A k - ϵ Turbulence Model for the Stable Atmosphere

XIPING ZENG, YANSEN WANG, AND BENJAMIN T. MACCALL

Army Research Laboratory, Adelphi, Maryland

(Manuscript received 4 April 2019, in final form 5 August 2019)

ABSTRACT

A realizable k - ϵ turbulence model of incompressible fluid is extended for the stable atmosphere after taking account of the buoyancy damping of gravity waves. The new model is consistent with the Monin–Obukhov similarity theory on the stable atmospheric boundary layer (ABL) over a horizontally uniform surface. The model is incorporated into an ABL model to simulate mean flow against observations. Its ABL-model output is compared with the Leipzig dataset, showing the turbulence model works well for a stable ABL. Specifically, the ABL model properly replicates 1) the mixing length, turbulent viscosity, and mean wind; 2) a significant decrease of the mixing length with height in the upper ABL and thus a reasonable altitude of the ABL top; and 3) a sensitivity of the mixing length and turbulent viscosity to atmospheric stability.

1. Introduction

It is still a challenge to model the turbulence in the atmospheric boundary layer (ABL), especially in the stable ABL [e.g., Kosović 1997; Kosović and Curry 2000; see Wilcox (2006) for a review of turbulence models]. Since the k - ϵ turbulence models are simple yet effective (where k is the turbulent kinetic energy and ϵ is its dissipation rate; Launder and Spalding 1974), they are discussed herein for a future application to the Atmospheric Boundary Layer Environment (ABLE) model in Army Research Laboratory (ARL) (Wang et al. 2019).

a. Uncertainty of the standard k - ϵ model

The standard k - ϵ model was widely used in many scientific and engineering simulations (Launder and Spalding 1974; Rodi 1987). It was also applied to the atmosphere after parameters were tuned or new processes were introduced (Detering and Etling 1985; Duynkerke 1988; Apsley and Castro 1997; Sogachev 2009; Sogachev et al. 2012).

On the other side, the standard k - ϵ model does not work well for flows with large mean shear, spreading of jets, or rotating turbulence (e.g., Yakhot et al. 1992,

hereafter Y92; Shih et al. 1995, hereafter S95). Its errors originate in its ϵ equation, because “the form of the equation for ϵ follows heuristic arguments” [see Sander (1998) for a review of the turbulent closures]. Specifically, its ϵ equation does not follow the exact dissipation rate equation [see (1a) of S95 for the equation expression], because several terms (including those of pressure) in the exact dissipation rate equation are unknown. Instead, a simple model dissipation rate equation (i.e., the ϵ equation) was created with a similar structure as the turbulent kinetic energy equation, by assuming that the source and sink terms of the dissipation rate are proportional to the source and sink terms of turbulent kinetic energy times the large eddy turnover time of k/ϵ , respectively (S95). Such assumption on the parallel structure between the k and ϵ equations, as an uncertainty of the standard model, brings about the relatively poor performance of the model in flows with strong mean shear.

To improve the standard model, Y92 introduced a new production term into the ϵ equation, and used the term to represent the contribution of all scales of motion besides the scale of isotropic turbulence length. To be specific, the standard model has a single fixed point, which is described as

$$d\eta/dt \propto \eta_0^2 - \eta^2 \quad (1.1)$$

at $\eta = \eta_0$, where η_0 is a constant [η is defined as k/ϵ times mean shear; see (2.10) for its expression]. When $\eta > \eta_0$

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Corresponding author: Xiping Zeng, xiping.zeng.civ@mail.mil

(or mean shear is strong), $d\eta/dt < 0$, which can bring about negative k or ε and thus violate the realizability conditions of $k \geq 0$ and $\varepsilon \geq 0$. In physics, such violation of the realizability conditions occurs because the normal stresses become negative and the Schwarz' inequality for shear stresses is violated (Schumann 1977; Shih et al. 1994).

The fixed point of η_0 in (1.1) corresponds to homogeneous shear flow with a single turbulence length scale; a model solution is attracted to the fixed point. To represent the contribution of all scales of motion to the turbulent diffusion, Y92 introduced a renormalization group method in terms of η around η_0 via a new source term of ε . Eventually Y92 constrained the new term by the log-wind profile with the von Kármán constant. As a result, Y92 removed the major errors of the standard model, such as the spreading rate anomaly of planar and round jets and the negative normal stresses (note that Y92 accounted for no effects of rotational strains).

b. A new approach from the vorticity/enstrophy equation

Different from Y92, S95 took a completely different approach to improve the standard k - ε model. S95 abandoned the ε equation of the standard model (and, of course, its assumption). Instead, S95 derived a new equation of ε based on the equation of enstrophy, where enstrophy is defined as the square of vorticity. Since all the terms of the enstrophy equation are clear in physics and none of them involves pressure (Tennekes and Lumley 1972, p. 86), S95 was built on a solid ground (see section 2 for further discussions). As a result, S95 removed the major errors of the standard model with large mean shear and jet spreading, too. In addition, S95 accounted for the effect of mean rotation on turbulent stresses.

S95, owing to its unique origin of the ε equation, is a new brand of k - ε model in comparison to the standard one and Y92. Since its $d\eta/dt > 0$ is always satisfied, it maintains the realizability conditions of $k \geq 0$ and $\varepsilon \geq 0$ and thus has no fixed point of (1.1). As a result, S95 needs no renormalization of Y92, because S95's ε equation contains the contribution of all scales of motion (see section 2 and Fig. 1 for the contribution of different scales of motion to ε).

c. Extension of S95 to the stable atmosphere

Both Y92 and S95 perform quite well in comparison to the laboratory observations with large mean shear and jet spreading, providing two good starting points to model the turbulent ABL. However, it is difficult to extend Y92 to account for atmospheric rotation and

FIG. 1. Schematic showing the spectra of turbulent energy $E(k)$ (thick) and enstrophy $k^2 E(k)$ (thin solid line) vs wavenumber k by linear-linear axis scales. The dashed line represents the enstrophy spectrum of the Kolmogorov $-5/3$ law over the inertial subrange ($\propto k^{1/3}$). Three mean eddy scales— λ_s , λ_m (or the Taylor microscale), and λ_l —are marked on the axis of eddy scale that is proportional to k^{-1} .

static stability, using a comparable scale expansion technique (Y92). Instead, S95 maintains an open framework to incorporate new processes through its equations of energy and enstrophy. In the present paper, the spectra of energy and enstrophy are analyzed to incorporate the atmospheric static stability into S95 first and then evaluate the new model with Monin–Obukhov similarity and observational data. Specifically, the extension of S95 to the stable atmosphere is implemented by incorporating the buoyancy damping of gravity waves (Bretherton and Smolarkiewicz 1989; Nilsson and Emanuel 1999; Raymond and Zeng 2000, 2005; Sobel and Bretherton 2000; Zeng 2001; Bretherton and Sobel 2002; Zeng et al. 2007, 2008). See section 2c and appendix A for additional discussions on buoyancy damping.

The paper is composed of five sections. This section introduced the motivation of turbulence modeling in the atmosphere. Section 2 discusses the spectra of energy and enstrophy first and then incorporates buoyancy damping into S95. Section 3 shows the consistency between the new turbulence model and the Monin–Obukhov similarity theory. Section 4 compares simulations with field observations for further model evaluation. Section 5 summarizes. Except for special illustrations, all symbols are defined in appendix D.

2. Formulation of the proposed k - ε model

In this section, S95 with kinetic energy and enstrophy as prognostic variables is summarized, beginning with

the conservations of energy and enstrophy. It is then used to incorporate the effects of buoyance on kinetic energy and enstrophy.

a. Conservations of energy and enstrophy

The Navier–Stokes equation shows that kinetic energy is conserved for inviscid flow and enstrophy is conserved for two-dimensional inviscid flow (Tennekes and Lumley 1972; Arakawa and Lamb 1981). Since kinetic energy and enstrophy are conservative quantities for inviscid flow, they can be used as prognostic variables in a turbulence model.

The kinetic energy k and enstrophy δ of turbulent fluctuations per unit mass of air are defined as

$$k = \frac{1}{2} \overline{u_i u_i}, \quad (2.1a)$$

$$\delta = \frac{1}{2} \overline{\omega_i \omega_i}, \quad (2.1b)$$

where overbar denotes the Reynolds averaging, u_i and ω_i are the components of turbulent velocity and vorticity in direction i ($i = 1, 2$, and 3), respectively. Theoretically, the kinetic energy and enstrophy are expressed with energy spectrum $E(\kappa)$ as (e.g., Lesieur 1990)

$$k = \int_0^\infty E(\kappa) d\kappa, \quad (2.2a)$$

$$\delta = \int_0^\infty \kappa^2 E(\kappa) d\kappa, \quad (2.2b)$$

where κ is wavenumber magnitude. The enstrophy expression, (2.2b), in contrast to the energy one, (2.2a), has κ^2 in its integrand (see Fig. 1 for illustration). If an incompressible liquid system has a down-scale energy cascade with the conservation of kinetic energy, its enstrophy increases correspondingly. Hence, an increase in enstrophy measures the energy cascade in a fluid system, providing a clue to develop a turbulence model.

To measure the energy and enstrophy spectra, three mean eddy scales are defined, two of which are

$$\lambda_l = \pi \left[\frac{\int_0^\infty \kappa E(\kappa) d\kappa}{\int_0^\infty E(\kappa) d\kappa} \right]^{-1}, \quad (2.3a)$$

$$\lambda_s = \pi \left[\frac{\int_0^\infty \kappa^3 E(\kappa) d\kappa}{\int_0^\infty \kappa^2 E(\kappa) d\kappa} \right]^{-1}. \quad (2.3b)$$

These two scales approximately represent the mean sizes of energy-containing eddies and enstrophy-containing ones, and, therefore, are sensitive to large and small eddies, respectively. The third mean eddy scale is defined as

$$\lambda_m = \pi \left[\frac{2 \int_0^\infty \kappa^2 E(\kappa) d\kappa}{\int_0^\infty E(\kappa) d\kappa} \right]^{-1/2}, \quad (2.3c)$$

which is equal to $\pi[k/(2\delta)]^{1/2}$. The scale λ_m is proportional to the Taylor microscale with a constant proportionality factor.¹ Different from λ_l and λ_s , λ_m (or the Taylor microscale) is sensitive to not only large but also small eddies (Wilcox 2006, p. 49). Hence, λ_m approximately measures the mean size of all turbulent eddies and thus falls between λ_l and λ_s (see Fig. 1 for their relative magnitudes).

To illustrate how λ_m changes with turbulence sources/sinks, Fig. 1 displays the spectra of turbulent energy $E(\kappa)$ and enstrophy $\kappa^2 E(\kappa)$ versus wavenumber κ with two solid lines, respectively. The figure employs linear–linear axis scales so that the area below the energy (or enstrophy) spectrum line represents energy (or enstrophy); specifically, the area below the energy (or enstrophy) spectrum line from κ to $\kappa + d\kappa$ is equal to $E(\kappa)d\kappa$ [or $\kappa^2 E(\kappa)d\kappa$]. The figure also marks the three mean eddy scales of (2.3), relative to energy-containing, inertial, and viscous subranges.

A turbulence process contributes to energy/enstrophy at a favorable scale. Mean wind shear generates energy/enstrophy at large scale whereas friction dissipates energy/enstrophy at small scale (Tennekes and Lumley 1972). Other processes (e.g., small-wake eddies past canopy elements such as tree branches and leaves) increase energy/enstrophy at their process scales, too. Since the processes impact λ_m so that λ_m shifts toward their characteristic scales, the sensitivity of λ_m to turbulence sources/sinks is used to judge whether a turbulence model works well. See appendix B for additional discussions.

b. Equations of the proposed model

The standard k – ε turbulence model uses k and ε as two prognostic variables. It has deficiencies to simulate flows with large mean shear, spreading of jets, or

¹ The time scale λ_m is expressed in terms of k , ε , and ν_0 in (2.11). Since the Taylor microscale $\lambda_T = (10\nu_0 k/\varepsilon)^{1/2}$ (Tennekes and Lumley 1972, p. 66; Wilcox 2006, p. 49), thus $\lambda_T = \sqrt{10}\lambda_m/\pi \approx \lambda_m$.

rotating turbulence, because its ε equation needs an improvement (Y92; S95).

At a high Reynolds number, ε is equal to the kinematic viscosity times the mean-square vorticity fluctuations (Tennekes and Lumley 1972, p. 88; Grinstein et al. 2011; Yeung et al. 2012) or

$$\varepsilon \cong \nu_0 \overline{\omega_i \omega_i} = 2\nu_0 \delta, \quad (2.4)$$

where ν_0 is the molecular viscosity. Since ε is directly proportional to the turbulent enstrophy δ , it has a similar spectral expression as (2.2b).

S95 redefined ε as (2.4) and then obtained a new equation for ε based on the enstrophy equation. It has three characteristics: (i) its budgets in k and ε are clear because both k and ε are conserved quantities; (ii) its k and ε equations are affected majorly by large and small eddies (with the mean scales of λ_l and λ_s), respectively, exhibiting no parallel structure between the two equations; and (iii) its results agree well with the laboratory observations in various benchmark flows, showing that it performs better than the standard one (S95).

Following the relationship between ε and δ in (2.4), the present paper extends S95 to the stable atmosphere. Its equation for turbulent kinetic energy, without involving canopy production of k and other similar turbulence processes, is expressed (Launder and Spalding 1974; Duynkerke 1988; S95) as

$$\frac{dk}{dt} - \nabla \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) = \nu_t (S^2 + G) - \varepsilon, \quad (2.5)$$

where the effective diffusivity of momentum (or turbulent viscosity) ν_t is specified as

$$\nu_t = C_\mu \frac{k^2}{\varepsilon}, \quad (2.6)$$

with constant C_μ to be determined. In (2.5), $\nu_t S^2$ and $\nu_t G$ represent the energy generation due to shear and buoyancy, respectively, where S and G are expressed in the Cartesian coordinate system $(x, y, z) = (x_1, x_2, x_3)$ as

$$S = \sqrt{2S_{ij}S_{ij}} \quad \text{with} \quad S_{ij} = \frac{1}{2} \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \quad (2.7a)$$

$$G = -\frac{g}{\sigma_\theta \theta_0} \frac{\partial \theta_0}{\partial z}, \quad (2.7b)$$

using potential temperature θ_0 and three mean velocity components $(U_1, U_2, U_3) = (U, V, W)$.

Based on the enstrophy equation, S95 obtained an equation of ε or

$$\frac{d\varepsilon}{dt} - \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) = C_1 S \varepsilon - C_2 \frac{\varepsilon^2}{k + \sqrt{\nu_0 \varepsilon / C_\mu}} + B, \quad (2.8)$$

where C_1 and C_2 are constants, and B represents the contribution of buoyancy to enstrophy. The first two terms on the right-hand side of (2.8) represent the contributions of shear and friction, respectively. The term $\sqrt{\nu_0 \varepsilon / C_\mu}$ is negligible at a high Reynolds number and works in the transition from turbulence to laminar flow.

For the neutral ABL at a high Reynolds number (or $B = 0$), S95 chose the following parameters: $C_2 = 1.9$, $\sigma_k = 1.0$, $\sigma_\varepsilon = 1.2$,

$$C_1 = \max[0.43, \eta/(\eta + 5)], \quad (2.9a)$$

$$C_\mu = \left[4.0 + \frac{k}{\varepsilon} \sqrt{6(S_{ij}S_{ij} + \Omega_{ij}\Omega_{ij}) \cos \phi} \right]^{-1}, \quad (2.9b)$$

$$\phi = \frac{1}{3} \arccos \left[\sqrt{6} \frac{S_{ij}S_{jk}S_{ki}}{(S_{ij}S_{ij})^{3/2}} \right], \quad (2.9c)$$

where Ω_{ij} is the mean-rotation-rate tensor and the dimensionless parameter

$$\eta = Sk/\varepsilon. \quad (2.10)$$

With the parameters, (2.8) and (2.5) can replicate the flows with strong shear and rotation well and therefore remove the deficiencies in the standard k - ε model (including the well-known spreading rate anomaly of planar and round jets).

c. Representation of buoyancy damping

In the present paper, the buoyancy term B in (2.8) is improved to incorporate the effect of gravity waves on ε . In a stable atmosphere, gravity waves damp temperature perturbations (excited by turbulent eddies) so that the perturbations disappear eventually, which is referred to as buoyancy damping (Bretherton and Smolarkiewicz 1989; Nilsson and Emanuel 1999; Raymond and Zeng 2000; Sobel and Bretherton 2000). The buoyancy damping is analogous to the surface waves excited when a pebble is tossed into a pond. Its time scale, as shown in appendix A, depends on the phase speed of gravity waves and the effective turbulent viscosity.

With the aid of an analytic model, appendix A shows buoyancy damping consists of two stages (Zeng 2001; Zeng et al. 2007). In the first stage, an initial temperature perturbation (generated by an eddy) propagates outward into a wide area via gravity waves, resulting in the

decrease of perturbation amplitude. In the second stage, the perturbation over a wide area is dissipated by molecular friction (or turbulent mixing).

In short, an eddy generates a temperature perturbation that in turn generates gravity waves, resulting in its energy drain. Obviously, such an energy drain depends on gravity waves, rather than on the energy cascade in turbulence. Since eddy scale (or perturbation scale) is associated with the phase speed of eddy-induced gravity waves, the energy drain depends on eddy scale. Hence, the mean eddy scale in (2.3c), expressed in terms of k , ε , and ν_0 as

$$\lambda_m = \pi(\nu_0 k / \varepsilon)^{1/2}, \quad (2.11)$$

is helpful to parameterize gravity waves. Furthermore, the scale in (2.11) is used to rewrite (2.6) as

$$\nu_t = C_\mu \pi^{-2} \lambda_m^2 k / \nu_0, \quad (2.12)$$

which shows λ_m functions similarly as the mixing length, to some extent.

The buoyancy term B in (2.8) is obtained by analyzing the energy and enstrophy of eddies. Consider an eddy with characteristic scale λ and velocity u . Its vorticity $\propto u/\lambda$ and thus enstrophy $\propto (u/\lambda)^2$. Given that all eddies have the same scale of λ_m , differentiating the enstrophy with respect to time and then summing the enstrophy change for all eddies yields $B \propto \nu_t G / \lambda_m^2 = \nu_t G \varepsilon / k$, which corresponds to the buoyancy term of $\nu_t G$ in the energy equation, (2.5).

However, eddies have different scales and thus generate gravity waves with different phase speeds. Since the gravity waves excited by large eddies travel faster than those by small eddies, larger eddies drain energy faster than smaller ones. As a result, a compensation term in B is needed to represent the dependence of energy drain on eddy scale,² which is expressed as

$$B = C_3 \frac{\varepsilon}{k} \nu_t G + C_4 \min \left(1, \sqrt{\frac{R_i}{C_5}} \right) N \varepsilon, \quad (2.13)$$

² A positive compensation term is necessary in (2.13) for a proper representation of buoyancy damping, which is illustrated using an ideal system with only two wavenumbers: 1 and 5. Suppose the decreases in energy at wavenumber 1 and 5 due to gravity waves equal 1.0 and 0.2, respectively. Thus, the real decrease in total enstrophy is $1.0 \times 1^2 + 0.2 \times 5^2 = 6.0$. However, since the first term on the right-hand side of (2.13) treats all eddies as one mean scale or a wavenumber of $(1 + 5)/2 = 3$, it computes the decrease in total enstrophy as $(1.0 + 0.2) \times 3^2 = 10.8$, which is larger than the real decrease of 6.0. Hence, a compensation increase of $4.8 = 10.8 - 6.0$ is needed that corresponds to the second term on the right-hand side of (2.13).

where C_3 , C_4 , and C_5 are constants to be determined, the gradient Richardson number

$$R_i = \frac{g}{\theta_0} \frac{\partial \theta_0 / \partial z}{|\partial \mathbf{V} / \partial z|^2},$$

and the Brunt–Väisälä frequency

$$N = \left(\frac{g \partial \theta_0}{\theta_0 \partial z} \right)^{1/2}. \quad (2.14)$$

On the right-hand side of (2.13), the first term represents the sink of ε due to static stability just as the standard k – ε model (Duykerke 1988; Apsley and Castro 1997). In contrast, the second term represents the buoyancy damping whose time scale is proportional to N^{-1} when $R_i > C_5$ (or ν_t is small), where C_5 functions similarly as the critical Richardson number. When $R_i < C_5$, turbulence is maintained by shear and consequently ν_t is large, bringing about a damping time scale longer than N^{-1} (see Fig. A2 for the effect of viscosity on buoyancy-damping time scale). Hence, $1/\min(1, \sqrt{R_i/C_5})$ is introduced into (2.13) to represent the effect of turbulent viscosity on the buoyancy-damping time scale while R_i is small. As for the expression of $\sqrt{R_i/C_5}$, it is chosen so that the model is consistent with Monin–Obukhov similarity, which is discussed next.

3. Consistency with Monin–Obukhov similarity

This section shows the consistency between the proposed turbulence model and Monin–Obukhov similarity and further uses the similarity to determine parameters C_3 , C_4 , and C_5 in (2.13). To be specific, the section introduces formulas from the similarity first, then re-expresses them in terms of the Obukhov length L , and finally expands them in the Taylor's series so as to determine C_3 , C_4 , and C_5 .

a. Formulas from Monin–Obukhov similarity

Consider a steady horizontally homogeneous surface boundary layer where Monin–Obukhov similarity is applicable. Its momentum flux does not change with height and is described by

$$u_*^2 = \nu_t S, \quad (3.1)$$

where $S = |d\mathbf{V}/dz|$ and u_* is the friction velocity. Using the universal similarity functions, the mean velocity and potential temperature profiles are expressed (Businger et al. 1971; Dyer 1974; Gibson and Launder 1978) as

$$\frac{k_0 z}{u_*} S = 1 + \beta \frac{z}{L}, \quad (3.2)$$

$$\frac{k_0 z}{\theta_*} \frac{d\theta_0}{dz} = \sigma_\theta + \beta \frac{z}{L}, \quad (3.3)$$

where the von Kármán constant $k_0 = 0.41$ (Dyer 1974; S95), $\sigma_\theta \approx 0.74$ (Gibson and Launder 1978), $\beta \approx 4.7$, θ_* is the surface-layer temperature scale, and L the Obukhov length that is associated with θ_* by

$$L = \frac{u_*^2}{k_0 g \theta_*/\theta_0}. \quad (3.4)$$

Substituting (3.4) and (2.7b) into (3.3) to remove θ_* and $d\theta_0/dz$ yields

$$G = -\left(\sigma_\theta + \beta \frac{z}{L}\right) \frac{u_*^2}{\sigma_\theta k_0^2 z L}. \quad (3.5)$$

Similarly, (3.2) is rewritten as

$$S = \left(1 + \beta \frac{z}{L}\right) \frac{u_*}{k_0 z}. \quad (3.6)$$

b. Reformulation in terms of the Obukhov length L

To compare the similarity with the turbulence model, (3.1)–(3.6) from the similarity are reexpressed in terms of L or the dimensionless height

$$\tilde{z} = \beta z/L. \quad (3.7)$$

Substituting (3.6) into (3.1) yields the expression of ν_t , or

$$\nu_t = u_* k_0 z (1 + \tilde{z})^{-1}. \quad (3.8)$$

Equation (2.6) times S and then substituting (2.10) and (3.1) into the resulting equation gives

$$k = \frac{u_*^2}{C_\mu \eta}, \quad (3.9)$$

which is used to rewrite (2.10) as

$$\varepsilon = \frac{S u_*^2}{C_\mu \eta^2}. \quad (3.10)$$

For the steady boundary layer, the energy equation, (2.5), is simplified to

$$\varepsilon = \nu_t (S^2 + G) + \frac{d}{dz} \left(\frac{\nu_t}{\sigma_k} \frac{dk}{dz} \right),$$

which is further changed to

$$\frac{\eta^{-2}}{C_\mu} = 1 + \frac{G}{S^2} + \frac{k_0^2 \tilde{z}}{\sigma_k C_\mu (1 + \tilde{z})} \frac{d}{d\tilde{z}} \left[\frac{\tilde{z} d\eta^{-1}}{(1 + \tilde{z}) d\tilde{z}} \right] \quad (3.11)$$

using (2.6), (2.10), (3.1), (3.5), and (3.6), where the gradient Richardson number divided by σ_θ

$$-\frac{G}{S^2} = \frac{\sigma_\theta + \tilde{z}}{(1 + \tilde{z})^2} \frac{\tilde{z}}{\sigma_\theta \beta}. \quad (3.12)$$

Similarly, the ε equation of (2.8) is simplified to

$$\begin{aligned} & -C_1 S \varepsilon + C_2 \frac{\varepsilon^2}{k} - C_3 \frac{\varepsilon}{k} \nu_t G - C_4 \min \left(1, \sqrt{\frac{R_i}{C_5}} \right) N \varepsilon \\ & = \frac{d}{dz} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{d\varepsilon}{dz} \right), \end{aligned}$$

which is further changed to

$$\begin{aligned} & -C_1 + C_2 \eta^{-1} - C_3 C_\mu \eta \frac{G}{S^2} - C_4 \min \left(1, \sqrt{\frac{R_i}{C_5}} \right) \frac{N}{S} \\ & = \frac{k_0^2 \eta^2 \tilde{z}^2}{\sigma_\varepsilon (1 + \tilde{z})^2} \frac{d}{d\tilde{z}} \left[\frac{\tilde{z} d}{(1 + \tilde{z}) d\tilde{z}} \frac{1 + \tilde{z}}{\tilde{z} \eta^2} \right] \end{aligned} \quad (3.13)$$

using (3.1), (3.6), and (3.10).

Equations (3.11) and (3.13) come from the energy and the ε equations, respectively. They have only one unknown variable, η , and thus can be solved for η independently. *If the two solutions of η from (3.11) and (3.13) are close to each other, the turbulence model is consistent with Monin–Obukhov similarity.* Otherwise, the model is inconsistent with the similarity and thus is ill posed.

Specifically, the consistency between (3.13) and (3.11) involves two issues: (i) the η – $\tilde{z}$ relationship that is determined by the proposed turbulence model and (ii) the selection of three constants C_3 , C_4 , and C_5 . The consistency cannot be achieved if the η – $\tilde{z}$ relationship is not accurate (or wrong), no matter how the constants of C_3 , C_4 , and C_5 are selected. In other words, the consistency between (3.13) and (3.11) implies both the η – $\tilde{z}$ relationship and the constants in (3.13) are correct. Next, the solutions of η from (3.13) and (3.11) are compared, showing that the proposed turbulence model is physically consistent with Monin–Obukhov similarity.

c. Consistency near $\tilde{z} = 0$

The two solutions of η from (3.11) and (3.13) near $\tilde{z} = 0$ are compared analytically to obtain two constraints

of model parameters. Expanding η in the Taylor series of $\tilde{z}$ first and then substituting it into (3.11) give

$$\eta = \frac{1}{\sqrt{C_\mu}} + \frac{1}{2\sqrt{C_\mu - k_0^2/\sigma_k}} \tilde{z} + \dots \quad (3.14)$$

with the aid of (3.12). Equation (3.14) is then substituted into (3.13), which gives

$$\begin{aligned} C_2\sqrt{C_\mu} - C_1 + \left(C_3\sqrt{C_\mu} - \sigma_\theta \frac{C_4}{\sqrt{C_5}} - \frac{C_\mu C_2}{2\sqrt{C_\mu - k_0^2/\sigma_k}} \right) \frac{\tilde{z}}{\beta} \\ = \frac{k_0^2}{\sigma_\varepsilon} + \frac{2k_0^2}{\sigma_\varepsilon} \left(\frac{\sqrt{C_\mu}}{2\sqrt{C_\mu - k_0^2/\sigma_k}} - \beta \right) \frac{\tilde{z}}{\beta}, \end{aligned}$$

after retaining the first two orders. The preceding equation requires

$$C_2\sqrt{C_\mu} - C_1 = \frac{k_0^2}{\sigma_\varepsilon}, \quad (3.15a)$$

$$\begin{aligned} C_3\sqrt{C_\mu} - \sigma_\theta \frac{C_4}{\sqrt{C_5}} - \frac{C_\mu C_2}{2\sqrt{C_\mu - k_0^2/\sigma_k}} \\ = \frac{2k_0^2}{\sigma_\varepsilon} \left(\frac{\sqrt{C_\mu}}{2\sqrt{C_\mu - k_0^2/\sigma_k}} - \beta \right), \end{aligned} \quad (3.15b)$$

where (3.15a) is a constraint of σ_ε for neutral ABL (S95) and (3.15b) is a constraint of C_3 , C_4 , and C_5 for stable ABL. In other words, when C_3 , C_4 , and C_5 satisfy (3.15), the two solutions of η from (3.11) and (3.13) are close to each other near $\tilde{z} = 0$.

d. Consistency at $\tilde{z} \gg 1$

The two solutions of η from (3.11) and (3.13) at $\tilde{z} \gg 1$ are compared analytically to obtain another constraint of model parameters. Just as shown by (3.6) and (3.12), S becomes constant and $G/S^2 \rightarrow -(\sigma_\theta\beta)^{-1}$ when $\tilde{z}$ is very large. Hence, $dk/dz \approx d\eta/dz \approx 0$ there, and consequently (3.11) and (3.13) become

$$\frac{\eta^{-2}}{C_\mu} = 1 - \frac{1}{\sigma_\theta\beta}, \quad (3.16a)$$

$$-C_1 + C_2\eta^{-1} + C_3C_\mu\frac{\eta}{\sigma_\theta\beta} - \frac{C_4}{\sqrt{\beta}} = 0, \quad (3.16b)$$

respectively. Eliminating η in (3.16) gives a constraint of C_3 , C_4 , and C_5 . That is,

$$C_2(\sigma_\theta\beta - 1) + C_3 = \left(C_1 + \frac{C_4}{\sqrt{\beta}} \right) [C_\mu^{-1}\sigma_\theta\beta(\sigma_\theta\beta - 1)]^{1/2}. \quad (3.17)$$

In short, when C_3 , C_4 , and C_5 satisfy (3.17), the two solutions of η from (3.11) and (3.13) are close to each other at $\tilde{z} \gg 1$.

e. Consistency in the vertical profile of η

Two vertical profiles of η from (3.11) and (3.13) are compared for consistency between the turbulence model and Monin–Obukhov similarity through the boundary layer (or $0 \leq \tilde{z} \leq \beta$), yielding the last constraint of model parameters. Equation (3.11) is solved numerically for η , using the boundary conditions $\eta = C_\mu^{-1/2}$ at $\tilde{z} = 0$ and $d\eta/d\tilde{z} = 0$ at very large $\tilde{z}$ (e.g., $\tilde{z} = \beta$ or $z = L$). To be specific, (3.11) is solved numerically using 1024 grid points with a second-order finite-difference scheme. Its result is displayed by the black line in Fig. 2. Since (3.11) involves none of the constants C_1 to C_5 , this vertical profile of η from (3.11) represents the result of Monin–Obukhov similarity and thus is used as a standard to test other profiles from (3.13).

Equation (3.13), just like (3.11), is solved numerically once C_3 , C_4 and C_5 are given. Its η – $\tilde{z}$ profile is then compared with that of (3.11), showing whether the turbulence model is consistent with Monin–Obukhov similarity.

Equation (3.13), with no buoyancy-damping term (or $C_4 = 0$), is discussed first, where the effect of static stability on ε is introduced just as it was in the standard k – ε model. To be specific, $C_3 = 1.44$ and $C_4 = 0$ (Duynerkerke 1988; Apsley and Castro 1997). Using the two parameters, (3.13) is solved numerically and its result is displayed in Fig. 2, showing that the turbulence model with $C_4 = 0$ has a significant error of η . In other words, the model with no buoyancy damping (or $C_4 = 0$) is not consistent with Monin–Obukhov similarity (see footnote 2 for physical explanation).

With increasing C_4 , the vertical profiles of η obtained from (3.13) and (3.11) become close to each other, which implies that the turbulence model becomes close to Monin–Obukhov similarity. When $C_4 = 0.44$, thus $C_3 = 1.46$ and $C_5 = 0.08$ are obtained based on (3.15b) and (3.17). Using these new parameters (see Table 1 for a complete list of model parameters), the η – $\tilde{z}$ profile of (3.13) is obtained and displayed by the red line in Fig. 2, showing that the profile is quite close to that of (3.11). In other words, $C_3 = 1.46$, $C_4 = 0.44$, and $C_5 = 0.08$ are determined so that the turbulence model agrees well with Monin–Obukhov similarity.

FIG. 2. Consistency in the η - z profile between the present turbulence model (red) and the result from Monin-Obukhov similarity (black line). For comparison, the profile with no buoyancy damping (or $C_4 = 0$) is represented by the blue line, and the Shih model for neutral ABL is presented by the green dot.

4. ABL simulations

To further evaluate the turbulence model in section 2, an ABL model is developed that incorporates the turbulence model. Its framework is introduced next, followed by a comparison between ABL-model results and field observations.

a. Equations of the ABL model

The ABL model considers a steady, horizontally homogeneous ABL with the Coriolis force. Its mean horizontal-wind components U and V in the (x, y) direction are governed by

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial z} \left(\nu_t \frac{\partial U}{\partial z} \right) + f(V - V_g), \quad (4.1a)$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial z} \left(\nu_t \frac{\partial V}{\partial z} \right) - f(U - U_g), \quad (4.1b)$$

where f is the Coriolis parameter and (U_g, V_g) are the two components of geostrophic wind in the (x, y) directions.

To properly simulate the transition from turbulent to laminar flow, $\ln k$ and $\ln \varepsilon$ (instead of k and ε) are

TABLE 1. Parameters in the turbulence model. The values for C_1 and C_μ are calculated with the given equation numbers.

Parameter	Value or equation number
C_1	(2.9a)
C_2	1.9
C_3	1.46
C_4	0.44
C_5	0.08
C_μ	(2.9b)
σ_k	1.0
σ_ε	1.2
σ_θ	0.74

used as prognostic variables because k and ε vary by orders. The prognostic equations of $\ln k$ and $\ln \varepsilon$ are derived from (2.5) and (2.8) with the aid of (2.6). That is,

$$\frac{\partial \ln k}{\partial t} = \frac{\partial}{\partial z} \left(\frac{\nu_t}{\sigma_k} \frac{\partial \ln k}{\partial z} \right) + \frac{\nu_t}{\sigma_k} \left(\frac{\partial \ln k}{\partial z} \right)^2 + \frac{\nu_t}{k} (S^2 + G) - \frac{\varepsilon}{k}, \quad (4.2a)$$

$$\begin{aligned} \frac{\partial \ln \varepsilon}{\partial t} = & \frac{\partial}{\partial z} \left(\frac{\nu_t}{\sigma_\varepsilon} \frac{\partial \ln \varepsilon}{\partial z} \right) + \frac{\nu_t}{\sigma_\varepsilon} \left(\frac{\partial \ln \varepsilon}{\partial z} \right)^2 \\ & + C_1 S - \frac{C_2 \varepsilon}{k + \sqrt{\nu_0 \varepsilon / C_\mu}} + C_3 \frac{\nu_t}{k} G \\ & + C_4 \min \left(1, \sqrt{\frac{R_i}{C_5}} \right) N. \end{aligned} \quad (4.2b)$$

Equations (4.1) and (4.2) are solved with the following boundary conditions. At the bottom boundary (or $z = 0$), $U = V = 0$. In addition, (3.9) and (3.14) show

$$k = \frac{u_*^2}{\sqrt{C_\mu}}, \quad (4.3a)$$

and (3.6), (3.10), and (3.14) show

$$\varepsilon = \frac{u_*^3}{k_0} \left(\frac{1}{z_0} + \frac{\beta}{L} \right), \quad (4.3b)$$

which are the same boundary conditions as Duynkerke (1988) and others used.

At the top model boundary (or $z = 3$ km), $U = U_g$ and $V = V_g$. Since the top model boundary is far above ABL, there is no turbulence (except for clear-sky turbulence). Hence, the turbulent viscosity ν_t is set to equal the molecular one ν_0 . That is,

$$k = k_{\min}, \quad (4.4a)$$

where $k_{\min} = \nu_0^2 / (C_\mu \lambda_m^2)$ is obtained from (2.12) and the Taylor microscale $\lambda_m = 3.14$ mm that is smaller than that in ABL (Wyngaard 1992). Since there is no turbulence, the mean eddy scale is quite small and stays small while

$$\varepsilon = k_{\min} \sqrt{\frac{C_\mu [S^2 + (C_3 - 1) \sigma_\theta^{-1} N^2]}{C_2 - 1}}, \quad (4.4b)$$

which is derived from (B.4) in appendix B. Using the boundary conditions of (4.3) and (4.4), (4.1) and (4.2) can be solved once (U_g, V_g) is given.

To avoid an inertial oscillation in ABL, the model takes an initial condition of $U = U_g$ and $V = V_g$ and assumes the initial ABL is laminar or the model follows (4.4) initially. Since the mean wind shear is strong near the surface, turbulence develops near the surface first and then propagates upward. As a result of such initial and boundary conditions, the model approaches a steady status without an inertial oscillation.

b. Model parameters

The famous ‘‘Leipzig wind profile’’ is used to test the ABL model just as in the previous studies (Blackadar 1962; Detering and Etling 1985; Riopelle and Stublely 1989; Apsley and Castro 1997). The profile came from 28 pilot-balloon observations in a uniform air mass with stable weather near Leipzig, Germany, at 0915–1615 LT 20 October 1933 (Mildner 1932; Lettau 1950). Since it represents the average wind distribution during the period, it is usually treated as a representation of neutral ABL with steady geostrophic wind and homogeneous surface (Blackadar 1962; Detering and Etling 1985; Apsley and Castro 1997).

However, the 1400 LT sounding (less than 100 miles from Leipzig) in the same air mass showed an increase of potential temperature of $0.35^\circ\text{C} (100\text{ m})^{-1}$ in the lower ABL (Mildner 1932; Lettau 1950). Hence, the profile was also treated as a representation of slightly stable ABL (Riopelle and Stublely 1989). In the present study, two numerical simulations are carried out that use two increases in potential temperature: 0 (neutral stability) and $\sim 0.35^\circ\text{C} (100\text{ m})^{-1}$. These are referred to as experiments NEUTRAL and STABLE, respectively.

Experiment STABLE assumes that the atmospheric stability (or G) satisfies Monin–Obukhov similarity or (3.3) in the surface layer. The stability above the surface layer (or $z > 100$ m) is close to zero by G obtained from (3.3) times $\exp[-(z - 100)/520]$, where z takes the unit of meters. In addition, the experiment sets the Obukhov length $L = 0.58$ km, which is close to the estimation of Sundararajan (1979). As a result, the vertical gradient of potential temperature $d\theta_0/dz$ in STABLE is obtained

FIG. 3. Vertical gradients of potential temperature vs height used in experiments STABLE (thick solid) and NEUTRAL (thin dashed line).

(and displayed in Fig. 3), showing that the average of $d\theta_0/dz$ in the lower ABL is close to $0.35^\circ\text{C} (100\text{ m})^{-1}$ and the magnitude of $d\theta_0/dz$ above 0.5 km is close to zero.

Experiment NEUTRAL takes the same setup as STABLE except that the Obukhov length $L = 4.8$ km. Since this Obukhov length is so large that $d\theta_0/dz \approx 0$ (see Fig. 3), NEUTRAL aims at the ABL with neutral stratification.

Both experiments use the following physical parameters: $U_g = 17.5\text{ m s}^{-1}$, $V_g = 0$, $f = 1.13 \times 10^{-4}\text{ s}^{-1}$, air density is 1.25 kg m^{-3} , roughness length $z_0 = 0.3$ m, and $u_* = 0.65\text{ m s}^{-1}$. In addition, (4.1) and (4.2) are discretized using 1001 grid points with a model top of 3 km and a grid size of 3 m. They are integrated for 30 h (see appendix C for the numerical scheme). Their variables are averaged over the last 15 h to obtain the steady status in each experiment.

c. Comparison with the Leipzig data

The results of experiments STABLE and NEUTRAL are compared with the Leipzig dataset, showing that STABLE performs better than NEUTRAL. Figure 4 displays the vertical distributions of turbulent viscosity obtained from STABLE, NEUTRAL, and the Leipzig dataset. The turbulent viscosity of STABLE is quite close to the Leipzig data, whereas the viscosity of NEUTRAL is much larger than the observed. Moreover, the turbulent

FIG. 4. Vertical profiles of turbulent viscosity from experiments STABLE (blue) and NEUTRAL (green line) and the Leipzig observations (black circles).

viscosity of STABLE approaches zero at ~ 1.4 km, reasonably replicating the ABL top.

Using the data of STABLE, NEUTRAL, and Leipzig observations, the mixing length l is computed (Blackadar 1962) with

$$l = (\nu_t/S)^{1/2}.$$

The mixing lengths from STABLE, NEUTRAL, and Leipzig observations are displayed in Fig. 5. Generally speaking, the mixing length of STABLE is closer to the Leipzig data, but l in NEUTRAL is much longer. In other words, the weak stability in STABLE restrains the mixing length in contrast to NEUTRAL, which functions similarly as the limit of the mixing length imposed by Apsley and Castro (1997). Furthermore, the mixing length in STABLE decreases with height in the upper ABL, especially near the ABL top, which is different from that in the standard k - ϵ model (Detering and Etling 1985; Apsley and Castro 1997).

Figure 6 displays the velocity components obtained from STABLE, NEUTRAL, and the Leipzig observations. Generally speaking, the velocity components of STABLE are closer to the observational data than those of NEUTRAL. Since the turbulent viscosity of

FIG. 5. As in Fig. 4, but for the mixing length.

STABLE is quite close to the Leipzig data (see Fig. 4), the small difference in velocity components between STABLE and the Leipzig data may come from the uncertainty in the Leipzig dataset because the Leipzig dataset was synchronized with three methods from the 28 pilot-balloon observations (Lettau 1950).

Figure 6 also shows STABLE is closer to the Leipzig data in the velocity components than NEUTRAL. This result supports Duynkerke (1988) and Riopelle and Stublely (1989), who showed the weak stability in the Leipzig observations effectively reduces the turbulent viscosity.

5. Concluding remarks

A new turbulence model for the stable atmosphere is developed based on S95. The model represents the static stability by taking account of the buoyancy damping of gravity waves (Bretherton and Smolarkiewicz 1989). A theoretical analysis shows the model is consistent with Monin–Obukhov similarity for stable ABL (see Fig. 2).

The turbulence model is incorporated into a horizontally homogeneous ABL model for model evaluation by comparing ABL-model results with the Leipzig observations. Generally speaking, the results of the ABL model agree well with the Leipzig data, showing the turbulence model works well for stable ABL.

FIG. 6. As in Fig. 4, but for the horizontal velocity components (left) U and (right) V .

The results also show that the weak stratification in ABL plays an important role in the turbulence observed at Leipzig, which supports [Duynerke \(1988\)](#) and [Riopelle and Stuble \(1989\)](#). The results also show that the weak stability restrains the mixing length, which functions similarly as the limited mixing length imposed by [Apsley and Castro \(1997\)](#).

The model, in comparison with the standard one ([Detering and Etling 1985](#); [Duynerke 1988](#); [Apsley and Castro 1997](#)), has the following characteristics in the ABL modeling:

- consistency with Monin–Obukhov similarity;
- no parameter tuning for neutral ABL, which maintains its consistency with the laboratory observations of incompressible fluid ([S95](#));
- turbulent viscosity approaches zero at ~ 1.4 km (see STABLE in Fig. 4), which reasonably replicates the ABL top; and
- mixing length decreases with height in the upper ABL, significantly near the ABL top (Fig. 5).

This turbulence model will be incorporated into the ABL model in ARL ([Wang et al. 2019](#)) for further evaluation over topography; special attention will be given to its sensitivity to model resolution. Furthermore, it will be extended to incorporate convective instability for a complete set of turbulence

parameterization, addressing cloud and boundary layer modeling.

On the other hand, the model needs further improvements for two specific cases. 1) When the Reynolds number is not high, the ratio of the energy–dissipation rate to turbulent enstrophy or $\varepsilon/(2\nu_0\delta)$ may deviate from one (e.g., [Yeung et al. 2012](#)). Hence, the last term of (2.5) needs an adjustment to represent the difference between the energy–dissipation rate and turbulent enstrophy. 2) Clear-sky turbulence occurs in a stratified atmosphere due to the Kelvin–Helmholtz instability. Although the instability-induced turbulence can be represented by the model via C_5 , its development modeled depends on the background values of k and ε in the stratified atmosphere that are chosen in (4.4). Hence, observations and explicit simulations of clear-sky turbulence are needed to test the background values of k and ε set for the stratified atmosphere.

Acknowledgments. The authors are grateful to the anonymous reviewers for their constructive comments and Martin Kufus for reading the paper. They also thank Dr. Tina Katopodes Chow for her kind comments on buoyancy damping. This research was supported by the ARL Basic Research Program. Dr. Zeng was also supported by the NASA

Precipitation Measurement Mission (PMM) project under Grant NNX16AE24G.

APPENDIX A

Time Scale of Buoyancy Damping

Buoyancy damping versus gravity waves resembles music versus sound waves. Although music is an ensemble effect of sound waves, a musician composes a song usually based on music terminology (e.g., rhythm, intonation, beat, and pitch) rather than sound waves. Similarly, buoyancy damping is an ensemble effect of gravity waves. Its effect on turbulent enstrophy is represented by the last term of (2.13) rather than gravity waves, which is discussed next.

Buoyancy damping, as a concept, was first proposed by Bretherton and Smolarkiewicz (1989) and its terminology by David J. Raymond (Zeng 2001). It was applied to studying the interaction between tropical convection and large-scale circulations (Nilsson and Emanuel 1999; Raymond and Zeng 2000, 2005; Sobel and Bretherton 2000; Bretherton and Sobel 2002). Its analytic solution was given by Zeng (2001) and Zeng et al. (2007, 2008). In this appendix, buoyancy damping is analyzed with a three-dimensional linear model of a stable dry atmosphere, focusing on how the time scale of buoyancy damping varies with viscosity coefficient and gravity wave phase speed. At the end of this appendix, the buoyancy damping in ABL is compared to that in the tropical troposphere with thermal forcing (or convective heating).

Consider an ABL in the Cartesian coordinate system (x, y, z) . Its perturbation status is described by (Zeng 2001; Zeng et al. 2008)

$$\partial_t(\rho u) + \partial_x p' - \nu \partial_{zz}^2(\rho u) = 0, \quad (\text{A.1a})$$

$$\partial_t(\rho v) + \partial_y p' - \nu \partial_{zz}^2(\rho v) = 0, \quad (\text{A.1b})$$

$$\partial_x(\rho u) + \partial_y(\rho v) + \partial_z(\rho w) = 0, \quad (\text{A.1c})$$

$$\partial_t(\rho \theta') + N^2 g^{-1} \theta_0 \rho w = Q, \quad (\text{A.1d})$$

$$\partial_z p' - g \rho \theta' / \theta_0 = 0, \quad (\text{A.1e})$$

where (u, v, w) are the three components of velocity vector, p' the pressure perturbation, θ' the potential temperature perturbation, Q the thermal heating rate, and ν the viscosity coefficient.

For the waves trapped in ABL, they are expanded vertically in Fourier series, or

$$\{\rho u, \rho v, p'\} = \sum_{m=1}^{\infty} [(\rho u)_m, (\rho v)_m, p'_m] \cos \frac{m\pi}{H} z, \quad (\text{A.2a})$$

$$\{\rho w, \rho \theta', Q\} = \sum_{m=1}^{\infty} [(\rho w)_m, (\rho \theta')_m, Q_m] \sin \frac{m\pi}{H} z, \quad (\text{A.2b})$$

where a symbol with subscript m denotes the expansion coefficient of its corresponding variable at mode m , and H is the thickness of ABL. Substituting (A.2) into (A.1) first and then eliminating $(\rho w)_m$ and $(\rho \theta')_m$ in the resulting equation gives

$$\partial_t(\rho u)_m = -\partial_x p'_m - \nu N^2 c_m^{-2}(\rho u)_m, \quad (\text{A.3a})$$

$$\partial_t(\rho v)_m = -\partial_y p'_m - \nu N^2 c_m^{-2}(\rho v)_m, \quad (\text{A.3b})$$

$$\partial_t p'_m + c_m^2 [\partial_x(\rho u)_m + \partial_y(\rho v)_m] = -c_m g(N\bar{\theta})^{-1} Q_m, \quad (\text{A.3c})$$

where $c_m = HN/m\pi$ is the phase speed of gravity wave at mode m .

Differentiating (A.3a) to x and (A.3b) to y , then summing the two equations and finally substituting (A.3c) into the resulting equation, yields a pressure equation. Since pressure perturbation is related to temperature perturbation by

$$N p'_m / c_m = -g(\rho \theta')_m / \theta_0, \quad (\text{A.4a})$$

the pressure equation is further changed to

$$\begin{aligned} & \partial_t^2(\rho \theta')_m + \nu N^2 c_m^{-2} \partial_t(\rho \theta')_m - c_m^2 [\partial_{xx}^2(\rho \theta')_m + \partial_{yy}^2(\rho \theta')_m] \\ & = \partial_t Q_m + \nu N^2 c_m^{-2} Q_m \end{aligned} \quad (\text{A.4b})$$

which can be used to analyze the buoyancy damping in ABL.

For a perturbation in (A.4b) that is described by $(\rho \theta')_m$, its other variables, such as $(p')_m$, $(\rho u)_m$, and $(\rho v)_m$, can be obtained from (A.4a), (A.3a), and (A.3b), respectively. For simplicity, the perturbation is discussed only in terms of temperature rather than other variables such as velocity vector or pressure.

Since (A.4b) is linear, the superposition principle works in buoyancy damping. Next, a special case is discussed to illustrate how buoyancy damping works without loss of generality.

a. Atmospheric response to an initial temperature perturbation

Suppose there is no thermal forcing in ABL or $Q_m = 0$. Consider an initial bell-shaped temperature perturbation

$$(\rho\theta')_m|_{t=0} = \frac{(\rho\theta^*)_m a^4}{(a^2 + x^2)(a^2 + y^2)}, \quad (\text{A.5a})$$

$$\partial_t(\rho\theta')_m|_{t=0} = 0, \quad (\text{A.5b})$$

with $-\infty < x < +\infty$ and $-\infty < y < +\infty$, where a represents the horizontal scale of the perturbation. The perturbation initiates gravity waves that propagate outward, leading to buoyancy damping. To quantify the buoyancy damping, the following dimensionless variables are used:

$$\begin{aligned} \hat{x} &= x/a, & \hat{y} &= y/a, & \hat{t} &= tc_m/a, \\ \hat{\theta} &= (\rho\theta')_m/(\rho\theta^*)_m, & \hat{\nu} &= aN^2\nu/2c_m^3. \end{aligned} \quad (\text{A.6})$$

With the aid of Fourier transform, (A.4b) and (4.5) are solved by

$$\hat{\theta}(\hat{x}, \hat{y}, \hat{t}) = \text{Re} \int_0^{+\infty} \int_0^{+\infty} \frac{1}{\omega_2 - \omega_1} e^{-(|\xi|+|\eta|)} (\omega_2 e^{\omega_1 \hat{t}} - \omega_1 e^{\omega_2 \hat{t}}) e^{-i(\xi\hat{x} + \eta\hat{y})} d\xi d\eta, \quad (\text{A.7})$$

where the operator Re chooses the real part of a complex expression and

$$\omega_{1,2} = -\hat{\nu} \pm \sqrt{\hat{\nu}^2 - (\xi^2 + \eta^2)}. \quad (\text{A.8})$$

$$\begin{aligned} \hat{\theta} &= \int_{\hat{\nu}}^{+\infty} \int_0^{\pi/2} e^{-r(\cos\alpha + \sin\alpha) - \hat{\nu}\hat{t}} \left[\cos(\hat{t}\sqrt{r^2 - \hat{\nu}^2}) + \hat{\nu} \frac{\sin(\hat{t}\sqrt{r^2 - \hat{\nu}^2})}{\sqrt{r^2 - \hat{\nu}^2}} \right] r \cos[r(\hat{x}\cos\alpha + \hat{y}\sin\alpha)] d\alpha dr \\ &+ \int_0^{\hat{\nu}} \int_0^{\pi/2} e^{-r(\cos\alpha + \sin\alpha) - \hat{\nu}\hat{t}} \left[\cosh(\hat{t}\sqrt{\hat{\nu}^2 - r^2}) + \hat{\nu} \frac{\sinh(\hat{t}\sqrt{\hat{\nu}^2 - r^2})}{\sqrt{\hat{\nu}^2 - r^2}} \right] r \cos[r(\hat{x}\cos\alpha + \hat{y}\sin\alpha)] d\alpha dr. \end{aligned} \quad (\text{A.9})$$

For a given $\hat{\nu}$, (A.9) is computed by numerical integration. Its result at $\hat{y} = 0$ and $\hat{\nu} = 0.1$ is displayed in Fig. A1. Obviously, the amplitude of temperature perturbation decreases with time, which is referred to as buoyancy damping. In other words, buoyancy damping is an ensemble effect of gravity waves, although it may have no symptom of a sine oscillation sometimes.

The first term on the right-hand side of (A.9) is important when $\hat{\nu}\hat{t} \ll 1$, whereas the second one is important when $\hat{\nu}\hat{t} \gg 1$. In other words, buoyancy damping consists of two stages. In the first stage (or $\hat{\nu}\hat{t} < 1$), an initial perturbation propagates outward into a wide area via gravity waves, resulting in the decrease of its amplitude. In the second stage (or $\hat{\nu}\hat{t} > 1$),

FIG A1. Potential-temperature perturbation $\hat{\theta}$ vs $\hat{x}$ at $\hat{y} = 0$ and $\hat{\nu} = 0.1$. The number beside a line indicates the dimensionless time $\hat{t}$.

After setting $\xi = r \cos\alpha$ and $\eta = r \sin\alpha$, the solution is expressed as (Zeng et al. 2007)

the perturbation over the wide area is dissipated due to molecular friction or turbulent mixing. Hence, when $\hat{\nu}$ is small, the amplitude of temperature perturbation decreases mainly in the first stage so that its decrease is almost independent of friction (Zeng et al. 2007).

b. Time scale of buoyancy damping

Buoyancy damping depends on not only the gravity wave speed c_m but also on the viscosity coefficient ν , especially for the modes with short vertical scale. In this subsection, its time scale is parameterized to take account of the frictional effect.

Figure A1 of Zeng et al. (2007) also displays the potential temperature perturbation versus $\hat{t}$ at $\hat{x} = \hat{y} = 0$,

showing that the time scale of buoyancy damping increases with $\hat{\nu}$. To quantify the time scale, the decrease of potential temperature is fitted with a formula of

$$\hat{\theta} = \exp(-\hat{t}/\hat{\tau}) \quad (\text{A.10})$$

to obtain a dimensionless damping time scale $\hat{\tau}$. Figure A2 displays the obtained time scales against $\hat{\nu}$, reaching the following approximate correlation between $\hat{\tau}$ and $\hat{\nu}$:

$$\hat{\tau} = (1 + 3\hat{\nu})/2. \quad (\text{A.11})$$

In summary, (A.10), (A.11), and (A.6) reveal the characteristics of the buoyancy-damping time scale. The time scale is inversely proportional to the gravity wave speed c_m or the vertical wavelength. It is also proportional to the viscosity coefficient, especially for the modes with short vertical scale. In ABL, a temperature perturbation generated by a larger eddy damps faster than that by a smaller eddy. Furthermore, a temperature perturbation near the surface damps slowly because of its short vertical scale and the large turbulent viscosity there, which is represented in (2.13) via $1/\min(1, \sqrt{R_i/C_5})$.

c. Comparison to the buoyancy damping in the tropical troposphere

The buoyancy damping in ABL shares the same physics with that in the convective troposphere. However, they are different in details, which is discussed below, beginning with a mathematical operation of (A.1).

Consider a variable array $\mathbf{A} = (u, v, w, \theta', p')$ and a forcing array $\mathbf{Q} = (0, 0, 0, Q, 0)$. Using $\mathbf{A}$ and $\mathbf{Q}$, the linear equation, (A.1), is expressed as

$$\mathbf{L}(\mathbf{A}) = \mathbf{Q}, \quad (\text{A.12a})$$

$$\mathbf{A}|_{t=0} = \mathbf{A}_0, \quad (\text{A.12b})$$

where $\mathbf{L}$ is a linear operator on $\mathbf{A}$, and $\mathbf{A}_0$ the initial value of $\mathbf{A}$. The solution of (A.12) consists of two parts: a forced solution of $\mathbf{A}_{\text{forced}}$ and a free solution of $\mathbf{A}_{\text{free}}$. That is,

$$\mathbf{A} = \mathbf{A}_{\text{forced}} + \mathbf{A}_{\text{free}}. \quad (\text{A.13})$$

The forced solution $\mathbf{A}_{\text{forced}}$ is a special solution of (A.12a) that satisfies

$$\mathbf{L}(\mathbf{A}_{\text{forced}}) = \mathbf{Q}. \quad (\text{A.14})$$

Zeng et al. (2008) showed that the amplitude of $\mathbf{A}_{\text{forced}}$ is directly proportional to the amplitude of the thermal forcing $\mathbf{Q}$. In ABL without clouds, $\mathbf{Q} \approx 0$ and subsequently $\mathbf{A}_{\text{forced}} \approx 0$.

FIG A2. Dimensionless damping time scale $\hat{\tau}$ vs the dimensionless friction coefficient $\hat{\nu}$. Circles indicate the time scales estimated using (A.10), and solid line represents (A.11) that fits the circles.

Since $\mathbf{L}$ is a linear operator, $\mathbf{L}(\mathbf{A}) = \mathbf{L}(\mathbf{A}_{\text{forced}}) + \mathbf{L}(\mathbf{A}_{\text{free}})$. Thus, substituting this equation and (A.14) into (A.12) gives

$$\mathbf{L}(\mathbf{A}_{\text{free}}) = 0, \quad (\text{A.15a})$$

$$\mathbf{A}_{\text{free}}|_{t=0} = \mathbf{A}_0 - \mathbf{A}_{\text{forced}}|_{t=0}. \quad (\text{A.15b})$$

Since $\mathbf{A}_{\text{free}}$ is the solution of (A.15) with an initial condition of $\mathbf{A}_0 - \mathbf{A}_{\text{forced}}|_{t=0}$, it accounts for all free gravity waves induced by the initial condition of (A.12b) and the forcing of $\mathbf{Q}$. Equation (A.9) and Fig. A1 show that, when $t \rightarrow \infty$, $\mathbf{A}_{\text{free}} \approx 0$; as a result, $\mathbf{A} \approx \mathbf{A}_{\text{forced}}$.

The preceding mathematical operation from (A.12) to (A.15) is summarized with one sentence: the solution of (A.12) approaches to the forced solution $\mathbf{A}_{\text{forced}}$ via the buoyancy damping of (A.15). In the convective troposphere, the thermal forcing from isolated convection has small horizontal scales in comparison to the Rossby radius of deformation and thus brings about the small amplitude of $\mathbf{A}_{\text{forced}}$ (Bretherton and Sobel 2002). Such small amplitude of $\mathbf{A}_{\text{forced}}$ is referred to as the weak-temperature-gradient approximation (Sobel and Bretherton 2000; Bretherton and Sobel 2002; Raymond and Zeng 2005), which contrasts to the terminology of buoyancy damping that emphasizes the process of gravity waves in approach to the approximation.

In addition, buoyancy damping depends on the dimensionality of (A.12); that is, buoyancy damps faster in a 3D (three-dimensional) space than in a 2D (two-dimensional) space (Zeng et al. 2007, 2008). Bretherton and Smolarkiewicz (1989) discussed the

buoyancy damping induced by a cloud in a 2D space (i.e., the buoyancy damping induced by a squall-line cloud in a 3D space). Although their buoyancy damping is different from that in a 3D space, its essence—an initial temperature perturbation propagates outward into a wide area via gravity waves first and then is dissipated by molecular friction or turbulent mixing over the wide area—has been adopted by the later studies (Nilsson and Emanuel 1999; Raymond and Zeng 2000, 2005; Sobel and Bretherton 2000; Zeng 2001; Zeng et al. 2007, 2008).

Return to the difference in buoyancy damping between small-scale turbulence and the convective troposphere. The time scale of buoyancy damping τ has two factors: the gravity wave phase speed c_m and the viscosity coefficient ν . Its relation to c_m and ν is obtained from (A.11), using (A.6). That is,

$$\frac{\tau c_m}{a} = \frac{1}{2} + \frac{3aN^2\nu}{4c_m^3}. \quad (\text{A.16})$$

In the tropical troposphere with deep convection, c_m is large due to the deep depth of the troposphere, and ν is small outside clouds for no turbulence. Hence, the effect of ν on τ or the second term on the right-hand side of (A.16) is negligible, and therefore, $\tau \approx a/2c_m$. Given a , τ is small for large c_m , suggesting that the buoyancy damping is strong in the convective troposphere and the effect of ν on τ is negligible.

In contrast, in small-scale turbulence, c_m is small due to the short scales of turbulent eddies, and ν is large due to turbulence. As a result, the effect of c_m on τ or the second term on the right-hand side of (A.16) becomes important. When c_m (or eddy scale) is very small and ν is very large (in comparison to the molecular viscosity), $\tau \propto \nu N^2/c_m^4$ or τ is very large, indicating that the buoyancy of small eddies does not damp so efficiently as that in the convective troposphere. Once $\tau \rightarrow \infty$, the concept of buoyancy damping disappears. This effect of ν on buoyancy damping is represented in (2.13) via $1/\min(1, \sqrt{R_i/C_5})$.

APPENDIX B

Constraint on the mean eddy scale

The mean eddy scale λ_m in (2.11) equals the Taylor microscale approximately. In this appendix, its sensitivities to shear and buoyancy are analyzed, showing whether the scale (or the mixing length) is limited in the proposed turbulence model. The tendency of λ_m is obtained by taking logarithm of (2.11) first, then differentiating the resulting equation with respect to time, and finally substituting (2.5), (2.6), (2.8), and (2.13) into the resulting equation. That is,

$$\begin{aligned} \frac{2d\lambda_m}{\lambda_m dt} - \frac{1}{k} \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \frac{1}{\varepsilon} \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) \\ = \left[C_\mu \left(\frac{kS}{\varepsilon} \right)^2 - C_1 \frac{kS}{\varepsilon} + (C_2 - 1) \right] \frac{\varepsilon}{k} \\ + (1 - C_3) C_\mu \frac{k}{\varepsilon} G - C_4 \min \left(1, \sqrt{\frac{R_i}{C_5}} \right) N, \end{aligned} \quad (\text{B.1})$$

where the first term on the right-hand side represents the contributions of shear and dissipation to λ_m , and consists of two parts: 1) the dissipation part $(C_2 - 1)\varepsilon/k > 0$, which leads to an increase in λ_m because molecular dissipation effectively weakens small eddies (see Fig. 1); and 2) the shear part $[C_\mu(kS/\varepsilon) - C_1(kS/\varepsilon)]\varepsilon/k > 0$ (or < 0), which leads to an increase (or decrease) in λ_m while shear is strong (or weak) because strong shear generates large eddies (or the downscale energy cascade becomes dominant when shear is weak).

The second and the third terms on the right-hand side of (B.1) represent the contribution of buoyancy to λ_m , which are analyzed below to obtain a constraint of C_3 , C_4 , and C_5 . Since $R_i = (N/S)^2$, then $G = -\sigma_\theta^{-1}N^2 = -\sigma_\theta^{-1}R_iS^2$. Substituting N and G into (B.1) gives

$$\begin{aligned} \frac{2d\lambda_m}{\lambda_m dt} - \frac{1}{k} \cdot \left(\frac{\nu_t}{\sigma_k} \nabla k \right) + \frac{1}{\varepsilon} \nabla \cdot \left(\frac{\nu_t}{\sigma_\varepsilon} \nabla \varepsilon \right) \\ = S \left\{ C_\mu [1 + (C_3 - 1)\sigma_\theta^{-1}R_i] \eta \right. \\ \left. - \left[C_1 + C_4 \min \left(1, \sqrt{R_i/C_5} \right) R_i^{1/2} \right] + (C_2 - 1)\eta^{-1} \right\} \end{aligned} \quad (\text{B.2})$$

with the aid of (2.10). When R_i is larger than the critical Richardson number, no new turbulent eddies are generated by mean flow, which implies $d\lambda_m/dt \leq 0$ or

$$\begin{aligned} C_\mu [1 + (C_3 - 1)\sigma_\theta^{-1}R_i] \eta \\ - \left[C_1 + C_4 \min \left(1, \sqrt{R_i/C_5} \right) R_i^{1/2} \right] + (C_2 - 1)\eta^{-1} \leq 0. \end{aligned} \quad (\text{B.3})$$

Figure B1 displays the region that satisfies (B.3) on the R_i - η plane, using the parameters presented in Table 1. The left-hand side of (B.3) reaches its minimum for a given R_i , when

$$\eta = \left\{ \frac{C_2 - 1}{C_\mu [1 + (C_3 - 1)\sigma_\theta^{-1}R_i]} \right\}^{1/2}. \quad (\text{B.4})$$

In summary, when (B.3) is satisfied, the mean eddy scale decreases with time, which can remove the error

of a too-large mixing length in the standard k - ε model (Detering and Etling 1985) and function similarly as the limited mixing length imposed by Apsley and Castro (1997). For a steady ABL, $d\lambda_m/dt = 0$, which is described approximately by the black solid line in Fig. B1.

APPENDIX C

Numerical scheme of the ABL model

The ABL model is discretized on a staggered grid with a standard second-order finite-difference scheme. Special attention is given to

$$\frac{\partial \ln k}{\partial t} = \frac{\nu_t}{\sigma_k} \left(\frac{\partial \ln k}{\partial z} \right)^2, \quad (\text{C.1})$$

which is a part of (4.2a). The equation can be treated as an advection equation

$$\frac{\partial \ln k}{\partial t} + v_{\text{diff}} \frac{\partial \ln k}{\partial z} = 0, \quad (\text{C.2})$$

where a “diffusive velocity” is defined as (Smolarkiewicz 1983)

$$v_{\text{diff}} = -\frac{\nu_t}{\sigma_k} \frac{\partial \ln k}{\partial z}. \quad (\text{C.3})$$

When $v_{\text{diff}} < 0$ (or $\partial \ln k / \partial z > 0$), an implicit upstream scheme of (C.1) is

$$\frac{\ln k_i^* - \ln k_i^n}{\Delta t} = \frac{\nu_{t,i+1/2}}{\sigma_k} \left(\frac{\ln k_{i+1}^n - \ln k_i^*}{\Delta z} \right)^2, \quad (\text{C.4})$$

where Δt and Δz are the time step and vertical grid size, respectively; superscript n and subscript i indicate time level and vertical grid, respectively; and superscript asterisk indicates a temporary variable at time level $n + 1$. Equation (C.4) is solved with

$$\ln k_i^* = \ln k_{i+1}^n + \chi - \sqrt{2\chi(\ln k_{i+1}^n - \ln k_i^n) + \chi^2}, \quad (\text{C.5})$$

where

$$\chi = \frac{\sigma_k \Delta z^2}{2\nu_{t,i+1/2} \Delta t}. \quad (\text{C.6})$$

When $v_{\text{diff}} > 0$ (or $\partial \ln k / \partial z < 0$), a similar scheme of (C.1) is obtained like (C.4)–(C.6) except that $i + 1$ and $i + 1/2$ are replaced by $i - 1$ and $i - 1/2$, respectively. That is,

$$\ln k_i^* = \ln k_{i-1}^n + \chi - \sqrt{2\chi(\ln k_{i-1}^n - \ln k_i^n) + \chi^2}. \quad (\text{C.7})$$

FIG B1. Region for negative tendency of λ_m (or the Taylor microscale) enclosed by the black solid line on the R_i - η plane. Shading density is directly proportional to the magnitude of the tendency, and dashed line corresponds to (B.4) or the maximum magnitude of the tendency for a given R_i .

Similarly, the scheme in (C.4)–(C.7) is applied to $\ln \varepsilon$ to guarantee stability of the model in case $k_i - 1/k_i$ and/or $\varepsilon_i - 1/\varepsilon_i$ are very large.

APPENDIX D

List of symbols

B	Buoyancy term of ε ; see (2.13)
g	Acceleration due to gravity
G	Static stability; see (2.7b)
k	Turbulent kinetic energy; see (2.5)
L	Obukhov length; see (3.4)
N	Brunt–Väisälä frequency; see (2.14)
R_i	Gradient Richardson number
S	Mean strain rate; see (2.7a)
S_{ij}	Mean-strain-rate tensor
t	Time
T	Air temperature
u_*	Friction velocity; see (3.1)
u_i	Velocity vector in tensor notation
(U, V, W)	Mean velocity components in x, y, z directions
(U_1, U_2, U_3)	Velocity vector
V	Velocity vector
(x, y, z)	Cartesian coordinates
(x_1, x_2, x_3)	Cartesian coordinates
$\tilde{z}$	Dimensionless height; see (3.7)
ε	Energy-dissipation rate; see (2.4)
δ	Turbulent enstrophy; see (2.1b)
θ_0	Potential temperature
η	Dimensionless variable; see (2.10)

λ_m	Mean eddy scale; see (2.11)
ν_0	Molecular viscosity of air
ν_t	Turbulent or effective viscosity; see (2.6)
σ_ε	Turbulent Prandtl number for ε
σ_θ	Turbulent Prandtl number for θ
ρ	Air density
ω_i	Vorticity vector in tensor notation
Ω_{ij}	Mean-rotation-rate tensor

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