The Tropical Cyclone as a Divergent Source in a Background Flow

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ABSTRACT: The interactions between the outflow of a tropical cyclone (TC) and its background flow are explored using a hierarchy of models of varying complexity. Previous studies have established that, for a select class of TCs that undergo rapid intensification in moderate values of vertical wind shear, the upper-level outflow of the TC can block and reroute the environmental winds, thus reducing the shear and permitting the TC to align and subsequently to intensify. We identify in satellite imagery and reanalysis datasets the presence of tilt nutations and evidence of upwind blocking by the divergent environmental winds, which are critical components of atypical rapid intensification. We then demonstrate how an analytical expression and a shallow water model can be used to explain some of the structure of upper-level outflow. The analytical expression shows that the dynamic high inside the outflow front is a superposition of two pressure anomalies caused by the outflow’s deceleration by the environment and by the environment’s deceleration by the outflow. The shallow water model illustrates that the blocking is almost entirely dependent upon the divergent component of the wind. Then, using a divergent kinetic energy budget analysis, we demonstrate that, in a full-physics TC, upper-level divergent flow generation occurs in two phases: pressure driven and then momentum driven. The change happens when the tilt precession reaches left of shear. When this change occurs, the outflow blocking extends upshear. We discuss these results with regard to prior severe weather studies.

KEYWORDS: Shallow-water equations; Wind shear; Tropical cyclones

1. Introduction

In a series of recent papers (Ryglicki et al. 2018a,b, 2019, hereinafter Part I, Part II, and Part III), a class of tropical cyclones (TCs) that undergo rapid intensification [RI; increase of 30 kt (1 kt $\approx$ 0.51 m s$^{-1}$) in 24 h of the 10-m winds] in seemingly hostile vertical wind shear conditions, henceforth called “atypical RI,” was described. Specifically, climatological studies (Kaplan et al. 2010; Rozoff et al. 2015) indicate that for their samples, the average vertical wind shear for TCs that undergo RI is approximately 4–5 m s$^{-1}$, depending on the basin, according to the Statistical Hurricane Intensity Prediction System (SHIPS; DeMaria et al. 2005) and its RI offshoot, SHIPS-RII (rapid intensity index: Kaplan et al. 2010). The aforementioned unexpected RI behavior occurs when the average vertical wind shear magnitude is 7.5 m s$^{-1}$, a value that is 1–2 standard deviations greater than climatologically favorable values. While shear is generally thought to be an overwhelming negative for TC intensification for a host of reasons (Riemer and Laliberté 2015, and references therein), atypical RI requires the presence of moderate shear for it to occur, as the shear excites a tilt-induced instability which then modulates convection.

During atypical RI, convection is modulated by nutations of the tilt of the vortex (Part II). A nutation is a smaller-scale, higher-frequency wobble on the larger and slower precession, which can be defined as a change in the orientation of the rotational axis of a rotating body, of the vortex tilt (Jones 1995, 2000; Schecter et al. 2002; Reasor et al. 2004; Schecter 2015; Reasor and Montgomery 2015). These nutations produce features in satellite imagery identified as tilt-modulated convective asymmetries (TCA). TCAs are distinct cloud structures in satellite imagery identified as tilt-modulated convective asymmetries (TCA). TCAs are distinct cloud structures that are separate from the diurnal signal (Kossin 2002), pulse with a periodicity of 4–8 h on the left-of-shear side of the TC core, and do not rotate around the core. Spatially, they are responsible for 10 000–15 000-km$^2$ changes in cloud coverage colder than $-70^\circ$C. They appear when the nutation and precession magnitudes are approximately equivalent: in Part II, this was 15 km. One of the critical effects of the TCAs is that their outflow blocks the environmental flow, rerouting the winds around the TC and reducing the shear over the center of the storm (Part III). This is then thought to permit the TC to realign itself vertically, ultimately leading to RI. The caveat is that the environmental flow must be limited to a 100-hPa layer under the tropopause for the blocking to be effective (Part III; Finocchio et al. 2016).

The concept of convection’s blocking of the environmental winds is not a completely new idea, although previous works have focused primarily on the convective scale. Klemp and Wilhelmson (1978), in their simulations of storm splitting,
noted that an individual updraft acts as an object in the flow, thus slowing down the flow upwind of the convective cell and causing sinking. They noted (Klemp and Wilhelmson 1978, p. 1105), “The weak downdraft to the west of the updraft $z = 2.25 \text{ km}$ ([their] Fig.12) is possibly due to blocking westerly flow by the storm forcing the approaching air to descend rather than flow around the storm.” Fritsch and Maddox (1981), in their analyses of upper-level observations of mesoscale convective complexes (MCC), noted the presence of mesohighs above the MCCs. While their analyses focused on the observations, they did hypothesize that, much like Klemp and Wilhelmson (1978), the updrafts act as obstacles to the flow (Fritsch and Maddox 1981, p. 17). Since the environmental flow decelerates due to these obstacles, Fritsch and Maddox suggested that the pressure might increase as per Bernoulli’s principle (Euler 1752; Darrigol and Frisch 2008):

$$\frac{1}{2} v^2 + gz + \frac{p}{\rho} = C,$$  

(1)

where $\rho$ is the density, $v$ is the wind speed, $g$ is gravity, $z$ is the physical height, $p$ is the pressure, and $C$ is a constant. Equation (1) is an expression of conservation of energy among kinetic, potential, and internal components following a streamline. If one ignores the geopotential term, which is the second term on the left-hand side, a decrease in the speed of the flow along a streamline must result in an increase in pressure. We will use this concept to explain some of the structures that develop at upper levels.

The subtle difference between the findings of Part III and the implications of Klemp and Wilhemson (1978) and Fritsch and Maddox (1981) is that Part III argues that it is the outflow that is causing the block and not the convective towers themselves. Elsberry and Jeffries (1996) were the first to indicate that TC outflow may act to block the environmental flow, but this idea had also been broached at the convective scale. Schmidt and Cotton (1990), Mapes (1993), and Trier and Sharman (2009) have all noted parts of the outflow blocking effect at the convective scale. The findings of Part III upscaled the conceptual model of outflow blocking from the convective scale, $O(10 \text{ km})$, to that of the TC scale, $O(100 \text{ km})$. The weak downdraft upwind noted by Klemp and Wilhelmson (1978) was also noted in Part III, except at much higher altitude in the troposphere and over a much broader horizontal area: from 8 km to the tropopause at 14 km, and from the outflow front (where the radial winds change direction aloft, usually 200 km upwind) to the bow wave (approximately 1000 km upwind). In addition, as a result of the blocking, Part III illustrated a localized high pressure region on the upwind side of the outflow. This dynamic high, also called a “Bernoulli high” in Part III, was argued to be a result of the dissipation of kinetic energy from both the environment and the outflow and could also be explained using (1). The effects of the high pressure then extend 1000 km upwind, which is approximately the Rossby deformation radius (Part III), thus slowing down the oncoming flow. Part III demonstrated that during the blocking process by TC outflow, the upper-level pressure field displays two lobes: a large high downshear that is the typical upper-level anticyclone associated with TCs adverted away by the environmental wind (Wu and Emanuel 1993) and the spatially smaller Bernoulli high 100–200 km upshear of the core.

The overarching goal of the original series was to document the markers of atypical RI in satellite imagery and reanalyses (Part I) and then to illustrate the basic physical underpinnings using an idealized model (Part II; Part III) while also being cognizant of the operational applicability of those findings. As a result, some of the finer mathematical details were overlooked for the sake of conveying the broader physical features. The goal of this particular study is then to explore some of the findings in more detail: specifically, the relationship of convection with the outflow and further details of the outflow itself. The long-standing assumption in vortex tilt analyses is that differential advection in the vertical by the background flow tilts the vortex, and an intrinsic resilience mechanism (i.e., the precession) prevents the vortex from being sheared apart (Jones 1995; Reasor et al. 2004). Simply put, the induced flows from the lower layers of the tilted TC serve to advect the upper levels back upwind, and the ability of the TC to perform this maneuver is related to its ability to redistribute angular momentum (Schecter et al. 2002; Reasor et al. 2004; Schecter 2015; Reasor and Montgomery 2015). Part III identified a second “defense mechanism” that TCs possess to maintain their coherence in shear: the outflow. Perhaps the most important detail from the analyses in Part III is that the blocking is associated with the divergent component of the wind. This finding allows us to focus our analyses in this paper on the divergent component of the flow and to explore some of the kinematics associated with atypical RI more carefully using a hierarchy of analyses: satellite observations, reanalyses, an analytical representation, a shallow water model, and a full-physics primitive equation model.

The paper is laid out as follows. Section 2 describes the data and method, including descriptions of the models used. Section 3 provides a review of the features of atypical RI, revisiting manifestations of its behavior in satellite observations, reanalyses, and a full-physics model. Section 4 provides deeper mathematical analyses of the outflow structures presented in section 3 using the model hierarchy. A summary and a discussion of the implications of our findings are presented in section 5.

2. Data and method

a. Observation, reanalysis, and full-physics model data

Infrared (IR; 10.7 μm) satellite observations from geostationary satellites are used. From the Geostationary Operational Environmental Satellite (GOES) line of satellites: GOES-13 and GOES-15; from the Himawari line: Himawari-8. The available temporal frequency of the scans is 30 min except for 2015 Joaquin, since we have access to rapid scan data and because it existed within the continental U.S. satellite scan area. Joaquin’s observations for its time period of interest range from 5 to 15 min. To augment the satellite data, we use the ERA5 (Hersbach et al. 2018) dataset whose horizontal grid spacing is 0.25° and is available every six hours. The wind fields of ERA5
are decomposed into divergent and rotational components using SPHEREPACK (Adams and Swarztrauber 1997).

Data from the idealized modeling simulations discussed in Part II and Part III are also used. Those data are from a slightly modified Cloud Model 1 (CM1; Bryan 2012), release 18. Full details of the simulations’ parameters can be found in Part II. A significant addition is the thermally balanced cloud and the TC itself, we apply the shallow water approximation order of magnitude shallower than the depth of the troposphere of homogeneous fluid on a plane with a meridional extent of 600 km, an initial RMW of 100 km, an initial surface maximum mean tangential wind of 20 m s\(^{-1}\), and an initial maximum height of 10 km. The thermodynamic base state is the “moist tropical” sounding (Dunton 2011). The tropopause \(N^2 = 2 \times 10^{-4} \text{s}^{-2}\) is at 14 km. Sea surface temperatures (SSTs) are horizontally homogenous and invariant in time with an average of 27.5°C (Part I). An \(f\) plane at 20°N is used. Three background wind profiles are used: a Gaussian profile with a 13–15-km (proxy for 200–850 hPa) shear value of 7.5 m s\(^{-1}\) (hence, G7.5), a cosine profile with a shear value also of 7.5 m s\(^{-1}\) (C7.5), and an additional Gaussian profile with a shear value of 11.5 m s\(^{-1}\) (G11.5). The shear is easterly and is initialized at the start of the simulation. The background environmental winds for the control are zero everywhere. For this paper, we focus on the G7.5 simulation and the control.

Multiple aspects of the simulations have already been discussed in Part II and in Part III, and we refer the reader to those articles for more details; however, we will provide a brief summary. The G7.5 and G11.5 TCs do not develop, the control TC develops, and the G7.5 TC develops. The G7.5 TC goes through two distinct intensification stages on its way to a Saffir–Simpson category-3-equivalent TC, as opposed to the control which continuously intensifies. The G7.5 TC exhibits tilt nutations of 9–10 h that exist on top of the larger, slower tilt precession. The magnitude of the nutations is approximately 15 km when analyzed using a center at 6-km height. After the initial 24 h during which the tilt grows, realignment occurs at 80 h into the G7.5 simulation, at which point the precession and the nutations both damp. Because of the outflow, vertical wind shear, calculated as the difference between winds at 13 and 1.5 km in a 200-km circle around the low-level center, reduces from 7.5 to 3.5 m s\(^{-1}\). An abrupt rapid intensification follows at 90 h. The explicit intensity traces are shown in section 4.3.

### Divergent shallow water model

Since the outflow of a TC is generally confined to a narrow 2-km-deep layer (Part III; Merrill and Velden 1996), which is an order of magnitude shallower than the depth of the troposphere and the TC itself, we apply the shallow water approximation (Pedlosky 1987) to reduce the dimensionality of the evolution, greatly simplifying it. We thus consider a shallow layer of homogeneous fluid on an \(f\) plane with a meridionally sloping bottom boundary that we describe using the standard nondimensional shallow water (DSW) equations (e.g., Cho and Polvani 1996); namely,

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\text{Ro}} \frac{\partial h}{\partial x} + d_u \nabla^2 u + S_u, \tag{2}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{\text{Ro}} \frac{\partial h}{\partial y} + d_v \nabla^2 v + S_v, \tag{3}
\]

\[
\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = d_s \nabla^2 h + S_h, \tag{4}
\]

where \(u, v, h\) and \(h\) are the zonal wind, meridional wind, and height of the fluid, respectively. Primes indicate flow that is strictly associated with the vortex. The nondimensionalization has been carried out as

\[
(x, y, h, h) \rightarrow (Lx, Ly, H, h) \quad \text{and} \quad (u, v) \rightarrow (Vu, Vv),
\]

where we have set \(L = 10^2\ m, H = 2000\ m,\) and \(V = 10\ m\ s^{-1}\). The Rossby and Froude numbers respectively become

\[
\text{Ro} = \frac{V}{fL} \approx 0.1 \quad \text{and} \quad \text{Fr} = \frac{V}{\sqrt{gH}} \approx 0.1,
\]

and for which we have centered the \(f\) plane at 20°N latitude (as in the CM1 simulation); g is gravity. The sloping bottom boundary implies a thermal wind relationship that allows the wind and height to be decomposed as

\[
h(x, y, t) = 1 - \frac{\text{Fr}^2}{\text{Ro}^2} U_0 y + h'(x, y, t) \quad \text{and} \quad u(x, y, t) = U_0 + u'(x, y, t).
\]

\(U_0\) can be set, for example, to \(-0.5\) to mimic weak easterly winds. Equations (2)–(4) include a simple eddy viscosity parameterization in the form of diffusion. This diffusion is implemented implicitly for stability, and the nondimensional parameters are set as

\[
d_u = d_v = d_s = \frac{k_v}{V^2 L},
\]

where \(k_v\) is set to \(5.5 \times 10^4\ \text{m}^2\ \text{s}^{-1}\).

The sink terms \(S\) in (2)–(4) are calculated to ensure that the specified vortex wind and height fields are exact solutions to the shallow water equations. This is critical to guarantee that any deviations we see from these fields is entirely a result of the background winds impinging on the vortex. We accomplish this task by simply brute-force numerically solving these equations for the sink terms after specifying the winds and the height of the vortex. It is important to note that we solve for the sink terms for the case in which \(U_0 = 0\) so as to obtain the exact forcing needed to render the vortex an exact solution of the equation in the absence of the background winds.

The model is solved in time with the third-order Adams–Bashforth method. The spatial domain is solved spectrally using Fourier methods with the nonlinear terms evaluated using the nonlinear transform method of Orszag (1970). Because of the
inherent periodicity of the spectral Fourier method, we employ a sponge boundary along all sidewalls to prevent reflections around the edge of the domain.

3. Review of atypical rapid intensification features

In this section, we will briefly summarize some of the key components of the original series by applying a new technique to observations and reanalysis (Part I). We will also revisit the most salient findings from the full-physics modeling results (Part II; Part III) to motivate the analyses presented in section 4: analytical representations, shallow water model results, and finally a divergent kinetic energy budget analysis of the full-physics outflow.

a. Observations and reanalysis review

Two fundamental findings from the original series were 1) the upshear movement and wobbling of the tilt as deduced from the cloud shield and as modeled explicitly in the CM1 and 2) the presence of a proximate upper-level anticyclone as the source of the shear. Four TCs are chosen to demonstrate these two aspects: 2015 northern Atlantic (NATL) Joaquin, 2016 NATL Matthew, 2017 northeastern Pacific (EPAC) Kenneth, and 2017 southern Indian (SIO) Ernie. While Joaquin (Part I; Part II; Part III) and Matthew (Part III) have been discussed previously, Kenneth and Ernie are shown to illustrate more examples of this process. Ernie also serves the purpose of demonstrating this phenomenon in a new basin (Courtney et al. 2019). Figure 1 illustrates the intensities and three shear calculations—two from SHIPS and one from the Cooperative Institute for Meteorological Satellite Studies (CIMSS)—for the four chosen TCs. All four of them endured sustained moderate shear (>5 m s⁻¹; Reasor et al. 2013; Rios-Berrios and Torn 2017) prior to their respective RI episodes.

If we assume that the coldest IR brightness temperatures are associated with the strongest convection that is in turn also associated with the tilt of the vortex (Part II), we can track the migration of the cloud shield upshear as a proxy for the movement of the tilt. We use a brightness temperature centroid (Ryglicki and Hart 2015) computed thusly:

\[
\langle x, y \rangle = \left( \frac{\int (T_B - T_B) r^2 \cos \lambda \, dr \, d\lambda}{\int (T_B - T_B) r \, dr \, d\lambda}, \frac{\int (T_B - T_B) r^2 \sin \lambda \, dr \, d\lambda}{\int (T_B - T_B) r \, dr \, d\lambda} \right),
\]

(9)
where $T_B$ is the IR brightness temperature in kelvin and the overbar indicates the average $T_B$ over a 500-km circle surrounding the interpolated best-track position from HURDAT (Landsea and Franklin 2013) as per the methods from Part I. The centroid weighting area is a 150-km circle. The first guess at the initial time is the location of the coldest brightness temperature in the domain; all subsequent first guesses are the location of the centroid from the previous time. Figure 2 demonstrates that for each case, the centroid tracks the upshear migration of the clouds from 200 km away to left (or right) of the shear vector prior to the appearance of the eye. In the instances of Matthew (Fig. 2b), Kenneth (Fig. 2c) and Ernie (Fig. 2d), one can clearly identify manifestations of the nutations as the centroid nears the TC center in the left-of-shear or right-of-shear position, depending on the hemisphere.

A fundamental finding of Part I was that the TCs in this class are sheared by anticyclones. Usually, but not always, this implies that the strongest environmental winds are confined to near the tropopause. This is the same layer as the outflow of the TC. This permits outflow blocking and shear reduction (Part III) that is then thought to permit the upshear movement of the tilt and subsequent realignment. Setting aside the expectation of capturing tilt nutations in reanalysis data, it is worth investigating how global reanalyses represent at least the outflow blocking portion of this process.

\[ \text{FIG. 2. Storm-relative tracks of the IR brightness temperature centroid, colored by day for each TC: (a) Joaquin, (b) Matthew, (c) Kenneth, and (d) Ernie. Angles are math convention: 0° is east, 90° is north, etc. Radial axes are distance in kilometers. Teal arrows are time-averaged shear direction from SHRD [defined as 850–200-hPa shear magnitude vs time (200–800 km)], SHDC (defined as being the same as SHRD but with vortex removed and averaged from 0 to 500 km relative to 850-hPa vortex center), and CIMSS methods for the time periods of the panels.} \]
Figure 3 is an illustration of the divergent and rotational winds at 200 hPa of the four TCs prior to their respective RIs. Per the rotational winds, upper-level anticyclones shear all four TCs. Generally, each TC exhibits divergent wind magnitudes upshear of at least 7 m s\(^{-1}\) in coherent arc shapes, and the rotational winds are rerouted around the TCs to varying degrees.

b. Full-physics model review

As a summary of Part III, we present Fig. 4 which is a collection of three important features described in Part III from the G7.5 simulation. Figure 4a highlights the tilt-relative upper-level wind structure around the core of the TC during a TCA. In a tilt-relative frame, the stagnation point (where the winds reduce to zero) is located northeast (upshear right) of the core at 200 km radius. The peak winds exiting the TC are between 25 and 30 m s\(^{-1}\). This has the consequence of creating a localized high pressure (Fig. 4b), the magnitude of which, approximately 45 Pa, is the same as the hydrostatic high pressure that is usually associated with the upper levels of TCs but is advected downshear in sheared cases (Wu and Emanuel 1993).

There is also a second small lobe of low pressure located south-southeast (left of shear) of the TC on the downshear side of the block.

Figure 4c presents a time series of three important quantities: shear magnitude, where shear is defined as 13-km winds minus 1.5-km winds in a 200-km circle surrounding the low-level center; the upshear radial wind minimum location at 13-km height; and the difference between volume-integrated upshear and downshear divergent kinetic energy. For the difference, positive indicates upshear-oriented DKE is greater than downshear-oriented DKE and vice versa, and the bounds of integration are between 9 and 16 km and over the Cartesian domain. While the relationship among the three has been discussed at length in Part III, we instead focus on the two apparent regime shifts. For the first 48 h of the simulation, the outflow front is largely confined to within 150 km and the shear fluctuates between 5 and 8 m s\(^{-1}\). At 48 h, the divergent kinetic energy difference shifts upshear, the outflow front expands to 250 km, and the shear drops to 3.5 m s\(^{-1}\). Later in the simulation, at 114 h, the shear returns to 6 m s\(^{-1}\) and intensification subsequently ceases. We will discuss this further in section 4d.
4. Hierarchical analyses

a. Helmholtz decomposition overview

In this section, we use a hierarchy of models to explore the features presented in section 3. The sequence is a simple 2D analytical expression, the nonlinear divergent shallow water simulations, and then finally the nonlinear full-physics simulation. Before we explore the analytical solutions, we will start with a basic analysis of the structure of the upper levels of the TC from the full-physics simulation by performing a Helmholtz decomposition of the flow to describe its basic characteristics using rotational streamfunction $\psi$ and divergent velocity potential $\chi$. Since the background flow is by definition initialized as neither irrotational nor nondivergent, it is completely filtered out by the Helmholtz decomposition.

Given the relative lack of idealized-TC-specific $\psi$ and $\chi$ fields in the literature, we begin by presenting snapshots of the structures of the two fields in the G7.5 simulation and in the control at 13-km height at 85 h (Fig. 5). As a reminder, positive (negative) $\psi$ is associated with anticyclonic (cycloonic) motion whereas positive (negative) $\chi$ is associated with convergence (divergence). The G7.5 $\psi$ field (Fig. 5a) illustrates the downwind advection of the upper-level anticyclone, as previously demonstrated by Wu and Emanuel (1993, 1994). Two large, weak cyclonic gyres are created—one upshear right and one downshear. The control $\psi$ field (Fig. 5c) is circular, as is expected for the upper levels of a TC in no background flow (Rappin et al. 2011). The $\chi$ fields for both simulations are similar in that they both expand radially from a small local area (Figs. 5b,d). In the G7.5 case, there is also weak convergence downshear. Figure 6 illustrates how the maximum $\psi$ and minimum $\chi$ values change in time at 13.4-km height. This demonstrates that even though the shapes of the $\psi$ field are noticeably different, the largest magnitude of the streamfunction is basically the same. For the $\chi$ field, the G7.5 TC is consistently more negative.

b. Analytical solutions

The simplest representation of the upper levels of the TC is an analytical representation of a point source collocated with a point vortex in two dimensions (Lamb 1932, 62–109). As noted in appendix A, one can prescribe a point source for both the anticyclonic rotational and divergent winds at the top of the TC in a background flow:

$$ u(x, y) = U_0 + \chi_m \frac{x}{x^2 + y^2} - \psi_m \frac{y}{x^2 + y^2} $$

and (10)
where \( x \) and \( y \) are coordinate locations from the origin and \( x_m \) and \( c_m \) are magnitudes of the divergent and rotational sources, respectively. Effectively, the decay of the winds is \( r^{-1} \). Figure 7 illustrates the simple examples here using 5 for \( x_m \) and \(-5\) for \( U_0 \) and \( c_m \). (Note the signs). It can be shown that, for this series of equations, the stagnation point will be at the coordinate \((-x_m/U_0, c_m/U_0)\). In our example that uses both wind components (Fig. 7a), the stagnation point is at \((1,1)\). Comparing Fig. 7a with the full-physics results of Fig. 4a, the analytical solution reasonably represents the nonlinear numerical solution.

The analytical expressions also permit direct solution of the pressure perturbations using Bernoulli’s equation (appendix A). Given the wind field, one can solve for the final pressure field directly. Stated another way, we can quantify the conversion from kinetic to internal energy. Conceptually, we will treat the streamlines as two independent sources: one set originating far from the vortex; the other originating from the vortex itself. Using Bernoulli’s equation, we can solve for the pressure perturbations associated with each source:

\[
\nabla(x, y) = \frac{\chi_m y}{x^2 + y^2} + \frac{\phi_m x}{x^2 + y^2},
\]

(11)

where \( \nabla \) is the pressure perturbation from either the background [(12a)] or from the vortex [(12b)], \( U_0 \) is the background wind, \( \psi \) is the vortex winds, and \( \chi \) is the total flow after the two winds interact as seen in Fig. 3. We set the density to \( 0.1 \text{ kg m}^{-3} \).

In Figs. 7d–f, the maximum pressure perturbation is located between the stagnation point and the point source, while a lower pressure exists downwind. The individual pressure anomalies for the case with rotational and divergent vortex winds (Figs. 7a,d) are shown in Fig. 8. When the two pressure

\[
\rho_p = \frac{\rho}{2} (\psi^2 - \chi^2),
\]

(12a)
components of Fig. 8 are added together, the resultant pressure field is Fig. 7d. The high pressure is located between the vortex and the outflow front, mimicking the results of the CM1 numerical solution (Fig. 4b). This also explains the “drafting” downwind of the block and the localized lower pressure found in the CM1 simulation south of the low-level center (Fig. 4b), as winds accelerate downwind of the block and lower the pressure. Fundamentally, the total pressure perturbation is the linear superposition of the two pressure anomalies from (12).

c. Divergent shallow water results

Increasing complexity in the model hierarchy brings us to the DSW simulations. We base the initial conditions of the DSW model on the time-averaged \( \psi \) and \( \chi \) fields from the

![Fig. 7](image1)

**FIG. 7.** (a)–(c) Winds (m s\(^{-1}\)) and (d)–(f) pressure fields (Pa) of the analytical equations using different \( \chi_m \) and \( \psi_m \) values. Streamlines are total winds. The green X in all plots is the location of the stagnation point, i.e., where the winds reduce to zero.

![Fig. 8](image2)

**FIG. 8.** Pressure perturbations (Pa) associated with (a) the background and (b) the vortex. Black lines are total winds. The green X is the stagnation point. A combination of these two pressure fields yields Fig. 7d.
control simulation at 12.6-km height (Fig. 9). These cross sections are essentially axially symmetric and do not show the marked asymmetry of the shear simulation. While we realize that the asymmetric structure of the $\chi$ field is essential for results in the real world and in the full-physics simulation, our demonstrations do not require this asymmetry. Initial conditions (appendix B) are thus

$$\chi(r) = x_m e^{-r^2/(2a)^2}$$ \hspace{1cm} (13)

$$\psi(r) = \frac{\psi_m (ar)^{k-1} e^{-a r^2}}{\Gamma(k) \theta^r}.$$ \hspace{1cm} (14)

Subsequently, we modify the inner cyclonic vortex region of the $\psi$ field:

$$\psi(r) = \psi(r) + \beta [\psi_m - \psi(r)]; r < RM \psi;$$ \hspace{1cm} (15)

where $\psi_m$ and $x_m$ control the peaks of the profiles, $\sigma_\chi$ controls the radial decay of the $\chi$ field, $\beta$ controls the strength of the inner vortex, and $RM \psi$ is the radius of maximum $\psi$. Further explanations of these expressions can be found in appendix B. The initial height field $h'$ is

$$h' = \frac{Fr^2}{U_0} \psi.$$ \hspace{1cm} (16)

Figure 10 illustrates that our initial conditions are reasonable when compared with the results from the CM1 control simulation (Fig. 9). All fields are initialized at the origin.

Figure 11 shows the results of nine simulations of the DSW model, with details found in Table 1. All of these simulations are initialized with $U_0 = -0.5$, which is also part of the initial flow state. Our basic approaches are to modify the inner cyclonic vortex (Figs. 11a–c), initialize with only the $\psi$ field (Figs. 11d–f), and initialize with only the $\chi$ field (Figs. 11g–i). The most important results are those of Figs. 11d–f without the divergent winds. The initialized $\psi$ field is advected downstream (not shown), and no traces of the vortex remain at the origin. Conversely, in the $\chi$-only simulations, the divergent field maintains coherence. This fully proves that the $\chi$ field and the divergent component of the outflow are primarily responsible for the outflow blocking phenomenon. In addition, in the DSW simulations that incorporate both $\psi$ and $\chi$, the strongest perturbation winds are “upshear” right, indicating that to match the complete structure of the full-physics simulation (Fig. 4a), both are required, but without the divergent component, no block occurs.

To continue to explore the evolution of the outflow front (the location at which the winds reduce to zero), we simply focus on the divergent component of the flow. Figure 12 highlights the location of the front as $x_m$ increases holding the background wind $U_0$ fixed (Fig. 12a) and its location as $U_0$ increases while holding $x_m$ fixed (Fig. 12b). The results are plotted as the ratio of the $x_m$ to $U_0$, both nondimensionalized, in order to relate the findings to the analytical solutions. For Fig. 12a, $x_m$ is increased by integer multiples over the initial value taken from the CM1, $-1.25$. For Fig. 12b, we take the value of $x_m$ of $-6.25$ ($x_m/U_0$ ratio of 12.5) and increase the background wind magnitude incrementally by 2.5 m s$^{-1}$ (0.25 nondimensional wind units), spanning from $-5.0$ to $-20.0$ m s$^{-1}$ (from $-0.5$ to $-2$ nondimensional wind units). As $x_m$ increases linearly and the ratio increases, the outflow front distance does not increase linearly, indicating diminishing returns with regard to the increase of $\chi$. If we apply the nondimensional scaling to the CM1 fields, the resulting ratio is approximately $-3.5/-0.75$, assuming Galilean invariance of the 13-km winds, or 4.66.

According to Fig. 12, this should result in a dimensionalized outflow front location near 260 km. According to Fig. 4c, the average location of the outflow front between 48 and 90 h is approximately 250 km.

There are two important caveats of the interpretation of these results. In addition to the magnitude, these results are also a function of the radial extinction as expressed by (13). As was shown in Fig. 11i, holding $x_m$ and $U_0$ fixed but increasing $\sigma_\chi$ does not result in an outflow front because the outflow is too weak as divergent winds are derived from the gradient of $\chi$. The radial decay characteristics of the $\chi$ fields, such as $r^{-1}$ vice $e^{-r}$, will control the location of the stagnation point upwind. In addition, in the G7.5 simulation, the $\chi$ field is asymmetric. We noted that prior to the second abrupt RI, the DKE is skewed upshear during the nutations before shifting downshear (Fig. 4c). The downshear shift reduces the blocking effect. While obviously important, exploring the entirety of the upper level wind phase space is beyond the scope of this paper.

**d. Full-physics divergent kinetic energy budget**

Having shown that it is indeed the divergent wind that is the most critical part of this evolution, we now turn to exploring the evolution of the divergent wind in the full-physics simulation explicitly. The metric used in Part III to quantify the outflow was the divergent kinetic energy. In this study, we expand those analyses to include the rotational and cross-term kinetic energies to quantify the total kinetic energy (see appendix C):

$$k = \frac{\mathbf{V} \cdot \mathbf{V}}{2} = \frac{\mathbf{V}_\psi \cdot \mathbf{V}_\psi}{2} + \frac{\mathbf{V}_\chi \cdot \mathbf{V}_\chi}{2} + \mathbf{V}_\psi \cdot \mathbf{V}_\chi.$$ \hspace{1cm} (17)

where $k$ is the total kinetic energy (KE), $\mathbf{V}$ comprises the components of the wind, and the subscripts indicate either rotational or divergent winds. The total kinetic energy can be divided into rotational kinetic energy (RKE), divergent kinetic energy (DKE), and the cross term (CrossKE). CrossKE can be
simply thought of as the interaction between the rotational and divergent winds. The reason that we have chosen energy over momentum is that it converts the divergent wind from a vector to a positive-definite scalar quantity. Given the highly tilted structure of the G7.5 TC for the first half of the model simulation (see Part II), this greatly simplifies the interpretation without having to account for flow direction or tilt location explicitly.

One of the critical elements to the evolution at upper levels is the orientation of the terms relative to the flow. Figure 13 shows the spatial characteristics of these expressions at 13.4 km for the control (Figs. 13a–c), the G7.5 TC during a TCA (Figs. 13d–f), and the G7.5 TC later in the simulation after its intensification has ceased (Figs. 13g–i). The comparison of the three columns highlights the asymmetric nature of the upper-level structure in the G7.5 TC. In addition, Figs. 13f and 13i illustrate the important role of CrossKE on the upshear side of the TC, as it alters the total KE asymmetrically. For this reason, we include its time tendency explicitly in our budget calculation:

\[
\frac{\partial k}{\partial t} + \frac{\partial (V_{\phi} \cdot V_{x})}{\partial t} = -c_\rho \theta \left( V_{x} \cdot \nabla \Pi \right) + (\zeta + f)(u_{x} V_{\phi} - v_{x} u_{\phi}) + V_{\phi} \cdot \left( \frac{\partial}{\partial t} - c_{\text{enu}} \right) V_{x} - V_{x} \cdot \nabla k_{\phi} - V_{\phi} \cdot \nabla k_{\phi} - V_{x} \cdot \nabla (V_{\phi} \cdot V_{x}) - w \left( V_{x} \cdot \frac{\partial V_{\phi}}{\partial z} \right) - w \frac{\partial k}{\partial z} + R. 
\]

(18)

The derivation of (18) can be found in appendix C. From left to right, the terms on the right-hand side are pressure gradient or baroclinic generation (pgrad); divergent-rotational wind conversion (conv); rotational flux across local changes in divergent winds (chitt); meridional Coriolis (mercor); divergent fluxes of divergent kinetic energy (hadvchi), rotational kinetic energy (hadvpsi), and their cross term (hadvcross); divergent flux of the vertical flux of rotational winds (vadvpsi); and vertical advection of divergent kinetic energy (vadvkechi). We have grouped all sub-grid-scale effects into the residual $R$ (Res). All terms of (18) are mass weighted during integration. For all
following line plots, we smooth the data with a 2-h (nine point) running mean to aid in visualization and interpretation. An a posteriori analysis indicates that the three largest terms on the right-hand side ultimately become the divergent flux of divergent kinetic energy, the pressure gradient term, and the conversion term, but, as we shall show, there are important details in this evolution reflecting the transition shown in Fig. 4c at 48 h.

We first illustrate the evolution of the three energy components for the two simulations in Fig. 14. The integration constraints are different from Part III: they are limited to 10–14 km in the vertical direction and a 500-km circle around the low-level center. The DKE in the G7.5 simulation is approximately a factor-of-2 larger than the DKE in the control simulation. The RKE in the control simulation is...
larger, but Fig. 5 illustrated that the $\psi$ field of the Gaussian-7.5 simulation is advected downshear. Accordingly, the energy associated with that will also be advected downshear. CrossKE is comparatively smaller in the G7.5 simulation than the other two components, but it is practically zero in the control simulation.

For our analysis of (18), we separate the time series into two overlapping time periods for each simulation: 6–60 h (Fig. 15), and then from 48 h to the end of the simulation at 144 h (Fig. 16). There is no significant DKE prior to 6 h. To aid in visualization, we have incorporated the three smallest terms into the residual: meridional Coriolis, rotational flux across local changes in divergent winds, and divergent flux of the vertical flux of rotational winds. The easiest way to interpret this evolution is by focusing on the G7.5 TC (Fig. 15b) from 24 to 30 h. The two largest positive terms are the pressure gradient and the horizontal divergent flux of RKE, while the largest negative term (aside from the residual) is the vertical advection of DKE. We can thus greatly simplify (18) for this time period:

$$\frac{\partial k}{\partial t} + \frac{\partial \left( \mathbf{V} \cdot \mathbf{V} \right)}{\partial t} = -c_p \rho (\mathbf{V} \cdot \nabla) - \nabla \cdot \mathbf{k} - w \frac{\partial k}{\partial z} + R. \tag{19}$$

A change occurs at 42 h. From this point until the end of the simulation (Fig. 16b), the horizontal divergent flux of DKE becomes the largest positive term as it, along with the conversion term, largely competes with the negative effects of the pressure gradient term. We can simplify (18) again for the second period:

$$\frac{\partial k}{\partial t} + \frac{\partial \left( \mathbf{V} \cdot \mathbf{V} \right)}{\partial t} = -c_p \rho (\mathbf{V} \cdot \nabla) + (f + \sigma_z) w \mathbf{V} \cdot \nabla k + R. \tag{20}$$

In recalling Fig. 4c, note this change occurs at the same time as the DKE shifts upshear, the outflow front gets pushed out to 250 km, and the shear drops to 3.5 m s\(^{-1}\). A similar shift happens in the control simulation (Figs. 15a and 16a), albeit at smaller magnitudes. Simply stated, the DKE generation in the outflow can be divided into two phases: a pressure-driven phase and then a subsequent divergent-flux-driven phase. The second phase is largely what is responsible for the outflow’s pushing the environmental winds back. We also note that this change occurs after the precession of the tilt has reached left of shear (Part II).

From a spatial perspective, Figs. 17 and 18 illustrate these changes relative to convection by analyzing the vertically integrated total condensed water of the column (TCWC), DKE, CrossKE, the pressure gradient term, the conversion term, and all of the horizontal flux terms grouped together at 28 h and then again at 64 h 45 min. From Fig. 17a, it is clearly seen that the convection is displaced downshear, while the pressure gradient generation is collocated with the convective maxima (Fig. 17d) with modest upwind advection (Fig. 17f). The CrossKE and conversion terms are small. At 64 h 45 min, during a TCA, Fig. 18a shows how convection is now left of shear. The DKE and positive CrossKE (Figs. 18b,c) are now in phase. While the pressure gradient is still locally positive over convection, it is weakly negative everywhere else (Fig. 18d) while the conversion term is negative over convection (18e). The flux terms are largely focused upshear and are predominantly positive (Fig. 18f), and this corroborates the time evolution seen in Fig. 16b.

Since the whole of this evolution is strongly tied to convection, we believe it would be worthwhile to investigate the evolution of convection and updraft distribution. Figure 19 is a collection of log-scale contour-frequency-by-altitude diagrams (CFADs; Yuter and Houze 1995) during a tilt nutation in the
G7.5 simulation (Fig. 19a), when the tilt temporarily decreases to near zero in G7.5 (Fig. 19b), and a comparable time in the control (Fig. 19c). The CFADs indicate that there is substantially more coverage of updrafts above 20 m s$^{-1}$ when a TCA is present. If we use 20 m s$^{-1}$ as a threshold, Fig. 19d illustrates the evolution in time of the strongest updrafts. There are two clusters of strong updrafts in the G7.5 simulation prior to 42 h, and both of those can be related to increases in DKE via the pressure gradient generation term. The time period from 48 to 96 h indicates consistently strong updrafts, and this results in an increase of DKE aloft through upshear horizontal fluxes. Interestingly, after 100 h, the strong updrafts all but cease. This also corresponds to a leveling off of intensification and an increase of the shear from 3.5 to 6 m s$^{-1}$ (Fig. 4c). All of this would indicate that it is indeed the strongest convection that is driving the DKE generation and the blocking but in two different phases.

Fig. 13. Spatial structure at 13 km of (top) rotational kinetic energy, (middle) divergent kinetic energy, and (bottom) cross-term energy for (a)–(c) the control at 84 h, (d)–(f) the G7.5 simulation at 65 h 30 min, and (g)–(i) the G7.5 simulation at 120 h. The black X is the low-level TC center. The black arrow in (d) is the direction of flow for all sheared images in (d)–(i).
5. Summary and discussion

a. Summary

In this study, we have performed a more thorough, mathematical analysis of the outflow to complement the original series on atypical RI (Part I; Part II; Part III) using a hierarchy of analyses and models. We first reviewed the fundamental components of atypical RI: the return of the midlevel center upshear and outflow blocking. We showed how the movement of the midlevel center, based on findings from Part II, can be abstracted from infrared (IR) satellite imagery. The outflow blocking that permits this evolution (Part III) was then demonstrated structurally in ERA5. This highlighted that the divergent winds in ERA5, localized in an arc structure upwind, serve to block and reroute the environmental rotational winds. The environmental rotational winds are associated with upper-level anticyclones. We then revisited findings from Part III: the structure of the upper-level winds as the block occurs, the presence of an upshear Bernoulli high, and the time-varying shear and outflow front location.

Since the outflow is vertically confined to a layer approximately 2 km deep, we were able to reproduce facsimiles of the flow using an analytical 2D point-source solution and a forced divergent shallow water model. The simple analytical solution allowed us to demonstrate that the location of the Bernoulli high is a superposition of two pressure anomalies: one associated with the winds of the vortex, the other associated with the background winds. The shallow water model used initial conditions based on the time-averaged $\psi$ and $\chi$ fields from the control simulation. The shallow water model simulations demonstrated that the existence of the $\chi$ field is critical to the maintenance of the block, as the $\psi$ field by itself is advected downwind and offers little resistance to an environmental flow; however, we note that if we were using a two-layer model at minimum such as in Smith et al. (2000) then the vortex would exhibit more resistance to the flow through vertical coupling, and the single-layer shallow water model is in no way meant to model the full vertical structure of a TC. Nevertheless, based on our results of the simpler models, the steady-state outflow front location, while dependent upon the radial decay of $\chi$, can be meaningfully assessed using the non-dimensional ratio of the peak value of $\chi$ to the magnitude of the background wind: $\chi_{\text{peak}}/U_0$.

Given that the divergent wind is the most important part of the outflow for blocking, we derived a divergent kinetic energy budget equation to analyze the evolution of the divergent outflow of the CM1 simulation more carefully. We showed that the evolution of the divergent outflow can be divided into two regimes: a pressure-gradient-driven one and a divergent-flux-driven one. Both are strongly tied to convection. When the shift occurs at 48 h into the simulation, which is also the time the precession reaches left of shear (Part II), the outflow front

FIG. 14. The volume integral between 10- and 14-km heights and in a 500-km circle centered on the low-level TC center of the three kinetic energy components for each simulation. All terms are smoothed with a 2-h running mean.

FIG. 15. Time series of the time-tendency terms of the divergent and cross-term kinetic energy budget equation for the (a) control simulation and (b) Gaussian-7.5 shear simulation from 6 to 60 h integrated between 10- and 14-km height and within a 200-km circle of the low-level center. All are smoothed with a 2-h running mean. The terms are defined in the text.

FIG. 16. As in Fig. 15, but for 48–144 h and for a change in the y axis.
is subsequently pushed out to 250 km. The divergent kinetic energy and the positive lobe of cross-term energy are in phase. Later in the simulation, after the abrupt rapid intensification at 90 h, the divergent kinetic energy maximum and the cross term are no longer in phase, the strongest vertical motions cease, the outflow front retreats to 150–200 km, the shear increases from 3.5 to 6 m s$^{-1}$, and the TC ceases its intensification.

b. Discussion

At the most basic level, what we have described herein is the Magnus effect (Magnus 1853) which is a specialized interpretation of the Bernoulli effect (Euler 1752). As a flow interacts with a rotating object, a high–low pressure couplet forms on either side of the object. This phenomenon has previously been used to explain the movement of tornadoes (Fujita 1965). The two differences here are that tornadoes are primarily rotational without a prominent divergent component and that the pressure anomaly caused by a modification of the tornado’s rotation was neglected. A TC possesses a large and significant divergent flow field, and this divergent flow is what is necessary to divert the environmental flow around the TC upwind. This creates an interesting paradox with regard to shear calculations used both operationally and academically. Many works have tried to remove the vortex in various ways to ascertain the environmental flow, but the original series plus this paper demonstrate that one cannot remove the divergent component of the flow at upper levels. This could be a significant explanation as to why operational shear calculations can be misleading, such as in 2016 Matthew (Stewart 2017; Part III). This also highlights the necessity of using shear calculations that incorporate the vertical structure of the environmental winds, as discussed by Finocchio et al. (2016), Onderlinde and Nolan (2016), Velden and Sears (2014), and Knaff et al. (2018), since outflow blocking is only effective between 150 and 250 hPa, and measures of vortex resilience for other wind profiles would fall to previously established research (Jones 1995; Reasor et al. 2004; Schechter 2015).

We also draw attention to the fact that, based on the time series of the $\chi$ minimum in the CM1 simulations, the only way to create a situation in which a weak TC can reduce oncoming moderate shear is by paradoxically shearing the TC, all else being equal thermodynamically. The minimum value of $\chi$ in the G7.5 simulation is 1.5–2 times as large as in the control when both are borderline hurricanes. In the control, the $\chi$ minimum is $-1.0 \times 10^6$ m$^2$ s$^{-1}$ while it is the equivalent of a category-1 hurricane. According to the nondimensionalized shallow water results in this case, to create an outflow front at a distance of 250 km the environmental shear can only be 2.6 m s$^{-1}$. We hypothesize that this stark contrast is due to the thermal anomaly caused by the tilt and the subsequent importance of buoyancy in modulating updraft characteristics (Part II) that in turn modulate the structure of $\chi$, but we leave a
more detailed investigation of this phenomenon for a future study.

An understated appeal of the simpler models—both shallow water and analytical—is that we can generalize our results beyond the realm of TCs. As noted in the introduction, various severe storm studies have noticed high pressures aloft upwind of severe weather and tried to attribute its existence to various mechanisms. Using our nondimensional studies as a framework, we have shown that the divergent component of any flow will act as a barrier to the oncoming winds; therefore, the high pressures and the blocking aloft noted by Mapes (1993), Schmidt and Cotton (1990), and Fritsch and Maddox (1981) are likely a result of the divergent outflow’s colliding with the environmental winds. The anticyclonic rotation of the winds around the system (Trier and Sharman 2009) is then associated with the rotational component of the outflow. Determining the extent of the blocking and the pressure perturbation is thus controlled by setting the nondimensionalization parameters such as length scale, wind scale, and so on to parameters that match the scale of whatever phenomenon one wishes to study.

More work still needs to be done. Throughout our simulations, we have neglected variations in radiative properties and SSTs, both of which would alter the thermodynamics of the situation considerably. This would then alter the structure and strength of the χ field through modulation of the updrafts. It has been shown that the two initial convective pulses in the G7.5 simulation are important to divergent kinetic energy generation prior to full pushback by the flow as the tilt reaches left of shear. We have limited our full-physics simulation to 27.5°C SSTs and a background shear magnitude of 7.5 m s⁻¹, as these were parameters motivated by observations. All else being equal, if SSTs play an important role in modulating the χ field through buoyant updrafts (Hendricks et al. 2004), then warmer SSTs would also enhance the outflow blocking mechanisms described throughout this paper and the original series. In addition, since convection is also favored diurnally at night (Kossin 2002), that will also modulate outflow associated with these features. Future work will address these details.

As a final note, we close by discussing the motivation behind this entire series. This study was motivated in large part by the Office of Naval Research Tropical Cyclone Intensification Defense Research Initiative (Doyle et al. 2017). One of the key science goals was to “understand the coupling of TC outflow with inner-core convection and its implications for intensity change.” We believe that we have firmly established a link between the two, albeit under specific circumstances and in an unexpected way. Convection in sheared TCs projects onto the divergent component of outflow, and this divergent flow serves to block the environmental flow, to lower the shear, and to permit realignment and subsequent vortex intensification if the TC is sheared by an anticyclone. These studies highlight the paramount importance of a new shear calculation: one that factors in both the vertical structure of the environmental winds and the ability of the TC to defend itself via its divergent outflow.

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Data availability statement. ERA5 data can be accessed from the Copernicus Climate Data Store. Geostationary satellite imagery for GOES-13 has been obtained from NOAA’s Comprehensive Large Array-Data Stewardship System. All additional modeling data are available from the authors upon request.

APPENDIX A

Analytical Point Vortex and Point Source Equations

a. Derivation of Bernoulli’s equation

Euler’s equation can be written with no approximation as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{1}{2} \mathbf{v}^2 + \phi + \frac{\rho}{\rho_0} \right) = -\frac{\rho}{\rho_0} \nabla \phi + \mathbf{v} \times (\nabla \times \mathbf{v}).$$

(A1)

where $\mathbf{v}$ is the total wind, $\phi$ is the geopotential, and pressure and density are their usual symbols. We note that

Fig. 19. (a)–(c) Log10 CFADs of vertical velocity (a) during a TCA in the Gaussian-7.5 simulation, (b) after a TCA after the tilt has reached a relative minimum, and (c) during the control at a time when the low-level intensity was similar. Values are smoothed with a 2-h running mean. (d) Total volume covered by updrafts greater than 20 m s$^{-1}$ for each simulation (bars; left axis) and the low-level maximum mean tangential wind of each simulation (lines; right axis).
\[ \mathbf{v} \cdot [\mathbf{v} \times (\nabla \times \mathbf{v})] = 0, \quad (A2) \]
since the expression inside the parentheses is orthogonal to \( \mathbf{v} \). Applying the dot product of \( \mathbf{v} \) with \( (A1) \) and noting \( (A2) \), \((A1)\) becomes
\[ \frac{1}{2} \frac{\partial v^2}{\partial t} + \mathbf{v} \cdot \nabla \left( \frac{1}{2} v^2 + \phi + \frac{p}{\rho} \right) = -\frac{p}{\rho^2} \mathbf{v} \cdot \nabla p. \quad (A3) \]
Assuming steady state, the time tendency vanishes, and \((A3)\) becomes
\[ \mathbf{v} \cdot \nabla \left( \frac{1}{2} v^2 + \phi + \frac{p}{\rho} \right) = -\frac{p}{\rho^2} \mathbf{v} \cdot \nabla p. \quad (A4) \]
Assuming that the winds \( \mathbf{v} \) can be described by a stream-function, then
\[ u = -\frac{\partial \phi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial x}. \quad (A5a) \]
Using \((A5)\) in \((A4)\) results in
\[ -\frac{\partial \phi}{\partial y} \frac{\partial B}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial B}{\partial y} = -\frac{p}{\rho^2} \left[ \frac{\partial \phi}{\partial y} \frac{\partial \rho}{\partial x} + \frac{\partial \phi}{\partial x} \frac{\partial \rho}{\partial y} \right]. \quad (A6) \]
where
\[ B = \frac{1}{2} v^2 + \phi + \frac{p}{\rho} \quad (A7) \]
A simple solution to \((A6)\) is
\[ B = B(\psi) \quad \text{and} \quad \rho = \rho(\psi). \quad (A8a) \]
Using the chain rule, it can be shown that using \((A8)\) in \((A6)\) is identically zero.
We can now describe a steady flow in which \( \rho \) and \( B \) are functions of streamlines only. Given a situation where we know \( B \) initially, we can then use \((A7)\) to solve for \( B \) everywhere. For example, for a zonal flow (as indicated by subscript 0), we obtain
\[ \frac{1}{2} v_0^2 + \phi + \frac{p_0}{\rho_0} = \frac{1}{2} U_0^2 + \phi_0 + \frac{p_0}{\rho_0}. \quad (A9) \]
Assuming that the geopotential and density both remain constant along a streamline, we can then solve for the new pressure:
\[ p = p_0 + \frac{\rho_0}{2} (U_0^2 - v^2). \quad (A10) \]
In our situation, we can solve a similar expression for the initial vortex as well. As a result, we can specify a point vortex in terms of \( \psi \) and solve for pressure perturbations anywhere in the flow.

b. Point vortex with a divergent source in a uniform flow
The total streamfunction for the analytical field of a collocated point vortex and point source in a background flow is (e.g., Lamb 1932, 62–109):
\[ \psi(x, y) = -U_0 y - D \arctan \left( \frac{y}{x} \right) + \frac{K}{2} \ln(x^2 + y^2). \quad (A11) \]
The first term describes the background flow, the second is the “divergent” source for \( D > 0 \), and the third is the anti-cyclonically rotating field for \( K < 0 \). Using \((A5)\), the wind field derived from the streamfunction in \((A11)\) is
\[ u(x, y) = U_0 + \frac{D}{x^2 + y^2} - \frac{K y}{x^2 + y^2} \quad \text{and} \quad (A12a) \]
\[ v(x, y) = \frac{D y}{x^2 + y^2} + \frac{K x}{x^2 + y^2}. \quad (A12b) \]
As a small note, to maintain consistency with the strict assumptions used in deriving Bernoulli’s equation in section \( a \) of appendix \( A \), the \( D \) terms technically have no divergence since they are scaled by \( r^2 \) and would thus technically be considered diffusiveness. The interpretation of the result, however, remains the same. Additionally, in the main body of the paper, \( D \) and \( K \) are replaced by \( \chi_m \) and \( \psi_0 \) to maintain consistency with the shallow water model notation.

APPENDIX B

Shallow Water Model Initial Conditions
To permit maximum flexibility for the shallow water model simulations, we create analytical profiles based on time-averaged \( \psi \) and \( \chi \) fields from the CM1 control simulation. All values are nondimensional. We begin with the \( \chi \) field, since its general shape is relatively straightforward and can be simply approximated with a Gaussian:
\[ \chi(r) = \chi_m e^{-r^2/(2\sigma_r)^2}; \quad (B1) \]
\( \chi_m \) is the peak magnitude, \( r \) is radius, and \( \sigma_r \) is the radial width. To closely match the time-averaged results from the CM1, the peak magnitude is set to \(-1.25\) and the radial width is set to \(1.0\). These values are then augmented for subsequent simulations.
The \( \psi \) field is considerably more complicated, as it features a sharp rise in the core and then gradual roll-off outside its peak. We approximate the structure of the \( \psi \) field using the probability density function of the gamma distribution as a basis (Wilks 2011, p. 97). The \( \psi \) is constructed thusly:
\[ \psi(r) = \frac{\psi_m (ar)^{\psi - 1} e^{-ar\theta}}{\Gamma(k)\theta^k}. \quad (B2) \]
This is a gamma distribution scaled in both distance and in magnitude; \( \psi_m \) is the user-defined peak magnitude equal to \(3.5\), and \( \psi \) is the peak magnitude of the original gamma distribution; \( k \) and \( \theta \) are shape parameters and are set a priori to \(3.0\) and \(4.0\), respectively; \( r \) is radius; and \( \alpha \) is the radial scaling, set a priori to \(3.5\). We then implement a radially confined scaling:
\[ \psi(r) = \psi(r) + \beta \left[ \psi_m - \psi(r) \right]; \quad r < \text{RM}_\psi, \quad (B3) \]
where \( \beta \) is a scaling constant and \( \text{RM}_\psi \) is the radius of maximum \( \psi \). This scaling controls the strength of the inner cyclonic vortex. The peak magnitude and the strength of the inner cyclonic vortex are what are modulated in our simulations.
Each profile is set to zero beyond a radius of \(20\). They are smoothly transitioned to zero over a span of six radii using the
hyperbolic tangent function. We also note that, in the outer regions of the ψ field beyond 900 km in the control simulation, ψ becomes negative. We keep ψ small and positive in the shallow water simulations to ensure there is no weak cyclonic circulation during the transition to zero at outer radii.

APPENDIX C

Derivation of Divergent Kinetic Energy Budget Equation

Making use of Helmholtz’s theorem, we can decompose the horizontal wind \( \mathbf{V} \) (where boldface font signifies a vector) into rotational (\( \mathbf{V}_r \)) and divergent (\( \mathbf{V}_s \)) vector fields as

\[
\mathbf{V} = \mathbf{V}_r + \mathbf{V}_s,
\]

where

\[
\mathbf{V}_r = \mathbf{k} \times \nabla \psi \quad \text{and} \quad \mathbf{V}_s = \nabla \chi;
\]

\( \psi \) is the streamfunction; and \( \chi \) is the velocity potential. It follows as a result of this decomposition that divergence \( \delta \) and vorticity \( \zeta \) can be defined:

\[
\delta = \nabla \cdot \mathbf{V} = \nabla \cdot \mathbf{V}_s \quad \text{and} \quad \zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \mathbf{V}_r \times \mathbf{k}.
\]

The kinetic energy per unit mass is

\[
k = \frac{\mathbf{V} \cdot \mathbf{V}}{2} = \frac{\mathbf{V}_r \cdot \mathbf{V}_r}{2} + \frac{\mathbf{V}_s \cdot \mathbf{V}_s}{2} + (\mathbf{V}_r \cdot \mathbf{V}_s).
\]

If we let \( k_r = (\mathbf{V}_r \cdot \mathbf{V}_r)/2 \) and \( k_s = (\mathbf{V}_s \cdot \mathbf{V}_s)/2 \), then the total integrated kinetic energy is

\[
K = \int k \, d\Omega = \int k_r \, d\Omega + \int k_s \, d\Omega + \int (\mathbf{V}_r \cdot \mathbf{V}_s) \, d\Omega,
\]

where \( \int d\Omega = \iint \rho \, dx \, dy \, dz \), and \( \rho \) is density. The final term on the right-hand side of (C4) and (C5) is the cross term of rotational and divergent kinetic energy and can be thought of as the nonlinear interaction between the two energy components.

For the problem at hand discussed in this paper, we are interested in the evolution of the divergent kinetic energy:

\[
\frac{\partial k_s}{\partial t} = \int k_s \, d\Omega = \int \left( \frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} - w \frac{\partial u}{\partial z} - w \frac{\partial v}{\partial z} + \frac{\partial V_r}{\partial z} - \frac{\partial V_s}{\partial z} + V_s \cdot F \right) \, d\Omega.
\]

We use the same approach as Chen and Win-Nielsen (1976), except we apply it to the modified equations of horizontal motion from the CM1 (Bryan 2017; Part II):

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -c_\rho \theta_r \frac{\partial \mathbf{V}_r}{\partial x} + f v - e w + F_x
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -c_\rho \theta_r \frac{\partial \mathbf{V}_r}{\partial y} - f u + F_y,
\]

where \( \theta_r \) is density potential temperature, \( \Pi \) is the Exner function, \( f \) and \( e \) are the vertical and meridional components of the Coriolis acceleration, and \( F \) contains all of the subgrid terms (friction, turbulence, etc.), respectively. Utilizing the identities

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial k}{\partial x} - \xi v
\]

\[
\frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = \frac{\partial k}{\partial x} + \xi u,
\]

we rewrite (C7) and (C8) as

\[
\frac{\partial k}{\partial t} + \frac{\partial k_r}{\partial x} + \frac{\partial k_s}{\partial x} + \frac{\partial (\mathbf{V}_r \cdot \mathbf{V}_s)}{\partial x} + \frac{\partial u}{\partial z} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x} = (\zeta + f) v - \frac{\partial \Pi}{\partial x} \quad \text{and}
\]

\[
\frac{\partial k_s}{\partial t} + \frac{\partial k_r}{\partial y} + \frac{\partial k_s}{\partial y} + \frac{\partial (\mathbf{V}_r \cdot \mathbf{V}_s)}{\partial y} + \frac{\partial u}{\partial z} + w \frac{\partial u}{\partial z} + \frac{\partial u}{\partial y} = -(\zeta + f) u + \zeta f - \frac{\partial \Pi}{\partial y}.
\]

We then multiply (C10) and (C11) by their respective divergent wind components and add them together, yielding

\[
\frac{\partial k_s}{\partial t} - c_\rho \theta_r (\mathbf{V}_s \cdot \nabla \Pi) + (\zeta + f)(u V_r - \mathbf{V}_r \cdot \mathbf{V}_s) - \mathbf{V}_s \cdot \nabla k_r - \mathbf{V}_r \cdot \nabla k_s
\]

\[
= w (\mathbf{V}_s \cdot \nabla \mathbf{V}_r) - w \frac{\partial k}{\partial z} + \mathbf{V}_s \cdot \mathbf{F}.
\]

From left to right, the terms on the right-hand side can then be described as pressure-work or baradonic generation; divergent-rotational wind conversion; divergent flux across local changes in rotational winds; meridional Coriolis; divergent fluxes of divergent kinetic energy, rotational kinetic energy, and their cross term; divergent movement of the vertical flux of rotational winds; vertical advection of divergent kinetic energy; and the subgrid-scale terms. Given the spatial structure of the energy cross term with respect to the divergent kinetic energy in the Gaussian-7.5 sheared TC, we can modify (C12) using the definition of the time tendency of the cross-term energy, resulting in the final budget equation evaluated herein:

\[
\frac{\partial k_s}{\partial t} + \frac{\partial (\mathbf{V}_r \cdot \mathbf{V}_s)}{\partial t} = -c_\rho \theta_r (\mathbf{V}_r \cdot \nabla \Pi) + (\zeta + f)(u V_r - \mathbf{V}_r \cdot \mathbf{V}_s) + \mathbf{V}_r \cdot \frac{\partial \mathbf{V}_r}{\partial t} - w (\mathbf{V}_s \cdot \nabla \mathbf{V}_r) - w \frac{\partial k}{\partial z} + \mathbf{V}_s \cdot \mathbf{F}.
\]
Krishnamurti and Ramathan (1982) also combined the divergent kinetic energy and the cross term in their analyses. We do not make the same assumptions about the domain as they did; therefore, the cancellations and mathematical reorganization that they used are not present here.

REFERENCES


