is $q_{\circ} = Q_{\circ\circ}$ and to be cloudy (so the gap ends) is $1 - q_{\circ} = Q_{\circ\bullet}$.

The distributions described by Eqs. (29) and (30) are normalized as

$$\sum_{m=1}^N n_i(m) = 1, \quad (31)$$

and have the means

$$\bar{m}_i = [1 - q_i + q_i N^{-1}]^{-1}. \quad (32)$$

In the limit of infinitely long observation chain ($N \rightarrow \infty$, while the measurement resolution l_p remains finite) this expression takes the simple form

$$M_i = \bar{m}_i^{(\infty)} = \frac{1}{1 - q_i}. \quad (33)$$

The mean observed length for finite N from Eq. (32) can be expressed through this quantity as

$$\bar{m}_i^{-1} = M_i^{-1} (1 - N^{-1}) + N^{-1}, \quad (34)$$

and conversely, the infinite-array parameters can be expressed in terms of those obtained using finite-length observational sets:

$$M_i = \frac{1 - N^{-1}}{\bar{m}_i^{-1} - N^{-1}}. \quad (35)$$

Once an ensemble of N -point observational arrays is available the observed mean lengths $\bar{m}_{\bullet}$ and $\bar{m}_{\circ}$ of respectively cloudy and clear intervals can be determined and their infinite-sample analogs $M_{\bullet}$ and $M_{\circ}$ can be computed according to Eq. (35). These parameters directly depend on the elements of the Markov transition matrix of the observation chain defined by Eq. (27):

$$M_{\bullet} = Q_{\bullet\bullet}^{-1} \quad \text{and} \quad M_{\circ} = Q_{\circ\circ}^{-1}. \quad (36)$$

We also introduce M_* defined by analogy with Eq. (10):

$$M_*^{-1} = M_{\bullet}^{-1} + M_{\circ}^{-1} = Q_{\bullet\circ} + Q_{\circ\bullet}, \quad (37)$$

and use these parameters to define the observed cloud and gap lengths:

$$L_{\bullet}^{(\text{obs})} = M_{\bullet} l_p \quad \text{and} \quad L_{\circ}^{(\text{obs})} = M_{\circ} l_p, \quad (38)$$

as well as the observed scale (correlation) length

$$L_{*}^{(\text{obs})} = M_{*} l_p = \frac{M_{\bullet} M_{\circ}}{M_{\bullet} + M_{\circ}} l_p = \frac{l_p}{Q_{\bullet\circ} + Q_{\circ\bullet}}. \quad (39)$$

The observed cloud fraction is defined by analogy with Eq. (9):

$$\bar{c}_{\text{obs}} = \frac{M_{\bullet}}{M_{\bullet} + M_{\circ}} = \frac{Q_{\circ\bullet}}{Q_{\circ\bullet} + Q_{\bullet\circ}}. \quad (40)$$

The cloud-observation system has two scales: natural L_{*} and instrumental l_p . The observational statistics presented below will depend on the ratio of these scales

$$r = \frac{l_p}{L_{*}} \quad (41)$$

that we will call the “scale ratio,” as it is a specific case of Eq. (15) for $L = l_p$. It is also convenient to introduce the observed analog of this parameter

$$r_{\text{obs}} = \frac{l_p}{L_{*}^{(\text{obs})}} = M_{*}^{-1} = Q_{\bullet\circ} + Q_{\circ\bullet}. \quad (42)$$

4. Discrete-point sampling

We start with the simplest of the idealized observation procedures which samples the simulated 1D cloud field a discrete set of N observation points with the regular spacing l_p . In this case the transition matrix [Eq. (27)] of the observation Markov chain is directly related to the transition matrix of the actual cloud field [Eq. (24)] for $L = l_p$:

$$\mathbf{Q} = \mathbf{P}(l_p) = e^{-r} \mathbf{I} + (1 - e^{-r}) \mathbf{C}, \quad (43)$$

where r is defined by Eq. (41). Thus,

$$Q_{\bullet\circ} = \bar{s}(1 - e^{-r}) \quad \text{and} \quad Q_{\circ\bullet} = \bar{c}(1 - e^{-r}). \quad (44)$$

The r -dependent factor cancels from Eq. (40) for the observed cloud fraction. Thus,

$$\bar{c}_{\text{obs}} = \frac{Q_{\circ\bullet}}{Q_{\bullet\circ} + Q_{\circ\bullet}} = \frac{\bar{c}}{\bar{c} + \bar{s}} = \bar{c}; \quad (45)$$

that is, the observational estimate of $\bar{c}$ is unbiased in this case (that can be expected given the nature of the sampling). The scale length from Eq. (39) takes the form

$$L_{*}^{(\text{obs})} = \frac{l_p}{Q_{\bullet\circ} + Q_{\circ\bullet}} = \frac{l_p}{1 - e^{-r}}, \quad (46)$$

which is independent from CF. The relation between the ratios r_{obs} and r is particularly simple:

$$r_{\text{obs}} = 1 - e^{-r}. \quad (47)$$

The structure of the expressions for infinite-sample mean cloud and gap lengths derived using Eqs. (36) and (38) is similar to that of their counterparts for the actual cloud field [Eq. (11)]:

$$L_{\bullet}^{(\text{obs})} = \frac{L_{*}^{(\text{obs})}}{\bar{s}} \quad \text{and} \quad L_{\circ}^{(\text{obs})} = \frac{L_{*}^{(\text{obs})}}{\bar{c}}. \quad (48)$$

It follows from Eq. (46) that

$$L_{*}^{(\text{obs})} (r \rightarrow 0) \approx L_{*} + \frac{l_p}{2} \approx L_{*} \quad (49)$$

in the dense sampling limit; that is, when the observation resolution is much finer than the characteristic variability scale of the observed cloud system, $l_p \ll L_{*}$. In the opposite case of coarse-resolution observations ($l_p \gg L_{*}$) the value of L_{*} “falls through the net” of the measurements: $L_{*}^{(\text{obs})} \approx l_p$ and its dependence on L_{*} becomes very weak. The plot of the dependence [Eq. (46)] of $M_{*} = L_{*}^{(\text{obs})}/l_p$ on the corresponding actual ratio L_{*}/l_p is presented in Fig. 1. We see from Fig. 1 (top) that this dependence reaches its asymptotic regime [Eq. (49)] already at $l_p \leq L_{*}/3$. On the other hand, Fig. 1 (bottom) showing the same function for $l_p \leq L_{*}$ demonstrates that the dependence of M_{*} on L_{*} practically vanishes when $l_p \gtrsim 5L_{*}$.

Equation (46) can be used for derivation of the actual universal scale parameter L_{*} from the observed value as

$$L_{*} = -l_p \left\{ \ln \left[1 - \frac{l_p}{L_{*}^{(\text{obs})}} \right] \right\}^{-1}. \quad (50)$$

This means that both parameters of the underlying continuous-space cloud mask can be derived from observations given a large statistical ensemble of discrete-point observations (sufficient for accurate estimation of $M_{\bullet}$ and $M_{\circ}$) and also sufficiently high resolution of the observational grid ($l_p < 5L_{*}$).

5. Pixel observations

The second and more realistic (while still idealized) observation procedure uses a 1D array of N subsequent intervals (pixels) of the length l_p (with no gaps between them). This is an idealization of observations made by

FIG. 1. Dependence of the ratio of the observed scale length to the grid resolution $L_*^{(\text{obs})}/l_p = 1/r_{\text{obs}}$ on its actual counterpart $L_*/l_p = 1/r$ in a discrete-point observation scenario [from Eq. (46)]. (top) Asymptotic regime for $l_p \ll L_*$. (bottom) As in the top panel, but for a shorter x -axis range showing the lack of dependence of $L_*^{(\text{obs})}$ on L_* when $l_p \geq 5L_*$.

CCD arrays in actual satellite imagers. Similarly to the discrete-point case described above, the result of the observation is a sequence of N binary values (● or ○) attributed to the pixels in the array. The interval in the underlying continuous-space cloud field observed by a single pixel has some cloud fraction $c \in [0, 1]$, and the value of the observation (● or ○) is based on this number. The value-designation procedure is specified by SAFs having the meaning of the probability that the interval with certain CF is declared cloudy or clear in the observed cloud mask. As in the discrete-point case, an array of subsequent cloudy pixels will be called “observed cloud,” while an array of subsequent clear pixels will be called “observed gap.”

a. State attribution functions

SAFs relate the actual cloud fraction in the sample interval (a real number) to a binary result of the

observation. They can be chosen to be deterministic (e.g., equal to ● for CF larger than 50% and to ○ otherwise) or stochastic (e.g., proportional to the CF). The stochasticity of SAF may reflect the influence of external factors (solar-viewing geometry, detector sensitivity to low light, etc.) on the cloud–clear attribution in real satellite datasets.

Let us denote by $\eta_\bullet(c)$ the probability of the sample with cloud cover c to be declared cloudy. Then the complementary probability of declaring the same sample clear is

$$\eta_\circ(c) = 1 - \eta_\bullet(c). \quad (51)$$

Note that these probabilities depend only on the CF in the pixel’s field of view; thus, they are independent from the observation resolution l_p . Note also that the SAFs $\eta_\bullet(c)$ and $\eta_\circ(c)$ are not probability densities; thus, there are no conditions on their integrals. SAFs cannot be singular (i.e., cannot contain δ -function terms). The natural condition on SAFs is $0 \leq \eta_i \leq 1$. There are no additional mathematical restrictions on these functions; however, “physically sound” function $\eta_\bullet(c)$ may be expected to be nondecreasing, while $\eta_\circ(c)$ may be expected to be nonincreasing. Also, the sensor is expected to correctly interpret overcast and all-clear scenes:

$$\eta_\bullet(0) = 0, \quad \eta_\bullet(1) = 1; \quad \eta_\circ(0) = 1, \quad \eta_\circ(1) = 0. \quad (52)$$

One may also impose additional condition of symmetry with respect to cloud–clear interchange:

$$\eta_\bullet(1 - c) = 1 - \eta_\bullet(c) \quad \text{or} \quad \eta_\bullet(s) = \eta_\circ(c), \quad (53)$$

which implies $\eta_\bullet(1/2) = \eta_\circ(1/2) = 1/2$.

An obvious example of SAF is a step function:

$$\eta_\bullet(c) = \begin{cases} 0 & \text{for } c \leq c_{\text{th}} \\ 1 & \text{for } c > c_{\text{th}} \end{cases}. \quad (54)$$

where c_{th} is some threshold cloud fraction. In more realistic situation when the cloud mask is derived based on remote sensing measurements, there is a “gray area” between decisively clear and decisively cloudy estimates. This corresponds to smoothing of SAFs. The simplest example of a smooth SAF is the linear one:

$$\eta_\bullet(c) = c, \quad \eta_\circ(c) = 1 - c, \quad (55)$$

which allows for analytical computations of the observed statistics. Functional shapes of SAFs can be constructed to model technical problems affecting the sensor, for example, by introducing some randomness into state attribution process, thus, making it less

dependent on the actual CF observed in the pixel. For example, completely random SAF is a constant.

The distribution $F_{cf}(c)$ of cloud fraction in an ensemble of samples of the length l_p (pixel footprint size) is specified by Eqs. (19) and (20). Then the probabilities for a sample randomly selected from this statistical ensemble to be declared cloudy or clear are

$$p_k = \int_0^1 \eta_k(c) F_{cf}(c) dc, \quad (56)$$

where $k = \bullet$ or $\circ$. Using Eq. (19) this expression can be written in the matrix form

$$p_k = \text{Tr}[\mathbf{U}\mathbf{H}^{(k)}]. \quad (57)$$

Here $\mathbf{U}$ is the initial state matrix and

$$\mathbf{H}^{(k)} = \int_0^1 \eta_k(c) \mathbf{F}(c) dc \quad (58)$$

is the matrix with the elements

$$H_{ij}^{(k)} = \int_0^1 \eta_k(c) F_{ij}(c) dc, \quad (59)$$

where $i, j, k = \bullet$ or $\circ$. Each $H_{ij}^{(k)}$ is the probability of the interval to be assigned the state k when the states of its first and last points are respectively i and j ; $\mathbf{H}^{(k)}$ are not transition matrices, since their rows do not sum up to unity [as in Eq. (2)]. Instead, they satisfy the condition

$$\mathbf{H}^\bullet + \mathbf{H}^\circ = \mathbf{P}, \quad (60)$$

where $\mathbf{P}$ is the transition probability matrix of the model.

Note about the notations in this paper, we will put upper indices in parentheses if they are variables denoted by letters (such as k) to avoid confusion with taking power, while explicit indices $\bullet$ and $\circ$ will be written without parentheses to make the expressions less cumbersome.

In the case of a symmetric SAF [Eq. (53)] the matrix elements of $\mathbf{H}^\bullet$ and $\mathbf{H}^\circ$ are CCI related:

$$H_{\bullet\bullet}^\bullet \leftrightarrow H_{\circ\circ}^\circ, \quad H_{\bullet\circ}^\bullet \leftrightarrow H_{\circ\bullet}^\circ, \quad (61)$$

$$H_{\circ\bullet}^\bullet \leftrightarrow H_{\bullet\circ}^\circ, \quad H_{\circ\circ}^\bullet \leftrightarrow H_{\bullet\bullet}^\circ, \quad (62)$$

because so are the integrands in Eq. (59): $\eta_\bullet \leftrightarrow \eta_\circ$, while $F_{\bullet\circ} \leftrightarrow F_{\circ\bullet}$ and $F_{\bullet\bullet} \leftrightarrow F_{\circ\circ}$.

b. Conditional transition probability matrix

Let us derive the transition probabilities $G_{ij}^{(k)}$ between the states i of the first point and j of the last point of the interval (pixel) conditioned on the attributed state k of

the interval itself. These probabilities form the conditional transition matrix

$$\mathbf{G}^{(k)} = \begin{pmatrix} G_{\bullet\bullet}^{(k)} & G_{\bullet\circ}^{(k)} \\ G_{\circ\bullet}^{(k)} & G_{\circ\circ}^{(k)} \end{pmatrix}. \quad (63)$$

To derive the explicit expressions for $G_{ij}^{(k)}$ we restrict our statistical ensemble by requiring starting points of the intervals to have a given state i [this is identified by the index (i) in definitions below]. Then $G_{ij}^{(k)} = p_{(i)}(j|k)$ is the probability of the end point to have state j conditional to attribution of state k to the whole interval. The state matrix $\mathbf{U} = \mathbf{U}^{(i)}$ [Eq. (5)] describes the state of the interval's starting point; thus, according to Eq. (19), the CF distribution function in this case has the following form:

$$F_{cf(i)}(c) = \text{Tr}[\mathbf{U}^{(i)}\mathbf{F}(c)] = F_{i\bullet}(c) + F_{i\circ}(c). \quad (64)$$

This formula can be written also in the following form:

$$F_{cf(i)}(c) = P_{i\bullet} F_{cf(i\bullet)}(c) + P_{i\circ} F_{cf(i\circ)}(c), \quad (65)$$

where P_{ij} are elements of the transition matrix with the meaning of probability of the interval's end-point state to be j , while

$$F_{cf(ij)}(c) = \frac{F_{ij}(c)}{P_{ij}} \quad (66)$$

are normalized [due to Eq. (23)] CF distribution functions corresponding to an ensemble of intervals with end-point state j . Then the probability of the state k to be attributed to an interval with end-point state j is

$$p_{(i)}(k|j) = \int_0^1 \eta_k(c) F_{cf(ij)}(c) dc = \frac{H_{ij}^{(k)}}{P_{ij}}, \quad (67)$$

while the total probability of the interval's state to be k is

$$p_{(i)}(k) = H_{i\bullet}^{(k)} + H_{i\circ}^{(k)}, \quad (68)$$

as it follows from Eq. (65). Using the Bayes formula

$$p_{(i)}(j)p_{(i)}(k|j) = p_{(i)}(k)p_{(i)}(j|k), \quad (69)$$

[where $p_{(i)}(j) = P_{ij}$] we find that

$$H_{ij}^{(k)} = (H_{i\bullet}^{(k)} + H_{i\circ}^{(k)}) G_{ij}^{(k)}, \quad (70)$$

and, therefore,

$$G_{ij}^{(k)} = \frac{H_{ij}^{(k)}}{H_{i\bullet}^{(k)} + H_{i\circ}^{(k)}}. \quad (71)$$

It is easy to see that this expression satisfies the transition matrix condition

$$G_{i\bullet}^{(k)} + G_{i\circ}^{(k)} = 1. \quad (72)$$

It is also not difficult to verify that the total transition probability unconditional with respect to k is indeed P_{ij} :

$$\begin{aligned} \sum_{k=\bullet,\circ} p_{(i)}(k) G_{ij}^{(k)} &= \sum_{k=\bullet,\circ} (H_{i\bullet}^{(k)} + H_{i\circ}^{(k)}) G_{ij}^{(k)} \\ &= \sum_{k=\bullet,\circ} H_{ij}^{(k)} = P_{ij} \end{aligned} \quad (73)$$

[here we used Eqs. (60), (68), and (71)].

c. Markov chain of pixel observations

The state of one pixel can influence that of the next (adjacent) one only through the state of its ending point, which also is the starting point of the second pixel. If the attributed state of the first pixel is k and the starting point state of this pixel is described by the matrix $\mathbf{U}$, then the state of its ending point is described by the matrix

$$\mathbf{U}' = \mathbf{U}\mathbf{G}^{(k)}, \quad (74)$$

which has the same structure as $\mathbf{U}$. This matrix defines CF distribution function in the second interval:

$$F_{\text{cf}}(c) = \text{Tr}[\mathbf{U}'\mathbf{F}(c)]. \quad (75)$$

Thus, the probability for the second pixel to be assigned with the state l is

$$p_l = \int_0^1 \eta_k(c) F_{\text{cf}}(c) dc = \text{Tr}[\mathbf{U}'\mathbf{H}^{(l)}], \quad (76)$$

which can be interpreted as the probability of transition from the state k of the first pixel to the state l of the second one:

$$Q_{kl} = \text{Tr}[\mathbf{U}\mathbf{G}^{(k)}\mathbf{H}^{(l)}]. \quad (77)$$

This probabilities form the transition matrix

$$\mathbf{Q} = \begin{pmatrix} Q_{\bullet\bullet} & Q_{\bullet\circ} \\ Q_{\circ\bullet} & Q_{\circ\circ} \end{pmatrix} \quad (78)$$

of the pixel-state Markov chain. This matrix satisfies the transition matrix property [Eq. (28)] that can be verified using Eqs. (60) and (74) for $k \neq l$:

$$\begin{aligned} Q_{kk} + Q_{kl} &= \text{Tr}[\mathbf{U}\mathbf{G}^{(k)}(\mathbf{H}^{(k)} + \mathbf{H}^{(l)})] \\ &= \text{Tr}(\mathbf{U}'\mathbf{P}) = 1, \end{aligned} \quad (79)$$

and noticing that $\mathbf{U}'\mathbf{P}$ is also a state matrix with unit trace [see Eq. (7)]. This condition means that only diagonal

elements of $\mathbf{Q}$ need to be computed explicitly, while the off-diagonal elements can be derived from Eq. (79).

The matrix $\mathbf{U}'$ defined by Eq. (74) is a state matrix corresponding to the probability

$$u' = uG_{\bullet\bullet}^{(k)} + vG_{\circ\bullet}^{(k)}, \quad (80)$$

while

$$\begin{aligned} Q_{kk} &= \text{Tr}[\mathbf{U}'\mathbf{H}^{(k)}] \\ &= u'[H_{\bullet\bullet}^{(k)} + H_{\circ\bullet}^{(k)}] + v'[H_{\circ\bullet}^{(k)} + H_{\circ\circ}^{(k)}], \end{aligned} \quad (81)$$

where

$$v' = 1 - u' = uG_{\circ\bullet}^{(k)} + vG_{\circ\circ}^{(k)} \quad (82)$$

[here we used Eq. (72)]. Substituting Eqs. (80) and (82) into Eq. (81) and using Eq. (71) we obtain

$$Q_{kk} = uH_{\bullet\bullet}^{(k)} + uH_{\circ\bullet}^{(k)} R^{(k)} + v \frac{H_{\circ\bullet}^{(k)}}{R^{(k)}} + vH_{\circ\circ}^{(k)}, \quad (83)$$

where

$$R^{(k)} = \frac{H_{\circ\bullet}^{(k)} + H_{\circ\circ}^{(k)}}{H_{\bullet\bullet}^{(k)} + H_{\circ\bullet}^{(k)}}. \quad (84)$$

It is imported to note that in this formulation pixel states do not form a proper Markov chain since the probability of the transition from the state k of the first pixel to the state l of the second one depends not only on k but also on the initial-state matrix $\mathbf{U}$ entering Eq. (77). However, we can make the model Markovian by assigning the value of the initial-state probability u (and, therefore, $\mathbf{U}$) depending solely on the state k . This is an approximation, the accuracy of which will be checked by numerical simulations. The assignment of u is to be determined from correct asymptotic behavior in the case when pixels are infinitesimally smaller than the characteristic scale of the cloud field ($l_p \ll L_*$, $r \rightarrow 0$). In this case almost all pixels have CF of either 0 or 1 [corresponding to the singular parts of Eqs. (13) and (20); see Part I for details], so no randomness is involved in the pixel-state attribution procedure [because of the conditions in Eq. (52)]. This means that observations with infinitely high resolution provide the true picture of the cloud field; thus, the observed statistics in this case should coincide with the actual ones: $r_{\text{obs}} = r$ and $\bar{c}_{\text{obs}} = \bar{c}$. These requirements together with Eqs. (40)–(42) impose the following conditions on the matrix elements of $\mathbf{Q}$:

$$Q_{\circ\bullet} + Q_{\bullet\circ} = r \quad \text{and} \quad \frac{Q_{\circ\bullet}}{Q_{\circ\bullet} + Q_{\circ\circ}} = \frac{Q_{\circ\bullet}}{r} = \bar{c}, \quad (85)$$

which yield $Q_{\bullet\bullet} = \bar{c}r$ and $Q_{\bullet\circ} = \bar{s}r$ and the asymptotic pixel transition matrix should have the following form:

$$\mathbf{Q}^{(0)} = \begin{pmatrix} 1 - \bar{s}r & \bar{s}r \\ \bar{c}r & 1 - \bar{c}r \end{pmatrix} = \mathbf{I} + (\mathbf{C} - \mathbf{I})r. \quad (86)$$

The expression for $\mathbf{Q}^{(0)}$ is derived in [appendix A](#) based on the general form [Eqs. (83) and (84)] of the matrix elements of $\mathbf{Q}$. It coincides with Eq. (86) if and only if we set $v = 0$ ($u = 1$) for $k = \bullet$ and $u = 0$ ($v = 1$) for $k = \circ$ [i.e., if $\mathbf{U} = \mathbf{U}^{(k)}$; Eq. (5)]. This choice of u means that we assume that the state of the starting point of the pixel is the same as the attributed state of this pixel as a whole.

Note that having different state matrices $\mathbf{U}$ for different attributed states k does not affect the Markovian property [Eq. (79)] as well as the asymptotic expression for $\mathbf{Q}$ in the large-pixel limit (see [appendix A](#)):

$$\mathbf{Q}^{(\infty)} = \begin{bmatrix} \eta_{\bullet}(\bar{c}) & \eta_{\circ}(\bar{c}) \\ \eta_{\bullet}(\bar{c}) & \eta_{\circ}(\bar{c}) \end{bmatrix} \quad (87)$$

[note that $\eta_{\bullet}(\bar{c}) + \eta_{\circ}(\bar{c}) = 1$]. The observed correlation length in this case $L_{*}^{(\text{obs})} = l_p$ (i.e., $r_{\text{obs}} = 1$, independent of r) meaning that the state of each pixel correlates only with itself; that is, all pixels are statistically independent. The observed cloud fraction has the form

$$\bar{c}_{\text{obs}} = \eta_{\bullet}(\bar{c}), \quad (88)$$

which is expected since each of the pixels has CF close to $\bar{c}$.

After setting $\mathbf{U} = \mathbf{U}^{(k)}$ in Eq. (77) as described above, we can write the explicit expressions for the diagonal elements of the pixel transition probability matrix in general form:

$$Q_{\bullet\bullet} = H_{\bullet\bullet}^{\bullet} + H_{\bullet\circ}^{\bullet} R^{\bullet} \quad (89)$$

and

$$Q_{\circ\circ} = \frac{H_{\circ\circ}^{\circ}}{R^{\circ}} + H_{\circ\bullet}^{\circ}. \quad (90)$$

In the case of symmetric SAF the matrix $\mathbf{Q}$ exhibits simple transformation properties under CCI. In this case the matrix elements $H_{ij}^{(k)}$ transform according to Eqs. (61) and (62); thus,

$$R^{\bullet} \leftrightarrow \frac{1}{R^{\circ}}, \quad (91)$$

so

$$Q_{\bullet\bullet} \leftrightarrow Q_{\circ\circ} \quad \text{and} \quad Q_{\bullet\circ} \leftrightarrow Q_{\circ\bullet}. \quad (92)$$

This significantly simplifies computations since only one diagonal matrix element (e.g., $Q_{\bullet\bullet}$) should be derived

directly, while the other diagonal element can be obtained using CCI. The off-diagonal elements are then computed using Eq. (79).

The statistics of Markov chain of pixel observations can be derived from the general expressions shown in Eqs. (36)–(40) in section 4 with

$$Q_{\bullet\bullet} = 1 - H_{\bullet\bullet}^{\bullet} - H_{\bullet\circ}^{\bullet} R^{\bullet}, \quad (93)$$

and

$$Q_{\circ\bullet} = 1 - \frac{H_{\circ\bullet}^{\circ}}{R^{\circ}} - H_{\circ\circ}^{\circ}. \quad (94)$$

The results of substitution of Eqs. (93) and (94) into Eqs. (36)–(40) are rather cumbersome and do not allow for any substantial simplification, so we leave the final general expressions to the reader.

d. Example of the linear SAF

The general method for derivation of the observed cloud-field statistics using the actual parameters $\bar{c}$ and L_{*} (or r) and the sensor resolution l_p can be illustrated on the example of SAF having simple linear form $\eta_{\bullet}(c) = c$. In this case all computations can be performed analytically using the results of [Part III](#). These computations (presented in [appendix B](#)) lead to the following expressions for the off-diagonal elements of the observational transition probability matrix $\mathbf{Q}$:

$$Q_{\bullet\circ} = \bar{s} - \bar{s} \frac{\bar{c}g + \bar{s}f}{\bar{c} + \bar{s}g}, \quad (95)$$

and

$$Q_{\circ\bullet} = \bar{c} - \bar{c} \frac{\bar{s}g + \bar{c}f}{\bar{s} + \bar{c}g}, \quad (96)$$

where

$$f(r) = e^{-r} \quad \text{and} \quad g(r) = \frac{1 - e^{-r}}{r} \quad (97)$$

are functions of the scale ratio r .

In the small pixels limit ($r \rightarrow 0$) these expressions take the forms

$$Q_{\bullet\circ}(r \rightarrow 0) \approx \bar{s}r \quad \text{and} \quad Q_{\circ\bullet}(r \rightarrow 0) \approx \bar{c}r, \quad (98)$$

the same as the corresponding matrix elements in Eq. (86). In this case all observed statistics coincide with the actual parameters of the cloud field.

In the opposite limit of large pixels ($r \rightarrow \infty$) both functions $f(r)$ and $g(r)$ vanish in Eqs. (95) and (96) yielding

FIG. 2. Comparison between observed and actual cloud-field statistics in the case of the linear SAF. (top left) Differences between the observed cloud fraction $\bar{c}_{\text{obs}}$ and the actual CF $\bar{c}$ as functions of the scale ratio r for various values of $\bar{c}$. (top right) As in the top-left panel, but as functions of $\bar{c}$ for various values of r . (bottom left) Observed scale ratios r_{obs} as functions of actual r for $\bar{c} = 0.5$ (solid) and $\bar{c} = 0$ or 1 (dashed). The latter curve coincides with that in the case of discrete sampling [Eq. (47)]. The curves corresponding to all other values of $\bar{c}$ would lie between these two curves. (bottom right) Relatively weak dependences of r_{obs} on the actual CF shown as the $\bar{c}$ -dependent differences between r_{obs} and its minimal values (at given r) corresponding to $\bar{c} = 0.5$. The CF and r values in the legend are the actual, not the observed, ones.

$$Q_{\bullet\bullet}(r \rightarrow \infty) = \bar{s} \quad \text{and} \quad Q_{\bullet\bullet}(r \rightarrow \infty) = \bar{c}; \quad (99)$$

thus, $\mathbf{Q}^{(\infty)} = \mathbf{C}$. This is in agreement with Eq. (87) since in this case $\eta_{\bullet}(\bar{c}) = \bar{c}$ and $\eta_{\circ}(\bar{c}) = \bar{s}$. Substitution of these values into Eqs. (40) and (42) results in

$$\bar{c}_{\text{obs}}(r \rightarrow \infty) = \bar{c} \quad \text{and} \quad r_{\text{obs}}(r \rightarrow \infty) = 1, \quad (100)$$

respectively, which is also in agreement with the results of the previous subsection.

Figure 2 presents a comparison between the observed $\bar{c}_{\text{obs}}$ and r_{obs} and the actual counterparts $\bar{c}$ and r . The observed statistics were computed according to Eqs. (40) and (42) using Eqs. (95) and (96) for the elements of the transition probability matrix. This figure shows that the statistics of pixel observations with the linear SAF do not significantly deviate from those in the discrete-sampling

scenario, where $\bar{c}_{\text{obs}} \equiv \bar{c}$ and r_{obs} is independent from the CF. The top panels of Fig. 2 show the difference between the observed and the actual CFs as a function of the actual scale ratio r and cloud fraction $\bar{c}$. As we demonstrated above the equality $\bar{c}_{\text{obs}} = \bar{c}$ holds in both asymptotic cases of small ($r \rightarrow 0$) and large $r \rightarrow \infty$ pixels, and also when $\bar{c} = 1/2$ or is close to either 0 or 1. The observed CF underestimates the actual one when $\bar{c} < 1/2$ and overestimates when $\bar{c} > 1/2$; however, the differences are always very small (below 0.005 in absolute value). The bottom panels of Fig. 2 show that the observed scale ratio r_{obs} depends mostly on the actual r , while its dependence on $\bar{c}$ is weak. The values of r_{obs} corresponding to all CFs are confined between the solid and dashed curves in Fig. 2 (bottom left). The solid curve here corresponds to $\bar{c} = 1/2$, where r_{obs} achieves its minimum for a given value of r , while the maxima of r_{obs} are at the

ends of the CF interval ($\bar{c} = 0$ or 1) corresponding to dashed curve in Fig. 2 (bottom left). Note that when $\bar{c} = 0$ or 1 the observed scale ratio $r_{\text{obs}}(r)$ can be computed using the same formula [Eq. (47)] as in the discrete-sampling approach (see appendix B). Figure 2 (bottom left) shows that for any value of $\bar{c}$ the behavior of the observed scale ratio is similar: $r_{\text{obs}} \approx r$ for small r and asymptotically approaches unity as r increases. The latter behavior is the result of statistical independence between states of large pixels, which prevents the observed cloud scale $L_*^{(\text{obs})}$ from being smaller than the pixel size l_p . In other words, subpixel variability cannot be directly observed. The deviation of r_{obs} from its minimal value at $\bar{c} = 1/2$ (at given r) is shown in Fig. 2 (bottom right) as a function of CF. This difference is rather small, being overall less than 0.04 (achieved at $r = 3$; solid curve), and vanishes in the asymptotic cases of small and large pixels.

We should note that the observed cloud-field statistics are not expected to coincide with the actual ones because of the role of the SAF in the pixel-observation model. This function is a characteristic of the observation process, not the cloud field being observed, and it can be chosen to be, for example, strongly asymmetric, thus, causing substantial bias in the observed CF. However, the dependences of the observed cloud-field statistics on the actual ones, such as these presented in Fig. 2, can be inverted numerically to obtain the latter from the former. Also, noting that pixel observations in the case of the linear SAF are so close to those in the discrete-sampling approach, the latter can be used as a convenient approximation yielding simple analytical inversion formula Eq. (50). The observation resolution requirement for the inversion ($l_p < 5L_*$) is expected to be also valid in the case of the linear SAF.

6. Numerical simulations

To verify our analytical expressions for the observed cloud-field statistics we performed numerical simulations of 1D cloud fields and observation protocols (sampling and pixel) applied to them. The cloud fields were generated according to the algorithm described in Part I (section 4a), which in fact is a binary Markov process technique [also used by Doicu et al. (2013, 2014a,b) and Efremenko et al. (2016)]. The CF values of 0.1, 0.2, 0.3, 0.4, and 0.5 were chosen, while the results for $\text{CF} > 0.5$ can be obtained from these for $\text{CF} < 0.5$ using cloud-clear interchange. For each value of CF we produced 5000 individual samples of the length 20, measured in units of “convective cell” length (denoted by l in Part I). Each cell consisted of 200 subcells. After the

samples were generated, the “observations” were made by dividing each sample into a number (5, 10, 20, 40, and 80) intervals (pixels) and applying linear SAF to them based on their CF; at the same time the midpoints of the intervals were used for construction of discrete-point sampling observations. Figure 3 presents two examples [corresponding to $\bar{c}$ of 0.1 (left column) and 0.5 (right column)] of actual cloud fields (top row) and those obtained as results of discrete-point (middle row) and pixel (bottom row) observations. The first 50 generated samples are shown. The number of pixels and sampling points in these examples is 20. The actual and observational statistics for these two cases (computed theoretically and derived from the simulations) are presented in Tables 1 and 2. Comparison of the statistics $\bar{c}_{\text{obs}}$ and $L_*^{(\text{obs})}$ of the simulated datasets with their theoretical expectations showed agreement within 0.005 for the CF, while for the scale length it is within 1% in the sampling case and within 4% in the pixel case. The latter slightly larger difference is likely to be associated with the fact that the analytical model of pixel Markov chain [and, therefore, Eqs. (89) and (90)] is approximate. No systematic dependence of the agreement on the pixel number/size has been detected. These comparisons confirm the adequacy of our analytical approach and high accuracy (at least for the linear SAF) of the approximation used in description of pixel observations as realizations of the Markov chain.

7. Concluding remarks

The goal of Part IV of the “Cellular Statistical Models of Broken Cloud Fields” series is to develop a mathematical framework in which coarse-resolution cloud masks can be used to predict true cloud-field statistics under “idealized” observing conditions. Assuming Markovian properties of the actual cloud field (which are described in detail in Part III of this series) we “make observations” of it according to two protocols: simple discrete-point sampling and a more realistic pixel-based method. In both cases the cloud masks resulting from the observations are described as Markov chains characterized by the same type of statistics (cloud fraction and correlation or scale length) as the actual cloud field. The real and observed cloud parameters appear to be related, making it possible to restore the actual statistics from observations (if the sensor’s resolution is not too coarse compared to the internal scale of the cloud field).

In this study, as in Part III, we assume “narrowly sampled” cloud fields characterized by exponential chord-length distributions for both clouds and gaps.

FIG. 3. Examples of (top) simulated 1D cloud masks and secondary cloud masks resulting from (middle) discrete-point sampling and (bottom) pixel observations of the original fields for simulations with (left) $\bar{\epsilon} = 0.1$ and (right) $\bar{\epsilon} = 0.5$. The number of pixels (separated by dashed lines in bottom panels) and sampling points (depicted by circles in middle panels) is 20.

In the opposite case of “diverse” sampling these distributions have a power-law form (see [Part II](#) for details). Here (as in the entire series) we also assume availability of an infinite ensemble of cloud fields and

observations so we do not have to address the issues of statistical uncertainties due to a limited number of samples (e.g., [Astin et al. 2001](#); [van de Poll et al. 2006](#)). Note that in our “ideal” mathematical world we are not

TABLE 1. Actual and observed cloud-field statistics for the simulation from Fig. 3 (left) with $\bar{c} = 0.1$.

	$\bar{c}$	L_*	$L_\bullet$	L_o	r
Actual					
Theory	0.10	0.52	0.58	5.17	1.93
Simulation	0.10	0.52	0.58	5.18	1.91
Sampling					
Theory	0.10	1.17	1.30	11.67	0.85
Simulation	0.10	1.17	1.30	11.51	0.85
Pixel					
Theory	0.10	1.19	1.31	12.37	0.84
Simulation	0.10	1.24	1.38	12.42	0.81

constrained by the spatial size of the samples (unlike in the real world; e.g., Dey et al. 2008), since the infinite-sample statistics can be derived from the finite-sample ensemble (see Part I for details).

The idealized satellite sensor in our second, more realistic, model has a pixel array, with each pixel taking the value “cloudy” or “clear” depending on CF in the scene in its field of view. The assignment of pixel value is generally assumed to be stochastic, representing a variety of optical effects affecting real satellite instruments. This procedure is characterized by the state attribution function (SAF). The transition probability matrix of the observational Markov chain [Eq. (27)] in the general case has matrix elements described by Eqs. (89), (90), (93), and (94), while the expression for the observed CF and scale length are given by Eqs. (40) and (39), respectively. The analytical derivation of these formulas is based on the notion of the master matrix (introduced in Part III), which unifies the statistics of the cloud fraction and the Markovian properties of the cloud field. In the case of SAF having a simple analytical form (linear in CF) we succeeded in analytical computation of the pixel-observation statistics (using the results obtained in Part III), which appeared to be not so different numerically from those in the discrete-point sampling case. The matrix elements of the transition probability matrix in these case are given by Eqs. (95) and (96). The observed cloud fraction appears to be an almost unbiased estimate of the real one, while the scale length can be significantly overestimated when pixel size exceeds the characteristic scale of the observed cloud system.

Our future plans include further testing of the presented cloud-observation model using other SAFs (possibly utilizing the second matrix integral derived in Part III), and, in general, working on adoption of our mathematical framework for analysis of real satellite datasets.

TABLE 2. Actual and observed cloud-field statistics for the simulation from Fig. 3 (right) with $\bar{c} = 0.5$.

	$\bar{c}$	L_*	$L_\bullet$	L_o	r
Actual					
Theory	0.50	0.72	1.44	1.44	1.39
Simulation	0.50	0.72	1.45	1.44	1.39
Sampling					
Theory	0.50	1.33	2.67	2.67	0.75
Simulation	0.50	1.34	2.67	2.65	0.75
Pixel					
Theory	0.50	1.38	2.77	2.77	0.72
Simulation	0.50	1.41	2.84	2.82	0.71

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APPENDIX A

Pixel Transition Matrix and Observed Cloud Statistics in Asymptotic Cases

We test our construction in the asymptotic cases when pixels are infinitesimally small or infinitely large (compared to the cloud field’s scale length).

a. Small pixel

In the small-pixel limit ($l_p \ll L_*$, $r \rightarrow 0$),

$$I_0(2\sqrt{z}) \approx 1 + z \quad \text{and} \quad \frac{I_1(2\sqrt{z})}{\sqrt{z}} \approx 1 + \frac{z}{2} \quad (\text{A1})$$

(Abramowitz and Stegun 1972). In our case,

$$2\sqrt{z} = Z = 2r\sqrt{\bar{c}s} \quad cs; \quad (\text{A2})$$

thus,

$$z = \bar{c}s \quad csr^2. \quad (\text{A3})$$

While in order to compute the observed cloud fraction and characteristic scale in this limit we need an expansion of $\mathbf{Q}$ up to the terms linear in r , we have to keep both linear and quadratic terms in Eq. (14) for master matrix $\mathbf{F}(c)$ (in addition to the δ -function terms). This is necessary because of division by r in the expressions for $1/R^\bullet$ and R° . In this expansion the matrix elements of $\mathbf{F}(c)$ have the following form:

$$F_{\bullet\bullet} = \bar{c}s \, cr^2 + \delta(s) \left(1 - \bar{s}r + \bar{s}^2 \frac{r^2}{2}\right), \quad (\text{A4})$$

$$F_{\bullet\circ} = \bar{s}r - \bar{s}(\bar{c}s + \bar{s}c)r^2, \quad (\text{A5})$$

$$F_{\circ\bullet} = \bar{c}r - \bar{c}(\bar{c}s + \bar{s}c)r^2, \quad (\text{A6})$$

$$F_{\circ\circ} = \bar{c}s \, sr^2 + \delta(c) \left(1 - \bar{c}r + \bar{c}^2 \frac{r^2}{2}\right). \quad (\text{A7})$$

One can verify using Eq. (23) that the transition matrix in this limit coincides with the direct expansion in r of Eq. (24),

$$\mathbf{P} = \left(1 - r + \frac{r^2}{2}\right)\mathbf{I} + \left(r - \frac{r^2}{2}\right)\mathbf{C}, \quad (\text{A8})$$

and satisfies the requirements of Eq. (2).

Now we can derive the matrices $\mathbf{H}^{(k)}$ according to Eq. (58). It follows from Eq. (52) that

$$\int_0^1 \eta_{\bullet}(c)\delta(c) \, dc = \eta_{\bullet}(0) = 0, \quad (\text{A9})$$

$$\int_0^1 \eta_{\bullet}(c)\delta(s) \, dc = \eta_{\bullet}(1) = 1; \quad (\text{A10})$$

$$\int_0^1 \eta_{\circ}(c)\delta(c) \, dc = \eta_{\circ}(0) = 1, \quad (\text{A11})$$

$$\int_0^1 \eta_{\circ}(c)\delta(s) \, dc = \eta_{\circ}(1) = 0. \quad (\text{A12})$$

We also introduce the integrals

$$h_k = \int_0^1 \eta_k(c) \, dc \quad (\text{A13})$$

($k = \bullet$ and $\circ$), which are related to each other [as it follows from Eq. (51)]:

$$h_{\bullet} + h_{\circ} = 1. \quad (\text{A14})$$

We will also need the following parameters:

$$y_k^{(c)} = \int_0^1 c\eta_k(c) \, dc \quad \text{and} \quad y_k^{(s)} = \int_0^1 s\eta_k(c) \, dc, \quad (\text{A15})$$

which by definition have the following relationships:

$$y_k^{(c)} + y_k^{(s)} = h_k \quad (\text{A16})$$

and

$$y_{\bullet}^{(c)} + y_{\circ}^{(c)} = y_{\bullet}^{(s)} + y_{\circ}^{(s)} = \frac{1}{2}. \quad (\text{A17})$$

Using the above definitions and relations we can write the expressions for matrix elements of $\mathbf{H}^{\bullet}$ and $\mathbf{H}^{\circ}$ as following:

$$H_{\bullet\bullet}^{\bullet} = 1 - \bar{s}r + \bar{s} \left(\frac{\bar{s}}{2} + \bar{c}y_{\bullet}^{(c)}\right)r^2, \quad (\text{A18})$$

$$H_{\bullet\circ}^{\bullet} = \bar{s}h_{\bullet}r - \bar{s}(\bar{c}y_{\bullet}^{(s)} + \bar{s}y_{\bullet}^{(c)})r^2, \quad (\text{A19})$$

$$H_{\circ\bullet}^{\bullet} = \bar{c}h_{\bullet}r - \bar{c}(\bar{c}y_{\bullet}^{(s)} + \bar{s}y_{\bullet}^{(c)})r^2, \quad (\text{A20})$$

$$H_{\circ\circ}^{\bullet} = \bar{c}s \, y_{\bullet}^{(s)}r^2, \quad (\text{A21})$$

and

$$H_{\bullet\bullet}^{\circ} = \bar{c}s \, y_{\circ}^{(c)}r^2, \quad (\text{A22})$$

$$H_{\bullet\circ}^{\circ} = \bar{s}h_{\circ}r - \bar{s}(\bar{c}y_{\circ}^{(s)} + \bar{s}y_{\circ}^{(c)})r^2, \quad (\text{A23})$$

$$H_{\circ\bullet}^{\circ} = \bar{c}h_{\circ}r - \bar{c}(\bar{c}y_{\circ}^{(s)} + \bar{s}y_{\circ}^{(c)})r^2, \quad (\text{A24})$$

$$H_{\circ\circ}^{\circ} = 1 - \bar{c}r + \bar{c} \left(\frac{\bar{c}}{2} + \bar{s}y_{\circ}^{(s)}\right)r^2. \quad (\text{A25})$$

While here we do not assume that SAF is symmetric, the above expressions exhibit symmetry under CCI of the form Eqs. (61) and (62) if we also assume that $h_{\bullet} \leftrightarrow h_{\circ}$, $y_{\bullet}^{(c)} \leftrightarrow y_{\circ}^{(s)}$, and $y_{\bullet}^{(s)} \leftrightarrow y_{\circ}^{(c)}$. This allows us to compute only the matrix element $Q_{\bullet\bullet}$, while the matrix elements $Q_{\circ\circ}$ can be obtained from CCI [Eq. (92)]. We start with straightforward computation of $R^{\bullet}$ according to Eq. (84) (with $k = \bullet$). Then, substituting the resulting expression together with Eqs. (A18)–(A21) into Eq. (83), and keeping terms up to the first order in r we obtain the following formula:

$$\begin{aligned} Q_{\bullet\bullet} &= uH_{\bullet\bullet}^{\bullet} + uH_{\bullet\circ}^{\bullet}R^{\bullet} + v\frac{H_{\circ\bullet}^{\bullet}}{R^{\bullet}} + vH_{\circ\circ}^{\bullet} \\ &= 1 - \bar{s}(1 - v\alpha_{\bullet})r, \end{aligned} \quad (\text{A26})$$

while CCI provides its counterpart

$$\begin{aligned} Q_{\circ\circ} &= uH_{\bullet\bullet}^{\circ} + uH_{\bullet\circ}^{\circ}R^{\circ} + v\frac{H_{\circ\circ}^{\circ}}{R^{\circ}} + vH_{\circ\bullet}^{\circ} \\ &= 1 - \bar{c}(1 - u\alpha_{\circ})r. \end{aligned} \quad (\text{A27})$$

In the above expressions we used the notations

$$\alpha_{\bullet} = h_{\bullet} - \frac{y_{\bullet}^{(s)}}{h_{\bullet}} \quad \text{and} \quad \alpha_{\circ} = h_{\circ} - \frac{y_{\circ}^{(c)}}{h_{\circ}}. \quad (\text{A28})$$

The expressions for off-diagonal matrix elements of $\mathbf{Q}$ follow from the Markovian property of the transition matrix [Eq. (28)]:

$$Q_{\bullet\circ} = \bar{s}(1 - v\alpha_{\bullet})r \quad \text{and} \quad Q_{\circ\bullet} = \bar{c}(1 - u\alpha_{\circ})r. \quad (\text{A29})$$

Finally, we can write the asymptotic expression for $\mathbf{Q}$ in the limit $r \rightarrow 0$ corresponding to infinitesimally small pixel size in the following form:

$$\mathbf{Q}^{(0)} = \mathbf{I} + \mathbf{A}r, \quad (\text{A30})$$

where

$$\mathbf{A} = \begin{bmatrix} -\bar{s}(1 - v\alpha_{\bullet}) & \bar{s}(1 - v\alpha_{\bullet}) \\ \bar{c}(1 - u\alpha_{\circ}) & -\bar{c}(1 - u\alpha_{\circ}) \end{bmatrix}. \quad (\text{A31})$$

To provide exact actual cloud-field statistics at infinitely fine resolution of the observations, Eq. (A30) should coincide with Eq. (86); thus,

$$\mathbf{A} = \begin{pmatrix} -\bar{s} & \bar{s} \\ \bar{c} & -\bar{c} \end{pmatrix}, \quad (\text{A32})$$

which means that we must set $v = 0$ in the first row of $\mathbf{Q}^{(0)}$ (for $Q_{\bullet\bullet}$ and $Q_{\bullet\circ}$) and $u = 0$ in the second row (for $Q_{\circ\bullet}$ and $Q_{\circ\circ}$). This condition leads to Eqs. (89), (90), (93), and (90) for the elements of the transition probability matrix $\mathbf{Q}$ in the general case.

b. Large pixel

In the large-pixel limit ($l_p \gg L_*$, $r \rightarrow \infty$) the master matrix has the form shown in Eq. (18):

$$\mathbf{F}^{(\infty)}(c) = \mathbf{C}\delta(c - \bar{c}), \quad (\text{A33})$$

so Eq. (58) for the matrices $\mathbf{H}^{(k)}$ can be written as

$$\mathbf{H}^{(k)} = \mathbf{C} \int_0^1 \eta_k(c) \delta(c - \bar{c}) dc = \eta_k(\bar{c}) \mathbf{C} \quad (\text{A34})$$

(for $k = \bullet$ and $\circ$), and their matrix elements are

$$H_{\bullet\bullet}^{(k)} = \eta_k(\bar{c})\bar{c}, \quad H_{\bullet\circ}^{(k)} = \eta_k(\bar{c})\bar{s}, \quad (\text{A35})$$

$$H_{\circ\bullet}^{(k)} = \eta_k(\bar{c})\bar{c}, \quad H_{\circ\circ}^{(k)} = \eta_k(\bar{c})\bar{s}. \quad (\text{A36})$$

Then $\eta_k(\bar{c})$ cancels in the expressions

$$R^{(k)} = \frac{H_{\circ\bullet}^{(k)} + H_{\circ\circ}^{(k)}}{H_{\bullet\bullet}^{(k)} + H_{\bullet\circ}^{(k)}} = \frac{\bar{c} + \bar{s}}{\bar{c} + \bar{s}} = 1,$$

yielding

$$\begin{aligned} Q_{kk} &= uH_{\bullet\bullet}^{(k)} + uH_{\bullet\circ}^{(k)} + vH_{\circ\bullet}^{(k)} + vH_{\circ\circ}^{(k)}, \\ &= u\eta_k(\bar{c})(\bar{c} + \bar{s}) + \zeta\eta_k(\bar{c})(\bar{c} + \bar{s}) = \eta_k(\bar{c}). \end{aligned} \quad (\text{A37})$$

This means that (for any value of u and $v = 1 - u$) the matrix $\mathbf{Q}$ is the state matrix [Eq. (4)] corresponding to the probability $\eta_{\bullet}(\bar{c})$:

$$\mathbf{Q}^{(\infty)} = \begin{bmatrix} \eta_{\bullet}(\bar{c}) & \eta_{\circ}(\bar{c}) \\ \eta_{\bullet}(\bar{c}) & \eta_{\circ}(\bar{c}) \end{bmatrix} \quad (\text{A38})$$

[note that $\eta_{\bullet}(\bar{c}) + \eta_{\circ}(\bar{c}) = 1$].

The scale ratio and correlation length in this case can be easily derived:

$$r_{\text{obs}} = Q_{\bullet\circ} + Q_{\circ\bullet} = \eta_{\bullet}(\bar{c}) + \eta_{\circ}(\bar{c}) = 1; \quad (\text{A39})$$

thus, $M_* = 1$ and $L_*^{(\text{obs})} = l_p$. This means that each pixel correlates only with itself; that is, all pixels are statistically independent. The observed cloud fraction in this case has the form

$$\bar{c}_{\text{obs}} = \frac{M_*}{M_{\circ}} = M_* Q_{\circ\bullet} = \eta_{\bullet}(\bar{c}), \quad (\text{A40})$$

which is expected since each large pixel has CF close to $\bar{c}$.

APPENDIX B

Example of the Linear SAF

In the case of the linear SAF defined by Eq. (55) the pixel-observation statistics can be derived analytically. In this case we can use the results of computations performed in Part III, where the matrix integral

$$\bar{\mathbf{F}}^{(c)} = \int_0^1 c \mathbf{F}(c) dc \quad (\text{B1})$$

was taken:

$$\begin{aligned} \bar{\mathbf{F}}^{(c)} &= \bar{c}\mathbf{C} + \bar{s}(\mathbf{I} - \mathbf{C})e^{-r} \\ &\quad + (2\bar{c}\bar{s}\mathbf{K} + \mathbf{L})\frac{1 - e^{-r}}{r}, \end{aligned} \quad (\text{B2})$$

where

$$\mathbf{K} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} 0 & \bar{s} \\ -\bar{c} & 0 \end{pmatrix}. \quad (\text{B3})$$

Comparing Eq. (B1) to Eq. (58) with $\eta_{\bullet}(c) = c$ [Eq. (55)] we find that for the linear SAF $\mathbf{H}^{\bullet} = \bar{\mathbf{F}}^{(c)}$, while $\mathbf{H}^{\circ}$ can be derived from Eq. (60). The explicit expressions for the matrix elements of $\mathbf{H}^{\bullet}$ are

$$H_{\bullet\bullet}^{\bullet} = \bar{c} - \bar{s}(\bar{c} - \bar{s}f - 2\bar{c}g), \quad (\text{B4})$$

$$H_{\bullet\circ}^{\bullet} = \bar{s}[\bar{c} - \bar{s}f + (\bar{s} - \bar{c})g], \quad (\text{B5})$$

$$H_{\bullet\bullet}^{\circ} = \bar{c}[\bar{c} - \bar{s}f + (\bar{s} - \bar{c})g], \quad (\text{B6})$$

$$H_{\circ\circ}^{\circ} = \bar{c}\bar{s}(1 + f - 2g), \quad (\text{B7})$$

where we introduced the following notations:

$$f(r) = e^{-r} \quad \text{and} \quad g(r) = \frac{1 - e^{-r}}{r}. \quad (\text{B8})$$

In these notation the transition probability matrix [Eq. (24)] can be written in the form

$$\mathbf{P} = \begin{pmatrix} \bar{c} + \bar{s}f & \bar{s} - \bar{s}f \\ \bar{c} - \bar{c}f & \bar{s} + \bar{c}f \end{pmatrix}, \quad (\text{B9})$$

and the matrix elements of $\mathbf{H}^{\circ}$ derived from Eqs. (60) and (B4)–(B7) are

$$H_{\bullet\bullet}^{\circ} = \bar{c}\bar{s}(1 + f - 2g), \quad (\text{B10})$$

$$H_{\bullet\circ}^{\circ} = \bar{s}[\bar{s} - \bar{c}f + (\bar{c} - \bar{s})g], \quad (\text{B11})$$

$$H_{\circ\bullet}^{\circ} = \bar{c}[\bar{s} - \bar{c}f + (\bar{c} - \bar{s})g], \quad (\text{B12})$$

$$H_{\circ\circ}^{\circ} = \bar{s} - \bar{c}(\bar{s} - \bar{c}f - 2\bar{s}g). \quad (\text{B13})$$

One can see that these expressions can be also obtained from Eqs. (B4)–(B7) using CCI Eq. (61), as is expected for a symmetric SAF.

Now let us compute the element $Q_{\bullet\bullet}$ of the pixel transition matrix:

$$Q_{\bullet\bullet} = H_{\bullet\bullet}^{\circ} + H_{\bullet\circ}^{\circ} R^{\circ}. \quad (\text{B14})$$

We start with computation of R° according to Eq. (84):

$$R^{\circ} = \frac{\bar{c}(1 - g)}{\bar{c} + \bar{s}g}. \quad (\text{B15})$$

We can now write the expression for $Q_{\bullet\bullet}$ as

$$Q_{\bullet\bullet} = \bar{c} - \bar{s}(\bar{c} - \bar{s}f - 2\bar{c}g) + \bar{s}[\bar{c} - \bar{s}f + (\bar{s} - \bar{c})g] \frac{\bar{c}(1 - g)}{\bar{c} + \bar{s}g}, \quad (\text{B16})$$

which can be reduced to the following simple formula:

$$Q_{\bullet\bullet} = \bar{c} + \bar{s} \frac{\bar{c}g + \bar{s}f}{\bar{c} + \bar{s}g}. \quad (\text{B17})$$

Derivation of the pixel transition matrix element

$$Q_{\circ\circ} = H_{\circ\circ}^{\circ} + \frac{H_{\circ\bullet}^{\circ}}{R^{\circ}} \quad (\text{B18})$$

is similar and can be performed using CCI, which in this case is simply the interchange of $\bar{c}$ and $\bar{s}$ in the rhs of Eq. (B17):

$$Q_{\circ\circ} = \bar{s} + \bar{c} \frac{\bar{s}g + \bar{c}f}{\bar{s} + \bar{c}g}. \quad (\text{B19})$$

The off-diagonal elements of the pixel transition matrix then are

$$Q_{\bullet\circ} = \bar{s} - \bar{s} \frac{\bar{c}g + \bar{s}f}{\bar{c} + \bar{s}g} \quad (\text{B20})$$

and

$$Q_{\circ\bullet} = \bar{c} - \bar{c} \frac{\bar{s}g + \bar{c}f}{\bar{s} + \bar{c}g}. \quad (\text{B21})$$

Let us check the asymptotic behavior of $Q_{\bullet\bullet}$ and $Q_{\circ\circ}$ in the limit $r \rightarrow 0$. To do this we expand functions $f(r)$ and $g(r)$ up to the terms linear in r :

$$f(r) \approx 1 - r \quad \text{and} \quad g(r) \approx 1 - \frac{r}{2}, \quad (\text{B22})$$

then substitution of these expressions into Eqs. (B20) and (B21) yields

$$Q_{\bullet\bullet}(r \rightarrow 0) \approx \bar{s}r \quad \text{and} \quad Q_{\circ\circ}(r \rightarrow 0) \approx \bar{c}r. \quad (\text{B23})$$

These expressions coincide with the corresponding matrix elements in Eq. (86), thus presenting a verification of our computations.

In the large-pixel limit $r \rightarrow \infty$ both $f(r)$ and $g(r)$ vanish yielding $\mathbf{Q} = \mathbf{C}$ that coincides with Eq. (87) since for the linear SAF $\eta_{\bullet}(\bar{c}) = \bar{c}$ and $\eta_{\circ}(\bar{c}) = \bar{s}$.

We would like also to note that in the limit $\bar{c} \rightarrow 0$ we have $Q_{\bullet\bullet} \rightarrow 1 - f$, while $Q_{\circ\bullet} \rightarrow 0$. In the opposite limit case of $\bar{c} \rightarrow 1$ we have $Q_{\bullet\bullet} \rightarrow 0$, while $Q_{\circ\bullet} \rightarrow 1 - f$. In both cases

$$r_{\text{obs}} = Q_{\bullet\bullet} + Q_{\circ\bullet} = 1 - e^{-r} \quad (\text{B24})$$

that coincides with Eq. (47) for r_{obs} in the discrete-sampling approach.

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