Diffusivity-Factor Approximation for Spectral Outgoing Longwave Radiation

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ABSTRACT

Accurate integration of directional radiance shows that the conventional diffusivity-factor approximation with a constant diffusivity angle results in an overestimation of the outgoing longwave radiation (OLR) in the window band and an underestimation in the absorption band. We propose an analytical estimation of a spectrally dependent diffusivity angle for clear-sky spectral OLR, considering actual atmospheric conditions and realistic optical path geometry. Beginning with the plane-parallel geometry, we present a new, physical explanation of the conventional diffusivity angle that applies to the gas absorption bands and derives an alternative solution for the window bands. Then a correction scheme is developed to account for the impact of the spherical Earth geometry on the diffusivity angle. The proposed method achieves higher accuracy, reducing biases to generally less than 2% in all spectral regions.

1. Introduction

The outgoing longwave radiation (OLR) is a critical variable as it balances the incoming solar (shortwave) radiation and shapes Earth’s climate. In essence, OLR is the irradiance flux $F$ contributed by the outgoing radiances $I$ emerging from the top of the atmosphere (TOA) at all possible angles. Their relationship can be expressed as Eq. (1), assuming azimuthal homogeneity:

$$F(v) = \pi \int_{0}^{2\pi} \int_{0}^{\pi/2} I(v, \theta) \sin(\theta) \cos(\theta) \, d\theta \, d\phi = 2\pi \int_{0}^{\pi/2} I(v, \theta) \sin(\theta) \cos(\theta) \, d\theta = 2\pi \int_{0}^{1} I(v, \mu) \mu \, d\mu,$$

(1)

where $\theta$ is the zenith angle, $\phi$ is the azimuth angle, $\mu = \cos(\theta)$, and $v$ denotes the frequency.

It would be computationally expensive to compute OLR by calculating and integrating all the directional radiances. A widely used technique, commonly referred to as the diffusivity-factor approximation (DFA) (Elsasser 1942), is to approximate the OLR as isotropic radiation emerging from an effective zenith angle $\theta_0$, so that we can obtain the OLR by simply multiplying the radiance at that angle with a constant $\pi I(v, \theta_0)$. (2)

Without offering a detailed explanation, Elsasser (1942) first proposed a diffusivity factor $f = 1.66$, corresponding to a diffusivity angle of about 53° according to the relationship $f = 1/\cos(0)$. As this method became widely used, a few authors explained its mathematical basis. Armstrong (1968) considered this result a feature of broadband OLR resulting from spectral integration, which thus would not apply to the case of spectrally resolved OLR that we are concerned here. Li (2000) showed this method could be considered a one-node Gaussian quadrature approximation (1GQ) of the angular integration. In this paper, we will show that this result can be alternatively derived based on a physical interpretation of the OLR.

With the advancement of measurement technology as well as the improvement of radiation codes, spectrally decomposed radiation has been increasingly used for weather and climate monitoring and model validation (Huang et al. 2007; Feldman et al. 2015; Huang 2013; Bani Shahabadi et al. 2016). When simulating spectrally decomposed OLR fluxes from an atmospheric profile, an adequate radiance-to-irradiance conversion scheme is required. The DFA method with a constant diffusivity angle is often used in such applications to reduce computing costs (e.g., Huang et al. 2013; Pan and Huang 2018). However, the DFA method, although widely used for the spectrally integrated OLR, is not validated.
for the spectrally decomposed irradiance fluxes. As shown in the following sections, using constant-angle DFA across the longwave spectrum (100–2800 cm⁻¹) would lead to considerable biases. Other authors have discussed relevant problems. For instance, Mehta and Susskind (1999) used empirically determined diffusivity angles for different spectral bands that are independent of atmospheric condition. Zhao and Shi (2013) proposed a method for determining the diffusivity angle according to the optical depth. In general, these methods were based on the consideration of angular integration of radiance in an isolated atmospheric layer and thus do not account the geometry of the line of sight, which in reality is complicated, for instance, by the curvature of Earth’s surface.

In this paper, based on our physical interpretation of the DFA method, we propose a method for determining the diffusivity angle at every frequency according to specific atmospheric conditions. This method treats the window band (weak atmospheric absorption) and gas absorption bands (strong atmospheric absorption) differently, which as shown below is necessary according to the physical interpretation of the DFA method. We also provide a scheme that corrects the bias due to the path geometry. The paper is structured as follows: in section 2, we present our physical interpretation of the DFA method; in section 3, we adjust the scheme to account for the effect of spherical Earth surface geometry; the performance and implication of this method are demonstrated in section 4.

2. Physical model

The monochromatic radiance from viewing angle θ at spectral frequency of ν is (Goody and Yung 1989)

\[ I(ν, θ) = I_{atm} + I_{ref} = \int_0^\tau B[ν, T(τ)]e^{-ν/μ}d(τ/μ) + B[ν, T(τ)]e^{-ν/μ}, \]

(3)

where τ is the nadir optical depth from TOA to a vertical level, and τ is the total optical depth from TOA to the surface. The term \( B[ν, T(τ)] \) is the Planck function of blackbody radiation at wavenumber ν. The first term of Eq. (3) corresponds to the atmospheric contribution \( I_{atm} \), and the second term corresponds to the surface contribution \( I_{ref} \). Note this equation holds in a non-scattering plane-parallel atmosphere when both path refraction and Earth surface curvature are neglected.

To proceed, let us first relate the source function \( B(ν, T) \) to the optical depth τ. According to Planck function, the blackbody emission \( B(ν, T) \) is

\[ B(ν, T) = \frac{2hν^3}{c^2} \frac{1}{e^{hν/(k_BT)} - 1} = B(ν, T_0) + \frac{\partial B(ν, T)}{\partial T} \bigg|_{T_0} (T - T_0) + O(T - T_0)^2 \]

(4)

where \( \partial B(ν, T)/\partial T \bigg|_{T_0} \) is the bias due to the path geometry. The paper is structured by the Doppler method. We also provide a scheme that corrects the bias due to the path geometry. The paper is structured as follows: in section 2, we present our physical interpretation of the DFA method; in section 3, we adjust the scheme to account for the effect of spherical Earth surface geometry; the performance and implication of this method are demonstrated in section 4.
The term $W$ is the so-called weighting function, which is the derivative of transmission function with respect to altitude $z$:

$$W = \frac{\partial e^{-\tau}}{\partial z}. \quad (9)$$

The weighting functions in the gas absorption bands are typically bell shaped and peak at a level where $\tau = 1$ (Goody and Yung 1989; Huang and Bani Shahabadi 2014). The outgoing radiance largely originates from this level. Therefore, we consider the reference atmospheric temperature $T_0$ in Eq. (7) to be the temperature at this emission layer, so that Eq. (7) becomes

$$B[v, T_0] = B(v, T_0)[1 + \alpha \ln(\tau)]. \quad (10)$$

Following the emission-layer displacement model (ELDM) (Huang and Bani Shahabadi 2014), let us consider a set of weighting function in the nadir case: $W_1, W_2, \ldots, W_n$, which correspond to optical depths $\tau_1, \tau_2, \ldots, \tau_n$. Because $W$ is only a function of $\tau$, in the limb-view case, the same set of $W_i$ values are taken at $\mu \tau_1, \mu \tau_2, \ldots, \mu \tau_n$; that is, the layer with the same weighting function value is displaced to a higher altitude. We rewrite the Eq. (8) as

$$I(v, \mu) = \sum_i B[v, T(\mu \tau_i)]W_i \quad (11)$$

so that

$$I(v, \mu) = B(v, T_0)\sum_i [1 + \alpha \ln(\tau_i) + \alpha \ln\mu]W_i. \quad (12)$$

Based on Eq. (1), the OLR at this frequency can be obtained by integration of Eq. (12) over $\mu$:

$$F(v) = 2\pi \int_0^1 I(v, \mu) \mu d\mu \quad (13)$$

Combining Eqs. (2), (12), and (13),
Because the optical depth $\tau_i$ in the window band is typically much less than unity, the atmospheric effective emission temperature $T_e$ is close to surface temperature $T_s$. Hence, we can set the reference temperature $T_0$ in Eq. (7) to be $T_s$ and the reference optical depth $\tau_0 = \tau_s$:

$$B[v, T(\tau)] = B[v, T_s] \left[ 1 + \alpha \ln\left(\frac{\tau}{\tau_s}\right) \right],$$

$$B[v, T_a(\mu)] = \int_0^{\tau_s} B[v, T(\tau)] \frac{e^{-\tau/\mu}}{1 - e^{-\tau/\mu}} d\tau = B[v, T_s] \int_0^{\tau_s} \left(1 + \alpha \ln\frac{\tau}{\tau_s}\right) \frac{e^{-\tau/\mu}}{1 - e^{-\tau/\mu}} d\tau.$$

(17)

This integration cannot be analytically solved. However, it can be approximated as a linear function on $\tau_i/\mu$ if we linearized the term $e^{-\tau_i/\mu}/(1 - e^{-\tau_i/\mu})$, considering both $\tau_i/\mu$ and $\tau_i$ are small:

$$B[v, T_a(\mu)] \approx B[v, T_s] \int_0^{\tau_s} \left(1 + \alpha \ln\frac{\tau}{\tau_s}\right) \frac{e^{-\tau_i/\mu}}{\mu} d\tau = B[v, T_s] \int_0^{\tau_s} \left(1 + \alpha \ln\frac{\tau}{\tau_s}\right) \left(\frac{\mu}{\tau_s} + \frac{1}{2} - \frac{\tau}{\tau_s}\right) d\tau.$$

(18)

It means that, considering the limb-view path and inhomogeneity in the atmospheric column, the effective atmospheric emission is affected by $\alpha$ and $\tau_i/\mu$. To demonstrate the validity of this result, we set up a simulation test, using the atmospheric condition described in Eqs. (5) and (6), to compute the limb-view radiance from $0^\circ$ to $89^\circ$ and angularly integrate them to obtain the irradiance at 100 km above the ground. Prescribed in the computation are the surface temperature of 300 K and atmospheric temperatures continuously decreasing from this value at a lapse rate $\Gamma$ of 6.5 K km$^{-1}$. As shown in Fig. 1c, this approximation [Eq. (18)] generally yields error within 5% in the window band compared to the truth [$I_{\text{sim}}$ computed according to Eq. (3)]. The residual becomes larger with increasing viewing angle $\theta$ and the total optical depth $\tau_i$ because of the small $\tau_i/\mu$ assumption in Eq. (18).

If we substitute Eq. (18) back into Eq. (16), the window band directional radiance and flux become
\[ I(v, \mu) = B(v, T_s, \mu)(1 - e^{-\tau_s/\mu}) + B(v, T_s)e^{-\tau_s/\mu} \]

\[ = B(v, T_s) \left[ -\frac{1}{4\mu} \alpha \tau_s + (1 - \alpha) + \frac{1}{4\mu} \alpha e^{-\tau_s/\mu} + \alpha e^{-\tau_s/\mu} \right], \quad (19) \]

\[ F(v) = 2\pi \int_{0}^{1} I(v, \mu) d\mu \]

\[ = \pi B(v, T_s) \left[ 1 - \alpha - \frac{1}{2} \alpha \tau_s + \frac{\alpha \tau_s^3}{6} \right] \]

\[ + \alpha e^{-\tau_s} - \frac{1}{2} \alpha \tau_s e^{-\tau_s}, \quad (20) \]

where \( E_1 = \int_{0}^{1} \mu e^{-\tau_s/\mu} d\mu \). For the simplicity of the solution, we apply a Taylor series expansion at \( \tau_s = 0 \) to both \( I(v, \mu) \) and \( F(v) \) and keep the third-order term based on validation against \( \tau_s \) value in the range of [0, 1.5]; meanwhile, we eliminate the third-order term of \( \mu \) in a Taylor series expansion at \( \mu = 0.5 \).

\[ \frac{I(v, \mu)}{B(v, T_s)} \approx 1 - \frac{\alpha \tau_s^2}{\mu} + \frac{\alpha \tau_s^3}{24\mu^2} + O(\tau_s^4) \]

\[ \approx 1 - \frac{1}{3\alpha \tau_s} + \frac{\alpha \tau_s^3 - 2\tau_s}{2\mu} + \alpha \tau_s - \frac{\tau_s^3}{4\mu^2}, \quad (21) \]

\[ \frac{F(v)}{\pi B(v, T_s)} \approx 1 - 2\alpha \tau_s - \frac{1}{2} \alpha \tau_s^2 [\gamma + \ln(\tau_s) - 2] \]

\[ + \frac{1}{12} \alpha \tau_s^3 + O(\tau_s^4), \quad (22) \]

where \( \gamma \approx 0.5772 \) is the Euler’s constant. We validate the two approximations in Fig. 1d. It shows that, in the window band, both approximations yield errors less than 4%, based on which an accurate diffusivity angle can be inferred.

Combining Eqs. (2) and (18), we obtain an equation for the diffusivity angle \( \mu_0 \):

\[ \frac{I(v, \mu_0)}{B(v, T_s)} = \frac{F(v)}{\pi B(v, T_s)} \]

\[ = 1 - \frac{1}{3\alpha \tau_s} + \frac{\alpha \tau_s^3 - 2\tau_s}{2\mu} + \alpha \tau_s - \frac{\tau_s^3}{4\mu^2} \]

\[ = 1 - 2\alpha \tau_s - \frac{1}{2} \alpha \tau_s^2 [\gamma + \ln(\tau_s) - 2] + \frac{1}{12} \alpha \tau_s^3, \quad (23) \]

The problem then becomes solving a quadratic equation of \( \mu_0 \):

\[ a\mu_0^2 + b\mu_0 + c = 0, \]

\[ a = -2\alpha \tau_s - \frac{1}{2} \alpha \tau_s^2 (\gamma + \ln(\tau_s) - 2) + \frac{5}{12} \alpha \tau_s^3, \]

\[ b = \alpha \tau_s - \frac{1}{2} \alpha \tau_s^3, \]

\[ c = -\frac{\alpha \tau_s^3}{4} + \frac{\alpha \tau_s^3}{4}. \]

\[ \cos(\theta_0) = \mu_0 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \tau_s \leq 1.4705. \quad (24) \]

The limit, \( \tau_s \leq 1.4705 \), is chosen so that the solution is continuous with our absorption band solution, \( \mu_0 = e^{-1/2} \) when \( \tau_s = 1.4705 \). The solution can be further simplified to

\[ \mu_0 = \frac{1}{2} - \frac{\tau_s}{8} (\gamma + \ln(\tau_s)) \quad (25) \]

when optical depth \( \tau_s \leq e^{-1-\gamma} = 0.2066 \).

To validate the above result [Eq. (24)], we set a simple radiative transfer model that accurately computes the source function of each atmospheric layer and vertically integrates it according to Eq. (3). The directional radiances are obtained at each \( 0.01 \) zenith angle between \( 0^\circ \) and \( 89.99^\circ \), from which the irradiance flux is integrated. We then match the directional radiances and irradiances to find the \( \theta_0 \) exhibiting the smallest error according to Eq. (2). As shown in Fig. 2a, we affirm that the spectrally dependent DFA obtained in Eq. (24) has a noticeable improvement compared to the constant-angle DFA method, especially in the extremely transparent spectral regime. In contrast to the constant-angle DFA in absorption bands, it is now dependent on the total optical depth \( \tau_s \), although the atmospheric properties, such as the lapse rate \( \Gamma \), are still not relevant.

### 4. Spherical correction

In the previous discussion, a plane-parallel geometry is assumed, so that the zenith angle \( \theta \) from the surface level to the TOA is constant along the line of sight. As shown in Fig. 3, in reality, the effective zenith angle \( \theta' \) varies with altitude \( z \) and becomes larger farther away from the observer, resulting in an extended geometry path compared to plane-parallel geometry, because of the curvature of Earth’s surface.

Considering the spherical geometry, the effective zenith angle \( \theta' \) can be expressed as a function of height \( z \):
\[ m'_{0} = \frac{1}{2} \cos(u_{0})^{2} - \frac{1}{2} \left( \frac{H}{R + H} \right)^{2}. \]  

(27)

Because the TOA irradiance flux is mostly contributed by emission layers at height \( z_{0} \), we can approximate the zenith angle \( u_{0} \) with its value at this layer, so that

\[ m'_{0} = \frac{1}{2} \cos(u_{0})^{2} - \frac{1}{2} \left( \frac{H}{R + H} \right)^{2}. \]  

(26)

The emission level, as discussed in the previous section, is at \( \tau = 1 \) in the absorption band and at the surface in the window band. According to the \( \tau - z \) relationship described in Eq. (5), we have \( z_{0}(\tau = \tau_{s}) = 0 \) in the window band and \( z_{0}(\tau = 1) = H_{s} \ln(\tau_{s}) \) in the absorption band. Note that the latter one can be directly referred from the optical depth profile with a forward model.

Substituting Eq. (27) into Eq. (1), we find the spectral flux in Eqs. (22) and (13) can be adjusted with a factor \((R + z_{0})/(R + H))^{2}\) to account for the effect of Earth geometry, as shown below:

\[ F_{c}(\nu) = \frac{2\pi}{(R + H)^{2}} \int_{0}^{1} I(\nu,\mu') \mu' d\mu' \]

\[ \approx \left( \frac{R + z_{0}}{R + H} \right)^{2} F(\nu). \]  

(28)
It means that the surface geometry generally decreases the irradiance flux at TOA, resulted from an elevated emission layer, or a decreased contribution from the surface in the window band. This does not apply to the strong absorption band as the emission layer approaches the stratosphere, where the surface geometry leads to a slightly negative or even positive difference.

Now the solution for window band in Eq. (24) becomes

$$\mu_0' = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \quad \tau_s \leq 1.4705,$$

$$\cos(\theta_0) = \mu_0 \approx \sqrt{1 - \left(1 - \mu_0^2(0)\right)\left(1 - \frac{H}{R + H}\right)^2}, \quad (29)$$

$$a = \left\{1 - 2\alpha\tau_s - \frac{\alpha\tau_s^2}{2}[r + \ln(\tau_s) - 2] + \frac{\alpha\tau_s^3}{12}\right\}\left(1 - \frac{H}{R + H}\right)^2 - 1 + \frac{\alpha\tau_s^3}{3},$$

$$b = \alpha\tau_s - \frac{1}{2}\tau_s^2,$$

$$c = -\frac{\alpha\tau_s^2}{4} + \frac{\alpha\tau_s^3}{4}.$$
the conventional approach, the diffusivity angle in an extremely transparent band can be as high as 80°, especially when the lapse rate becomes smaller, such as in the Arctic. This significant difference in diffusivity angle may result in a highly overestimated window band OLR if the conventional diffusivity angle (52.66°) is used. Although some errors still exist in the spectrally dependent approximation, this approach accounts for the effect of such physical attributes as the total optical depth $\tau_s$ and lapse rate and exhibits errors within 5%.

5. Validation tests and discussion

In this section, we perform a series of tests with different atmospheric profiles to verify the accuracy of our revised DFA algorithms, in comparison with the conventional constant diffusivity-angle approach.
In these tests, we calculate the optical depths using a radiative transfer model MODTRAN 5.2, from 100 to 2800 cm\(^{-1}\) at 0.1-cm\(^{-1}\) resolution, and then apply a path geometry model that we develop to generate the directional radiances. The path geometry model here can implement either plane-parallel geometry or the realistic path that combines refraction and spherical Earth surface. The realistic path geometry model has been validated against the geometry package of MODTRAN. Note that the realistic path geometry option considers refraction, although its impact is insignificant compared to the spherical geometry, which we focus on here. We angularly integrate the directional radiances from 0° to 90° zenith angle at a 0.1° interval following Eq. (2), at each wavenumber, to generate the reference spectral flux (truth).

In addition to the U.S. standard profile (McClatchey et al. 1972), errors averaged in five bands with different atmospheric profiles are shown in Table 1. Besides the two DFA approaches, we also include a double Gaussian quadrature (2GQ) method (Li 2000) in the comparison. The result (Table 1) shows that the 2GQ method generally achieves better accuracy than the constant-angle DFA method (equivalently a 1GQ method) at the price of doubled computation. However, there is a similar pattern of overestimation in the window band to the conventional constant-angle DFA works in the absorption bands and an analytical solution to improve the accuracy in the window bands. This solution takes account of both optical and atmospheric properties while keeping the simplicity of the expression.

Table 2. As in Table 1, except that a correction to account for Earth’s surface geometry is applied.

<table>
<thead>
<tr>
<th>Atmosphere type</th>
<th>Method</th>
<th>[760, 1000]</th>
<th>[1061, 1230]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>(\mu = e^{-1/2})</td>
<td>–0.35</td>
<td>–0.13</td>
</tr>
<tr>
<td></td>
<td>2GQ</td>
<td>–0.65</td>
<td>–0.20</td>
</tr>
<tr>
<td>Tropical</td>
<td>(\mu = e^{-1/2})</td>
<td>–0.40</td>
<td>–0.11</td>
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<tr>
<td></td>
<td>2GQ</td>
<td>–0.65</td>
<td>–0.48</td>
</tr>
<tr>
<td>Midlatitude summer</td>
<td>(\mu = e^{-1/2})</td>
<td>–0.47</td>
<td>–0.15</td>
</tr>
<tr>
<td></td>
<td>2GQ</td>
<td>–0.51</td>
<td>–0.17</td>
</tr>
<tr>
<td>Midlatitude winter</td>
<td>(\mu = e^{-1/2})</td>
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<td>–0.21</td>
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<td></td>
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</tr>
<tr>
<td>High latitude summer</td>
<td>(\mu = e^{-1/2})</td>
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<td>–0.21</td>
</tr>
<tr>
<td></td>
<td>2GQ</td>
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</tr>
<tr>
<td>High latitude winter</td>
<td>(\mu = e^{-1/2})</td>
<td>–0.27</td>
<td>–0.29</td>
</tr>
<tr>
<td></td>
<td>2GQ</td>
<td>–0.61</td>
<td>–0.17</td>
</tr>
</tbody>
</table>

In conclusion, this study reinvestigates the diffusivity angle approach. We find a physical explanation of why the conventional constant-angle DFA works in the absorption bands and an analytical solution to improve the accuracy in the window bands. This solution takes account of both optical and atmospheric properties while keeping the simplicity of the expression.

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APPENDIX

Spherical Correction of Gaussian Quadrature Method

According to Li (2000), quadrature angles are solved from

\[
2 \int_0^1 e^{-t\mu} \mu \, d\mu = \sum_i b_i e^{-t\mu_i}.
\]

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A set of weight $b_i$ and $\mu_\omega$ are given in the referenced literature. When the curved Earth surface is considered, we use $\mu'$ as the cosine of the effective zenith angle:

$$2 \int_0^1 e^{-\tau \mu'} \mu' d\mu = \sum_i b_{ci} e^{-\tau \mu_\omega},$$

(A2)

where $b_{ci}$ is the corresponding weight. Based on Eq. (28),

$$2 \int_0^1 e^{-\tau \mu} \mu d\mu = 2 \left( \frac{R}{R + H} \right)^2 \int_0^1 e^{-\tau \mu'} \mu' d\mu',$

$$2 \left( 1 - \frac{H}{R + H} \right)^2 \int_0^1 e^{-\tau \mu'} \mu' d\mu' = \sum_i b_{ci} e^{-\tau \mu_\omega},$$

(A3)

Based on Eq. (27), the cosine of zenith angle at TOA and the adjusted weight $b_{ci}$ are therefore

$$\mu_{ci} = \sqrt{1 - (1 - \mu_\omega) \left( 1 - \frac{H}{R + H} \right)^2},$$

$$b_{ci} = \left( 1 - \frac{H}{R + H} \right)^2 b_i.$$

(A4)

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