Analytical Investigation of the Role of Lateral Mixing in the Evolution of Nonprecipitating Cumulus. Part I: Developing Clouds

M. PINsky AND A. KHAIN

Department of Atmospheric Sciences, Hebrew University of Jerusalem, Jerusalem, Israel

(Manuscript received 12 February 2019, in final form 11 August 2019)

ABSTRACT

Evolution of nonprecipitating cumulus clouds (Cu) at the developing stage under the influence of lateral entrainment and mixing is studied analytically using a minimalistic analytical model. We present a model of an ascending cloud volume (a model of developing Cu) whose structure is determined by the processes of droplet diffusion growth/evaporation and entrainment mixing in the horizontal direction. Spatial and time changes of liquid water content, the adiabatic fraction, droplet concentration, and the mean volume droplet radius are calculated. It is shown that the existence of a nondiluted core in a growing cumulus cloud significantly depends on the cloud width and vertical velocity. While at the updraft velocity of 2 m s\(^{-1}\) the core of a 400-m-wide cloud becomes diluted at distances of a few hundred meters above cloud base, the core of a cloud of 1000-m width remains nondiluted at distances up to 1500 m above cloud base. The explanation of this result is simple: the increase in cloud width and the decrease in the updraft velocity increase the time during which the cloud is diluted due to mixing. Since lateral mixing synchronously decreases both the cloud water content and droplet concentration, the variation of the mean volume droplet radius is low inside the cloud. The approximate quantitative condition for cloud formation in updraft is derived. It is shown that a cloud can arise when its vertical velocity exceeds a critical value. To produce clouds, narrow turbulent plumes should ascend at higher velocity as compared to wider plumes. High humidity of the environment air is favorable for formation of clouds from plumes. The comparison of the obtained results with previously published observational data indicates a reasonable agreement. The results can be useful for parameterization purposes.

1. Introduction

The main cloud parameters determining the effects of clouds on the radiative and energetic budget of Earth’s atmosphere are cloud cover, vertical profiles of liquid water content (LWC), droplet concentration, and the effective radius, which is directly related to the mean volume droplet radius. In large-scale models, small nonprecipitating cumulus clouds (Cu) are unresolvable and values of these parameters are calculated using simple parameterizations or are prescribed a priori (Doms and Schattler 2002; Plant and Yano 2015). Cloud microphysical properties are different at different stages of cloud evolution (developing, mature, and dissolving). In large-scale atmospheric models, the difference between cloud evolution stages is not taken into account. Convective parameterization schemes assume formation of a new cloud at each time step (which is typically of order 1 min) (Plant and Yano 2015). In large-eddy simulations (LESs), the Cu quantities are calculated directly using the model grid spacing of several tens to several hundred of meters (Siebesma et al. 2003; Kogan and Mechem 2014; Khain et al. 2015; Dagan et al. 2017; Park et al. 2017; Khain et al. 2019). In LES, the evolution of Cu fields is described by a complicated system of dynamical and microphysical equations, containing multiple parameters whose individual effects are not so easy to evaluate. Multiple recent in situ measurements (Gerber 2000; Gerber et al. 2008; Arabas et al. 2009; Katzwinkel et al. 2014; Schmeissner et al. 2015; Kumar et al. 2017) allow us to assume that evolution of nonprecipitating Cu is determined by two main processes: droplet growth in cloud updrafts and mixing of clouds with environment. According to measurements and theoretical analysis, the cloud–environment mixing takes place largely through the lateral cloud boundary (De Rooy et al. 2013; Bera et al. 2016). In situ measurements show that in convective clouds, especially in small cumulus clouds liquid water content may be substantially lower than the adiabatic value.
(Gerber et al. 2008). The decrease of the adiabatic fraction (AF) with increasing height indicates the importance of the process of entrainment and turbulent mixing.

There is a great number of studies dedicated to the process of turbulent mixing in clouds [see sections 5.9 and 5.10 in the book by Khain and Pinsky (2018)]. These studies can be separated into three groups. The first group focuses on analysis of mixing between two neighboring volumes of cloudy air and droplet-free air considering only the final states resulting from mixing. Since the resulting volume is closed, solution tends to a final stationary solution when the mixing ends. Parameters of the final state are determined from heat and moisture budget considerations and priory given assumptions regarding the mixing (e.g., Baker and Latham 1982; Burnet and Brenguier 2007; Lehmann et al. 2009; Korolev et al. 2016; Pinsky et al. 2016a).

Process of time-dependent mixing between two volumes was analyzed in the second group of the studies (e.g., Su et al. 1998; Pinsky et al. 2016b; Pinsky and Khain 2018a). This process also leads to a stationary state. Pinsky et al. (2016b) formulated diffusion–evaporation equation describing the change of droplet size distribution under two effects: 1) turbulent mixing induced by turbulent diffusion and 2) droplet evaporation. Evolution of mixing diagrams in this process was later analyzed by Pinsky and Khain (2018a).

The third group of studies focuses on entrainment and turbulent mixing with dry environment air through the lateral boundary of a cloud (Baker et al. 1980; Barahona and Nenes 2007; Krueger et al. 1997; Su et al. 1998; Pinsky and Khain 2018b, 2019). The specific feature of these studies is that the nonstationary problem is considered that does not lead to an equilibrium state; that is, mixing can continue infinitely long. In some studies belonging to this group, air volumes from environment penetrate an ascending cloud parcel continuously or at several time instances. Then immediate homogeneous mixing of the dry volumes with the entire parcel is assumed (Baker et al. 1980; Barahona and Nenes 2007). Other studies calculate the detailed process of mixing of air volumes penetrated by surrounding cloudy air (Krueger et al. 1997; Su et al. 1998).

Pinsky and Khain (2018b, 2019) theoretically investigate the effect of lateral mixing on the microphysical structure of the interface near cloud edges using the 1D diffusion–evaporation equation under the assumption of zero vertical velocity. It was shown that the interface zone where the effect of mixing is pronounced consists of two zones: the dilution zone where liquid water content decreases toward cloud edge, and a humid shell where LWC = 0, but air humidity exceeds that in the environment (see also Bar-Or et al. 2012). The internal boundary of the interface zone always propagates toward cloud center, while external boundary of the dilution zone (i.e., cloud boundary), which is also an internal boundary of the humid shell can move both toward cloud body or outward, depending on the value of the potential evaporation parameter defined as $R = S_2/(A_Sq_1) < 0$, where $S_2 < 0$ is supersaturation in cloud surrounding, $q_1$ is liquid water mixing ratio of cloud, and $A_S$ is a parameter slightly depending on temperature (see Table 1). At $R > -1$, cloud expansion takes place (i.e., the cloud grows); at $R < -1$, the cloud boundary shifts toward cloud center (i.e., the cloud dissipates). However, the studies mentioned above do not take into account the finite cloud width and the existence of vertical velocities within the cloud. Therefore, the results, while showing the important role of lateral turbulent mixing, do not allow us to evaluate the effect of mixing on the evolution of a cloud as a whole.

In this study we present a minimalistic analytical model of an ascending (descending) cloud volume—a model of developing (dissipating) Cu—whose structure is determined by the processes of droplet condensation (evaporation) and entrainment mixing. Hereinafter the model will be referred to as $\Gamma$–$K$ model. The meaning of this name will be clear from the following text. The analytical consideration can help to identify governing parameters as well as their role in cloud structure evolution. This is Part I of the study presenting $\Gamma$–$K$ model and analyzing the microphysics of Cu at their developing stage. The next part will be dedicated to analysis of the dissolving stage of Cu.

### 2. Minimalistic $\Gamma$–$K$ model of a cloud volume

According to Katzwinkel et al. (2014), the evolution of Cu includes 3 stages: (i) active growth Cu ($w > 0$, positive buoyancy force), (ii) decelerated clouds ($w > 0$, negative buoyancy force), and (iii) dissolving clouds ($w < 0$, negative buoyancy force). The conceptual scheme of an ascending cloud volume, corresponding to the active growth stage, considered in this study is presented in Fig. 1. A cloud volume of horizontal size $2L$ rises within an endless environment at constant vertical velocity $w$ directed upward and simultaneously mixes up through its lateral borders with unsaturated motionless surrounding air with the relative humidity of RH <100% (supersaturation $S_2 < 0$). The intensity of the turbulent mixing is characterized by the turbulent coefficient $K$. The relative humidity of the air inside the cloud is close to 100% ($S_1 \approx 0$).

The $\Gamma$–$K$ model has the following main simplifications:

- The model takes into account only two processes influencing droplet condensation/evaporation: (i) the updraft
Mixing caused by turbulence is carried out in the inertial range determine mutual diffusion of cloud and environment and lead to creation of the interface zone at lateral cloud boundaries. Turbulent intensity is characterized by the turbulent coefficient, which is assumed to have a uniform value over space and time.

- The surrounding dry air area is assumed to be infinitely large.

- The vertical velocity of the air inside cloud does not change over time and space.
- The microphysical processes, such as droplet nucleation, droplet collisions and droplet settling, are not considered.

The proposed $\Gamma$–$K$ model can be considered as an extension of widely used models of an ascending cloud parcel, in which entrainment mixing is described without applying the entrainment parameter (without parameterization) (Plant and Yano 2015; Khain and Pinsky 2018), but using the diffusion equation in which the parcel structure is a function of horizontal coordinates and time (height).

The $\Gamma$–$K$ model is based on the moisture balance in the mixing process. Using equations for mixing ratio of water vapor, temperature and the Clausius–Clapeyron equation, one can obtain the equation for supersaturation changes (Squires 1952; Pruppacher and Klett 1997; Korolev and Mazin 2003):

\[
\frac{dS}{dt} = (1 + S) \left( A_1 w - A_2 \frac{dq}{dt} \right).
\]  

(1a)

Since supersaturation in clouds usually obeys the condition $|S| \ll 1$, Eq. (1a) often can be rewritten in the simplified form

\[
\frac{dS}{dt} = A_1 w - A_2 \frac{dq}{dt}.
\]  

(1b)

In Eqs. (1a) and (1b), the values of $A_1$ and $A_2$ are calculated as

\[
A_1 = \frac{g}{T} \left( \frac{L_w}{c_p R_y T} - \frac{1}{R_y} \right), \quad A_2 = \frac{1}{\rho_v} + \frac{L_w^2}{R_v c_p T^2},
\]  

(2)

where $\rho_v$ and $q$ are water vapor mixing ratio and liquid water mixing ratio, respectively; $g$ is the acceleration of gravity; and $w$ is the updraft velocity of the cloud parcel. These coefficients slightly depend on temperature $T$. 

---

**TABLE 1. List of symbols.** In the units column, “nd” means nondimensional.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$[g/(R_y T)][(L_w R_y/c_p R_T) - 1]$</td>
<td>$m^{-1}$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$(1/q_r) + (L_w^2/c_p R_y T^2)$</td>
<td>$m\cdot s^{-1}$</td>
</tr>
<tr>
<td>$C$, $C_1$</td>
<td>Coefficients</td>
<td>nd</td>
</tr>
<tr>
<td>$c_p$</td>
<td>Specific heat of moist air at constant pressure</td>
<td>$J\cdot kg^{-1}\cdot K^{-1}$</td>
</tr>
<tr>
<td>$e_s$</td>
<td>Saturation vapor pressure above a flat water surface</td>
<td>$N\cdot m^{-2}$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational acceleration</td>
<td>$m\cdot s^{-1}$</td>
</tr>
<tr>
<td>$K$</td>
<td>Turbulent diffusion coefficient</td>
<td>$m\cdot s^{-1}$</td>
</tr>
<tr>
<td>$L$</td>
<td>Half of the cloud width</td>
<td>m</td>
</tr>
<tr>
<td>$L_w$</td>
<td>Latent heat for liquid water</td>
<td>$J\cdot kg^{-1}$</td>
</tr>
<tr>
<td>$L_1$</td>
<td>External turbulent scale</td>
<td>m</td>
</tr>
<tr>
<td>$N$</td>
<td>Droplet concentration</td>
<td>m$^{-3}$</td>
</tr>
<tr>
<td>$N_1$</td>
<td>Initial droplet concentration</td>
<td>m$^{-3}$</td>
</tr>
<tr>
<td>$q$</td>
<td>Liquid water mixing ratio in a cloud</td>
<td>kg$\cdot$kg$^{-1}$</td>
</tr>
<tr>
<td>$q_1$</td>
<td>Initial liquid water mixing ratio in a cloud</td>
<td>kg$\cdot$kg$^{-1}$</td>
</tr>
<tr>
<td>$q_{ad}$</td>
<td>Adiabatic liquid water mixing ratio</td>
<td>kg$\cdot$kg$^{-1}$</td>
</tr>
<tr>
<td>$q_C$</td>
<td>Total liquid water mixing ratio</td>
<td>kg$\cdot$kg$^{-1}$</td>
</tr>
<tr>
<td>$R$</td>
<td>$S_c^2/A_{eff}$, potential evaporation parameter</td>
<td>nd</td>
</tr>
<tr>
<td>$R_S$</td>
<td>Specific gas constant of moist air</td>
<td>$J\cdot kg^{-1}\cdot K^{-1}$</td>
</tr>
<tr>
<td>$R_v$</td>
<td>Specific gas constant of water-vapor</td>
<td>$J\cdot kg^{-1}\cdot K^{-1}$</td>
</tr>
<tr>
<td>$r_{eff}$</td>
<td>Effective radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_w$</td>
<td>Droplet mean volume radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_w$</td>
<td>MEAN value of mean volume radius</td>
<td>m</td>
</tr>
<tr>
<td>$r_{v,max}$</td>
<td>Maximal mean volume radius</td>
<td>m</td>
</tr>
<tr>
<td>$S$</td>
<td>$e/e_s - 1$, supersaturation over water</td>
<td>nd</td>
</tr>
<tr>
<td>$S_2$</td>
<td>Initial supersaturation in dry air</td>
<td>nd</td>
</tr>
<tr>
<td>$T$</td>
<td>Temperature</td>
<td>K</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
<td>s</td>
</tr>
<tr>
<td>$w(x)$</td>
<td>Horizontal profile of updraft velocity</td>
<td>m$\cdot$s$^{-1}$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>Updraft velocity inside cloud</td>
<td>m$\cdot$s$^{-1}$</td>
</tr>
<tr>
<td>$x$</td>
<td>Distance</td>
<td>m</td>
</tr>
<tr>
<td>$z$</td>
<td>Height above cloud base</td>
<td>m</td>
</tr>
<tr>
<td>$\Gamma(x, t)$, $\Gamma(x, z)$</td>
<td>Conservative function</td>
<td>nd</td>
</tr>
<tr>
<td>$e$</td>
<td>Turbulent dissipation rate</td>
<td>$m^2\cdot s^{-3}$</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>Air density</td>
<td>kg$\cdot$m$^{-3}$</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>Density of liquid water</td>
<td>kg$\cdot$m$^{-3}$</td>
</tr>
</tbody>
</table>
The first term on the right-hand side of Eq. (1) describes an increase in supersaturation due to adiabatic cooling of air during the parcel’s ascent; the second right-hand-side term describes the supersaturation sink caused by condensation of water vapor on droplets.

Assuming that coefficients \( A_1 \) and \( A_2 \) do not change with time, integration of Eq. (1b) with respect to time leads to the following equation describing the moisture balance in an ascending adiabatic parcel (Pinsky et al. 2013):

\[
S = A_1 z - A_2 q + C, \tag{3}
\]

where the integration constant \( C = A_2 q_0 + S_0 = 0 \) is determined by the initial conditions at \( z = 0 \).

Equation (3) was derived under condition \( |S| \ll 1 \), which is strictly valid within clouds. This assumption seems to be also valid within the part of humid shell surrounding cloud. Equation (3) becomes less accurate at significant distances from clouds.

Here we would like to add a small remark. Equation for supersaturation [Eq. (1)] is often written in the form that includes the droplet relaxation time (which can slowly change) and describes the exponential approach of supersaturation \([\text{Eq.}(1)]\) is often written in the form that includes the droplet relaxation time (which can slowly change) and describes the exponential approach of supersaturation to its quasi-equilibrium value (Korolev and Mazin 2003). It is difficult, however, to rewrite this equation in the balance form convenient for further analysis.

Equation (3) represents the total moisture balance that allows us to introduce the quantity \( \Gamma(x, t) \) (see also Pinsky and Khain 2018b, 2019):

\[
\Gamma(x, t) = S(x, t) + A_2 q(x, t) \tag{4}
\]

(where \( x \) is the horizontal coordinate and \( t \) is time), which is conservative with respect to condensation/evaporation taking place at a certain horizontal level. The conservation of total water means that evaporation of liquid droplets increases air humidity and, vice versa, condensation of water vapor decreases air humidity. This quantity was used by Pinsky and Khain (2018b, 2019) to determine the evolution of the interface zone around the lateral cloud edges.

Turbulent mixing in the horizontal direction between a motionless cloud volume and its environment can be described by the diffusion equation:

\[
\frac{\partial \Gamma(x, t)}{\partial t} = K \frac{\partial^2 \Gamma(x, t)}{\partial x^2}, \tag{5}
\]

where \( K \) is the turbulent coefficient. This coefficient is evaluated from the Richardson law as

\[
K = C \varepsilon^{1/3} L_1^{4/3}, \tag{6}
\]

where \( L_1 \) is the external turbulent scale equal to about \( L_1 = 2L/15 \) (Grabowski 1993, 1995; Benmoshe et al. 2012), \( \varepsilon \) is the turbulent dissipation rate, and \( C \) is a coefficient equal to 0.2 (Monin and Yaglom 1975; Boffetta and Sokolov 2002).

In case a cloud volume moves in the vertical direction, function \( \Gamma(x, t) \) is no longer conservative due to adiabatic changes in temperature. In this case, the changes in \( \Gamma(x, t) \) can be described by equation

\[
\frac{\partial \Gamma(x, t)}{\partial t} = K \frac{\partial^2 \Gamma(x, t)}{\partial x^2} + A_1 w(x), \tag{7}
\]

where the second term on the right-hand side of Eq. (7) describes the increase of supersaturation during cloud volume ascent. Since the main two quantities in the model are \( \Gamma(x, t) \) and \( K \), the minimalistic model is referred to as \( \Gamma-K \) model. Equation (7) is in accordance with the balance equation for a vertically moving adiabatic parcel. Indeed, neglecting the mixing term in Eq. (7) (i.e., setting the turbulent coefficient \( K \) equal to zero) and subsequent integration over time leads to the balance Eq. (3). We would like emphasize that solution of Eq. (7), characterizing phase transition in the course of cloud volume ascending, depends on horizontal coordinate \( x \). It means that the solution reflects microphysical structure inside the cloud volume in contrast to other entrainment-mixing models (e.g., Barahona and Nenes 2007).

We set the horizontal profile of the vertical velocity \( w(x) \) as a rectangle function:

\[
w(x) = \begin{cases} 
          w_1 & \text{if } |x| \leq L \\
          0 & \text{if } |x| > L,
        \end{cases} \tag{8}
\]

where \( w_1 > 0 \) corresponds to an updraft of cloud volume. The velocity of the surrounding air is equal to zero; that is, the cloud rises or descends within an unmovable environment. We also determine the initial conditions for function \( \Gamma(x, t) \) in the form of a rectangle function:

\[
\Gamma(x, 0) = \begin{cases} 
          \Gamma_1 & \text{if } |x| \leq L \\
          \Gamma_2 & \text{if } |x| > L,
        \end{cases} \tag{9}
\]

The initial conditions for a developing cloud are illustrated in Fig. 2. At cloud base, the LWC and supersaturation are equal to zero, so inside the cloud \( \Gamma(x, t = 0) = 0 \). Unsaturated air around the cloud is
characterized by negative supersaturation $S_2$ (i.e., $\Gamma_2 < 0$). These initial conditions are in accordance with Eq. (4).

\[
\Gamma(x, t) = \frac{1}{2} \left\{ 2\Gamma_2 + (\Gamma_1 - \Gamma_2) \left[ \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) - \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right] \right. \\
+ \frac{A_1 w_1}{2} \left\{ \frac{- (x + L)^2}{2K} \text{sgn}(x + L) + \frac{x + L}{\sqrt{\pi K}} \sqrt{t} \exp \left[ - \frac{(x + L)^2}{4Kt} \right] + \left[ t + \frac{(x + L)^2}{2K} \right] \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) \\
+ \frac{(x - L)^2}{2K} \text{sgn}(x - L) - \frac{(x - L)}{\sqrt{\pi K}} \sqrt{t} \exp \left[ - \frac{(x - L)^2}{4Kt} \right] + \left[ t + \frac{(x - L)^2}{2K} \right] \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right\} ,
\]

(10)

where \( \text{erf}(y) \) is the error function (see the solution in the appendix). Since the cloud volume rises at a constant velocity, one can consider the function of the two spatial coordinates \( \Gamma(x, z) \), where \( z = z_0 + w_1 t \) is the vertical coordinate, instead of function \( \Gamma(x, t) \). The solution in Eq. (10) represents a function symmetrical with respect to the vertical coordinate, instead of function \( \Gamma(x, t) \).

An analytical solution of Eq. (7) for the horizontal profile of the updraft velocity given by Eq. (8) with the initial conditions (9) has the form

\[
\Gamma[0, t] = \left\{ \Gamma_2 + (\Gamma_1 - \Gamma_2) \text{erf} \left( \frac{L}{2\sqrt{Kt}} \right) \right. \\
+ A_1 w_1 \left\{ \frac{- L^2}{2K} + \frac{L^2}{\sqrt{\pi K}} \sqrt{t} \exp \left[ - \frac{L^2}{4Kt} \right] + \left( t + \frac{L^2}{2K} \right) \text{erf} \left( \frac{L}{2\sqrt{Kt}} \right) \right\} .
\]

(13)

The zone where \( \Gamma_2 < \Gamma(x, z) < 0 \) is the zone of the humid shell, that is, of increased humidity (Pinsky and Khain 2018b).

We also present here two particular solutions of Eq. (10), which will be used in this study. The first one corresponds to the case of unmovable cloud volume \( w_1 = 0 \) when Eq. (10) takes the form

\[
\Gamma(x, t) = \frac{1}{2} \left\{ 2\Gamma_2 + (\Gamma_1 - \Gamma_2) \left[ \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) - \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right] \right. \\
+ \frac{A_1 w_1}{2} \left\{ \frac{- (x + L)^2}{2K} \text{sgn}(x + L) + \frac{x + L}{\sqrt{\pi K}} \sqrt{t} \exp \left[ - \frac{(x + L)^2}{4Kt} \right] + \left[ t + \frac{(x + L)^2}{2K} \right] \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) \\
+ \frac{(x - L)^2}{2K} \text{sgn}(x - L) - \frac{(x - L)}{\sqrt{\pi K}} \sqrt{t} \exp \left[ - \frac{(x - L)^2}{4Kt} \right] + \left[ t + \frac{(x - L)^2}{2K} \right] \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right\} ,
\]

(12)

This equation is analogous to Eq. (6) presented by Pinsky and Khain (2018b) in which mixing at a lateral cloud boundary was considered. The second one is the solution on the axis \( x = 0 \), that is, \( \Gamma(0, t) \) written as

\[
q(x, z) = \frac{1}{A_2} \Gamma(x, z)
\]

already at levels higher than \( \approx 200 \) m above cloud base. Although the accuracy of expression (11) decreases toward cloud edge, we apply Eq. (11) within the whole interval \( |x| \leq L \). Since \( q(x, z) > 0 \) in clouds, the cloud boundary corresponds to \( \Gamma(x, z) = 0 \). The statement that condition \( \Gamma(x, z) = 0 \) corresponds to cloud boundary is not exact. It is to certain extent justified by the following: (i) the condition \( (S = 0 \text{ and } q = 0) \) is consistent with the condition \( \Gamma = 0 \), and (ii) supersaturation in the vicinity of cloud boundary is close to zero (Pinsky and Khain 2018b). The relative error in determination of cloud boundary position using the condition \( \Gamma(x, z) = 0 \) can be estimated as \( \approx 13\% \) of thickness of the cloud dilution zone [see Fig. 4 in Pinsky and Khain (2018b)].
\( w_1 \), the relative humidity of surrounding air \( \text{RH}_2 = 1 + S_2 \), and temperature \( T \) (via coefficients \( A_1 \) and \( A_2 \)).

### 3. Analysis of an ascending cloud volume

In this section, we discuss the obtained results related to the evolution of cloud parcel microphysics. The evaluations will be performed for parameters typical of real nonprecipitating small cumulus clouds. The width of \( \text{Cu} \) varies from about 100 to 1500–2000 m, the vertical velocities in growing clouds range from 1 to 6 m s\(^{-1} \), with a typical value of 2 m s\(^{-1} \) (Gerber 2000; Gerber et al. 2008; Arabas et al. 2009; Katzwinkel et al. 2014; Kumar et al. 2017; Schmeissner et al. 2015). The values of turbulent kinetic energy dissipation rate \( e \) range from 10 to about 100 cm\(^2\) s\(^{-3} \) (Gerber et al. 2008; Katzwinkel et al. 2014; Schmeissner et al. 2015). Typically, the value of the turbulent coefficient is higher for larger clouds. Equation (6) allows us to evaluate the values of \( K \), which is ranged from about 1 to 20 m\(^2\) s\(^{-1}\).

#### a. Evolution of liquid water mixing ratio

Figure 3 illustrates the evolution of cloud structure, characterized by function \( q(x, z) \approx (1/A_2)\Gamma(x, z) \) calculated for three values of turbulent coefficient values \( K = 1 \text{ m}^2\text{s}^{-1}, \ K = 5 \text{ m}^2\text{s}^{-1} \), and \( K = 20 \text{ m}^2\text{s}^{-1} \) for two cloud widths of 400 and 1000 m at a vertical velocity of 2 m s\(^{-1}\).

In all the cases, liquid water mixing ratio increases with height; its maximum is reached at the cloud center. Influence of mixing at cloud boundaries is manifested in the decrease of the horizontal gradients of the liquid water mixing ratio, which indicates both a decrease of the liquid water mixing ratio maximum and cloud broadening (compare, for instance, contours \( q = 0.5 \text{ g kg}^{-1} \) at different \( K \)). For higher values of the turbulent coefficient, the cloud becomes more blurred with lower gradients at cloud edges. The cloud dilution (that can be characterized by a decrease in the liquid water mixing ratio as compared to the case of very low \( K = 1 \text{ m}^2\text{s}^{-1} \)) dramatically depends on the cloud width. While in case \( 2L = 1000 \text{ m} \) the cloud center remains undiluted up to heights of \( z = 3 \text{ km} \) at \( K = 20 \text{ m}^2\text{s}^{-1} \), in case of \( 2L = 400 \text{ m} \) mixing begins affecting the cloud center already at \( z = 700 \text{ m} \) above cloud base. It is remarkable that experimental measurement by Gerber et al. (2008) showed a deviation of the LWC from the adiabatic value in cloud of 400-m width at 500 m above cloud base, which agrees well with our results.

To better illustrate the horizontal structure of a cloud volume against the background of its environment, we present Fig. 4 showing the horizontal profiles of quantity \( \Gamma(x)/A_2 \) at fixed heights of 350 and 1000 m above cloud base (Figs. 4a–d). The condition \( \Gamma(x, z)/A_2 = 0 \) correspond to the cloud edge with the accuracy of a few meters. Under the assumption that this condition is valid, the positive values of quantity \( \Gamma(x, z)/A_2 \) correspond to liquid water mixing ratio, while the area of negative values is related to the humid shell around the cloud. The tails of the curves correspond to supersaturation of the environment air normalized on \( A_2 \).

Figures 4a–d show that the shapes of the horizontal profiles change from rectangular to the bell-like shape as the value of the turbulent coefficient \( K \) increases and the cloud width decreases. At \( z = 350 \text{ m} \) (Figs. 4a,b), the maximum values of \( \Gamma(x)/A_2 \) reached at the cloud center do not differ anyhow substantially from the case of negligibly weak mixing, that is, do not differ much from the adiabatic values for both 400- and 1000-m cloud widths. In clouds of 400-m width, the bell-shaped profile forms at \( z = 350 \text{ m} \) above cloud base only in case \( K = 20 \text{ m}^2\text{s}^{-1} \) (Fig. 4a). At \( z = 1000 \text{ m} \) there is a significant difference in the profiles depending on the cloud width. In the 1000-m-width cloud the initial rectangular profile does not change much even at \( K = 20 \text{ m}^2\text{s}^{-1} \). This result suggests that entrainment and mixing do not affect the cloud core of large clouds during their growing stage. In such clouds, the bell-like shape forms later, at the mature stage of cloud evolution. The profiles in Figs. 4a–d are plotted for \( w_1 = 2 \text{ m s}^{-1} \). A stronger mixing effect can be expected in cases of lower vertical velocity, when the updraft time is longer and mixing extends farther toward the cloud center.

We also present Figures 4e and 4f for comparison with measurements. Figure 4e taken from study by Heus and Jonker (2008) shows the horizontal in-cloud profiles of the total water content (including LWC and the mass of water vapor) averaged over all flights in different field investigations of small Cu (see also Rodts et al. 2003); Fig. 4f taken from the same study shows the horizontal profiles of the total water content in individual flights, measured in different small Cu. In Fig. 4e the mean value to the left of the cloud is subtracted. The cloud is centered at zero \( x \) coordinate and scaled with the cloud of size \( L_c \). The bars denote the root-mean-square spread of individual measurements. Only clouds with sizes exceeding 500 m were taken into account. Measurements were performed at about 1000 m above cloud base.

Comparison of the horizontal profiles obtained by the \( \Gamma-K \) model at \( z = 1000 \text{ m} \) (Figs.4c–d) with the measured profiles of the total water mixing ratio \( q_t \) (Fig. 4e) indicates a reasonable agreement between the values of \( q_t \) within the cloud and the environment: about 3.5–3.7 g kg\(^{-1}\) by the model versus 3 g kg\(^{-1}\) in the observations. Both Figs. 4e and 4d show significant variations of differences in the values of \( q_t \) in the cloud versus the
surrounding air. Figure 4d shows that this difference is caused largely by variations of absolute humidity in the cloud environment as found in various observations of small Cu, and varies from 2 to 5 g kg\(^{-1}\). In the model the environmental humidity is prescribed initially, so such variations in the \(q_t\) difference takes place due to change of LWC with the height (2.2 g kg\(^{-1}\) at \(z = 350\) m and 3.7 g kg\(^{-1}\) at \(z = 1000\) m). A reasonable similarity in the shapes of the profiles for the values of \(K = 10–20\) m\(^2\) s\(^{-1}\) is noteworthy, especially for clouds with large width (the experimental profiles are plotted for clouds with widths exceeding 500 m). The horizontal profiles of total water content in measurements are symmetric, which indicates a spatial homogeneity of the cloud environment, as

Fig. 3. Spatial distribution of the liquid water mixing ratio \(q(x, z)\) in cloud parcels at different values of the turbulent coefficients \(K\). Other parameters of the parcels are as follows: (a)–(c) \(2L = 400\) m and (d)–(f) \(2L = 1000\) m; \(T = 20^\circ\text{C}, w_1 = 2\) m s\(^{-1}\), and \(S_2 = -40\%\).
Fig. 4. The horizontal profiles of quantity $\Gamma(x)/A_2$ at the heights of (a),(b) 350 and (c),(d) 1000 m above cloud base at different values of the turbulent coefficients $K$. The cloud parameters are as follows: (a),(c) $2L = 400$ and (b),(d) $1000$ m; $T = 20^\circ$C; $w_1 = 2$ m s$^{-1}$, and $S_2 = -40\%$. (e),(f) From Heus and Jonker (2008), showing (e) the horizontal in-cloud profiles of the total water content (including LWC and the mass of water vapor) averaged over all flights in different field investigations of small Cu (see also Rodts et al. 2003) and (f) the horizontal profiles of the total water content in individual flights, measured in different small Cu. In (e) the mean value left of the cloud is subtracted. The cloud is centered at the zero $x$ coordinate and scaled with the cloud of size $L_c$. The bars denote the root-mean-square spread of the individual measurements. Only clouds with sizes exceeding 500 m were taken into account. Measurements were performed at about 1000 m above cloud base.
assumed in the \( \Gamma-K \) model. The vertical profiles of maximum LWC values obtained by the \( \Gamma-K \) model also agree well with the profiles of LESs (Khain et al. 2019).

As was mentioned above, negative values of \( \frac{\Gamma(x)}{A_2^2} \) can be interpreted as values of supersaturation in the humid shell where the humidity around the cloud increases due to turbulent diffusion and droplet evaporation. The horizontal parts of the negative tails correspond to the relative environmental humidity of 60%.

The width of the humid shell varies from several tens of meters at weak turbulence to a few hundred meters in case of \( K = 10 \text{ m}^2 \text{s}^{-1} \), which agrees well with the results of more detailed calculations performed by Pinsky and Khain (2018b, 2019).

We conclude, therefore, that the simple analytic solution in (10) provides realistic evaluations of the structure of a growing cloud and reflects well the comparative contribution of turbulent mixing and adiabatic ascent processes in formation of cloud microstructure.

**b. Adiabatic fraction**

There are both theoretical and practical questions regarding the role of adiabatic processes in clouds and the comparative contribution of adiabatic processes and mixing in formation of cloud microphysical structure. The measure of this contribution as well as the measure of cloud dilution is the value of the adiabatic fraction (AF) determined as the ratio of the liquid water mixing ratio to its adiabatic value: \( \text{AF}(x, z) = \frac{q(x, z)}{q_{\text{ad}}(z)} \). In our case AF is defined as a ratio of LWC in cloud to LWC, calculated in neglect by mixing between cloud and its environment. It is clear that \( \text{AF}(x, z) \leq 1 \).

Figures 3 and 4 show that the liquid water mixing ratio is distributed nonuniformly inside the cloud volume, reaching the maximum values at the cloud center and decreasing down to zero at lateral boundaries. The maximum possible value at the cloud center corresponds to the maximum value of the adiabatic fraction. We calculate the adiabatic vertical profile of the liquid water mixing ratio using Eq. (3) (see also Pinsky et al. 2013):

\[
q_{\text{ad}}(z) = \frac{A_1}{A_2} w_1 t = \frac{A_1}{A_2} z. \tag{14}
\]

In our study, coefficients \( A_1 \) and \( A_2 \) are assumed to be constant, so \( q_{\text{ad}}(z) \) is a linear function of distance above cloud base, which leads to some overestimation of the AF at significant distances above cloud base. Evaluating maximum adiabatic fraction at the cloud center using Eqs. (13) and (14) \( \text{AF}(z) = \text{AF}(0, z) \), we obtain

\[
\text{AF}(z) = \frac{1}{t} \left( \frac{S_2}{A_1 w_1} - \frac{L^2}{2K} \right) \text{erfc} \left( \frac{L}{2\sqrt{Kt}} \right) + \frac{L}{\sqrt{\pi Kt}} \exp \left( - \frac{L^2}{4Kt} \right) + \text{erf} \left( \frac{L}{2\sqrt{Kt}} \right), \tag{15}
\]

where \( t = z/w_1 \) and \( \text{erf}(y) \) is the complementary error function. The vertical profiles of the adiabatic fraction calculated for different values of the turbulent coefficient are shown in Fig. 5.

One can see that the adiabatic fraction decreases with increasing height, since dilution with dry air accumulates as the cloud volume ascends. The value of the turbulent coefficient has a crucial effect on AF. At \( K = 1 \text{ m}^2 \text{s}^{-1} \), there practically is no visible dilution effect regardless the cloud width. In the 400-m-width cloud, the adiabatic fraction deviates significantly from unity at the height of 300–500 m above cloud base at \( K = 10 \text{ m}^2 \text{s}^{-1} \) (Fig. 5a). A significant decrease of AF in Cu with widths of a few hundred meters at such heights was observed in several

---

**FIG. 5.** Vertical profiles of the maximum adiabatic fraction \( \text{AF}(z) \) at the cloud axis for different values of the turbulent coefficient \( K \). The cloud parameters are as follows: (a) \( 2L = 400 \) and (b) \( 1000 \) m; \( T = 20^\circ \text{C} \); \( w_1 = 2 \text{ m} \text{s}^{-1} \), and \( S_2 = -40\% \).
studies (Gerber 2000; Gerber et al. 2008; Schmeissner et al. 2015). According to Schmeissner et al. (2015), the maximum values of AF in Cu measured in the Cloud, Aerosol, Radiation and Turbulence in the Trade Wind Regime over Barbados (CARRIBA) field experiment exceeds about 0.6 at 100 m below cloud top. Since the cloud top was at about 2000 m high, the best agreement with the observations is reached at $K = 20 \text{ m}^2 \text{s}^{-1}$ (Fig. 5a).

In the cloud of 1000-m width the core remains actually undiluted at $K \leq 10 \text{ m}^2 \text{s}^{-1}$ (Fig. 5b). At $K = 20 \text{ m}^2 \text{s}^{-1}$, the AF slightly deviates from unity at 1000 m above the cloud base. But even at this value of turbulent coefficient, the maximum value of AF remains very close to unity. The high values of AF of 0.93 were measured in deep convective cloud of about 1200-m width at the height of 1500 m above cloud base by Prabha et al. (2011) and also reported by Khain et al. (2013). The existence of a nondiluted cloud core at 1000 m above cloud base in Cu of 2000-m width was simulated in LESs (Khain et al. 2019). Again, the value $K = 20 \text{ m}^2 \text{s}^{-1}$ seems to provide the best agreement with the observations. The shapes of the horizontal profiles of the adiabatic fraction AF($x$) are similar to those of the mixing ratio shown in the Fig. 4. At vertical velocities lower than 2 m s$^{-1}$, the AF deviates from unity at lower distances above cloud base and the deviation is stronger than that shown in Fig. 5. Therefore, our results demonstrate that even a very simplified the Γ–K model that considers only the growth of liquid water content by condensation and its decrease as a result of mixing allows getting reasonable estimations comparable with observations, at reasonable values of the turbulence intensity (the turbulent coefficients).

Hence, two processes control cloud growth: an adiabatic ascent of the volume leading to increase in LWC, and mixing of the rising volume with dry surrounding air, preventing the LWC increase. One can expect that in case of intensive mixing and small updraft velocity the cloud cannot develop at all. Equation (15) allows us to formulate the approximate condition for cloud development, namely, that a cloud will develop if the minimum value of function AF($z$) is positive. In this case, liquid water appears at the cloud center during the volume ascent. Using simplifying condition $\gamma = L/(2\sqrt{Kt}) \gg 1$ [which leads to an approximate representation $\text{erfc}(\gamma) = \exp(-\gamma^2)/\gamma\sqrt{\pi}$] allows one to reduce the inequality $\min[AF(z)] > 0$ to the condition

$$w_1 > w_{cr} = -C_1\frac{KS}{A_1L^2},$$

where $C_1 = 1.72$. Equation (16) shows that the minimum ascent velocity required decreases when the RH of the environment and cloud size increase. This velocity also decreases with decreasing mixing intensity. For a cloud with parameters $L = 200 \text{ m}$, $T = 20^\circ \text{C}$ and $S_2 = -40\%$, the velocity is $w_{cr} = 0.18 \text{ m s}^{-1}$ at $K = 5 \text{ m}^2 \text{s}^{-1}$ and 0.71 m s$^{-1}$ at $K = 20 \text{ m}^2 \text{s}^{-1}$. Turbulent plumes of several tens of meters in size can trigger cloud development only in a very humid surrounding.

c. Estimation of droplet concentration and the mean volume droplet radius

Two highly important microphysical parameters characterizing clouds are droplet concentration $N$ and the mean volume droplet radius $r_v$, which is a measure of the mean droplet size. These parameters are included in any expression of droplet size distributions used in bulk-parameterization schemes. The values of these parameters are the measure of aerosol effects on cloud microphysics, since an increase in aerosol concentration leads to increasing $N$, and decreasing $r_v$. A major parameter of cloud microphysics is the effective radius of droplet size distribution $r_{eff}$. The value of this parameter is necessary to evaluate cloud radiative properties. The mean volume radius is approximately linearly related to the effective radius, being about 10% lower (Martin et al. 1994; Brenguier et al. 2011; Freud and Rosenfeld 2012; Khain et al. 2019), which allows one to apply $r_v$ instead $r_{eff}$ in calculation of cloud radiative properties.

In large-scale models, Cu are nonresolvable and the decrease in $N$ with height due to entrainment and mixing is prescribed approximately, for example, by the exponential function (Khain et al. 2019). The crudeness of parameterization of liquid water mixing ratio and droplet concentration in Cu in the large-scale models leads to significant errors in determination of $r_v$ and other cloud properties. LESs show a strong dependence of $N$ and $r_v$, as well as their profiles, on aerosol concentration, environment humidity and atmospheric instability (i.e., the vertical velocity) (Khain et al. 2019). In this section of the paper, we reproduce the main features of $N$ and $r_v$ behavior in developing Cu by using the Γ–K model.

In our calculations of droplet concentration we neglect the decrease in droplet concentration caused by complete droplet evaporation in an ascending cloud volume experiencing mixing with environment, so the sole process leading to a decrease in droplet concentration inside cloud is turbulent mixing. The accuracy of calculations under this simplification is higher for the cloud core and decreases toward cloud edges. The comparison of the horizontal profiles of liquid water mixing ratio in Figs. 4a–d shows that the liquid water mixing ratio increases with height within
the segment $|x/L| < 1$, which likely can be interpreted as the existence of positive supersaturation and the absence of complete droplet evaporation within this zone.

The largest relative errors in droplet concentration calculations caused by neglecting complete droplet evaporation are expected near the cloud edges, where the droplet concentration is much lower than that within the cloud body. Crude evaluations of the relative error in the droplet concentration values at the external boundary of the cloud dilution zone (made under assumption $w = 0$) show that it may reach $50\%-100\%$ [see Fig. 7c in Pinsky and Khain (2018b)]. Despite this fact, we believe that the results discussed below concerning the cloud body correctly reflect the nature of a cloud at the developing stage. If complete droplet evaporation is neglected, the droplet concentration decreases only due to diffusion of droplets from the cloud. If mixing does not reach the cloud core, the droplet concentration remains constant. This assumption for Cu is supported by the measurements in CARRIBA field experiment (Schmeissner et al. 2015) as well as by LESs (Khain et al. 2019), where the maximum values of droplet concentration in the growing Cu were found to be close to the droplet concentration at cloud base.

In case droplet collisions and total droplet evaporation are neglected, the droplet concentration can be described by an equation similar to Eq. (2):

$$\frac{\partial N(x,t)}{\partial t} = K \frac{\partial^2 N(x,t)}{\partial x^2}. \quad (17)$$

This equation should be solved with the initial conditions given in the form of a rectangle function:

$$N(x,0) = \begin{cases} N_1 & \text{if } |x| \leq L \\ 0 & \text{if } |x| > L \end{cases} \quad (18)$$

where $N_1$ is the droplet concentration just after nucleation at the cloud base. Equations (17) and (18) lead to the solution

$$N(x,t) = \frac{N_1}{2} \left[ \text{erf} \left( \frac{x+L}{2\sqrt{Kt}} \right) - \text{erf} \left( \frac{x-L}{2\sqrt{Kt}} \right) \right]. \quad (19)$$

In a particular case corresponding to cloud center ($x = 0$), the solution is

$$N(0,t) = N_1 \text{erf} \left( \frac{L}{2\sqrt{Kt}} \right). \quad (20)$$

where $t = z/w_1$.

The vertical profiles of droplet concentration at the cloud axis at $2L = 400\, \text{m}$ and of $2L = 1000\, \text{m}$ are shown in Fig. 6. The initial droplet concentration at cloud base was set to $500\, \text{cm}^{-3}$. The contours of the droplet concentration field obtained using Eq. (19) at different values of the turbulent coefficient $K$ and cloud widths of $400$ and $1000\, \text{m}$ are shown in Fig. 7.

Comparison of the droplet concentration profiles shown in Fig. 6 with those of AF (Fig. 5) shows a significant similarity, since the decrease of AF($z$) and $N(z)$ at the cloud axis is caused by the same factor, namely, turbulent diffusion. Droplet concentration increases toward cloud center and decreases with increasing height. Increasing turbulent coefficient values lead to a decrease in droplet concentration within cloud body and causes cloud spreading. It is interesting that isolines $N = 200\, \text{cm}^{-3}$ are close to $|x/L| = 1$ in all the panels of Fig. 7. Increasing $K$ values and decreasing cloud
width lead to a decrease in the droplet concentration at $|x/L| < 1$, while increasing it at $|x/L| > 1$. The increase in the droplet concentration at $|x/L| > 1$ indicates cloud broadening. However, the shell of reduced droplet concentration at $|x/L| > 1$ is wider than that of liquid water mixing ratio. This is due to neglecting complete droplet evaporation. Nevertheless we believe that the obtained profiles agree well enough with the observational data and can be used to evaluate spatial changes of the droplet mean volume radius $r_v(x, z)$ at least inside a cloud at $|x| < L$. The reliability of these evaluations increases when considering wider clouds having larger liquid water content.

Evaluations of the droplet mean volume radius can be performed using its relationship with the liquid water mixing ratio $q(x, z) = (4\pi \rho_w/3\rho_a)N(x, z)r_v^3(x, z)$, where $\rho_w$ and $\rho_a$ are the densities of water and air, respectively. We obtain

![Figure 7. Spatial distribution of droplet concentration $N(x, z)$ at different values of the turbulent coefficients $K$. The cloud widths are (a)-(c) $2L = 400$ m and (d)-(f) $2L = 1000$ m. Other cloud parameters are $N_1 = 500$ cm$^{-3}$, $T = 20^\circ$C; $w_1 = 2$ m s$^{-1}$, and $S_2 = -40\%$.](image-url)
Figure 8 shows the profiles of the maximum values of $r_y$ at cloud axis in the $\Gamma$–K model (dashed and dotted black curves) at (a),(b) $N_1 = 100$ cm$^{-3}$ and (c),(d) $N_1 = 1000$ cm$^{-3}$ for updraft velocities of (a),(c) 0.5 and (b),(d) 2.0 m s$^{-1}$. The cloud parameters are as follows: $2L = 400$ m, $T = 20^\circ$C, and $S_2 = -40\%$. Solid blue curves show the profiles of $r_{y,\text{max}}(z)$ for the adiabatic case [profile $q_{\text{ad}}(z)$ is calculated using Eq. (14) and $N(z) = N_1 = \text{const}$]. Solid red lines in (b) and (d) show the maximum values of the mean volume radius obtained by horizontal averaging over a field of Cu in an LES (Khain et al. 2019).

\begin{equation}
\frac{r_y(x, z)}{N(x, z)} = \frac{1}{4\pi\rho\nu} \left(\frac{2\rho_x}{q(x, z)}\right)^{1/3}.
\end{equation}

Figure 8 shows the profiles of the maximum values of $r_y$ at cloud axis for $2L = 400$ m and the initial droplet concentrations at cloud base $N_1 = 100$ (maritime clouds) and 1000 cm$^{-3}$ (continental clouds), at updraft velocities of 0.5 and 2 m s$^{-1}$. In addition, Figs. 8a and 8c show the horizontally averaged values of $r_y$ observed in field experiments and obtained in the LESs at both low and high values of droplet concentration.

Analysis of results obtained by the $\Gamma$–K model, presented in Fig. 8, shows the following:

(i) The maximum values of $r_{y,\text{max}}(z)$ are lower at higher droplet concentration, which is in agreement with observations.

(ii) The vertical profiles of $r_{y,\text{max}}(z)$ significantly depend on the vertical velocity. At a comparatively large velocity (like 2 m s$^{-1}$ in Figs. 8b and 8d), the vertical profiles of $r_{y,\text{max}}(z)$ at the cloud axis coincide with the adiabatic ones even for the 400-m-width cloud. It is clear that for larger cloud widths and significantly higher updraft velocity $r_{y,\text{max}}(z)$ should remain closer to its adiabatic value. This result obtained by the $\Gamma$–K model coincides with the LESs (Khain et al. 2019) (solid red lines in Figs. 8b and 8d) and the in situ observations (Schmeissner et al. 2015).

(iii) Decreasing updraft velocity increases the role of mixing at the cloud growing stage. The maximum deviation from the adiabatic value is observed at $K = 20$ m$^2$ s$^{-1}$.

(iv) The closeness of $r_{y,\text{max}}(z)$ to its adiabatic profile can be attributed to the fact that mixing synchronously decreases both AF and N.

To compare the results obtained using the $\Gamma$–K model and the LES results, we present Fig. 9 showing the vertical profiles of the mean volume radius, averaged in the horizontal direction over the area where $q(z) > 0.1$ g kg$^{-1}$. The conditions of the simulations are similar to those shown in Fig. 8. The vertical profiles of $r_y(z)$ obtained by the horizontal averaging over the Cu fields in some field experiments and in the LES (Khain et al. 2019).
are also shown in Fig. 9. Figure 10 shows the horizontal profiles of the droplet mean volume radius \(r_v(x)\) at the height of 1000 m above cloud base at different updrafts, droplet concentrations and turbulence intensities.

Analysis of Figs. 9 and 10 shows that at low vertical velocities (0.5 m s\(^{-1}\); Figs. 9a,c) turbulent mixing decreases \(r_v(z)\) much stronger than at moderate velocity (2 m s\(^{-1}\); Figs. 9b,d). Comparison of profiles \(r_{v,\text{max}}(z)\) and \(r_v(z)\) shows that \(r_v(z)\) is quite close to \(r_{v,\text{max}}(z)\). The difference \(r_{v,\text{max}}(z) - r_v(z)\) increases with height and reaches a few micrometers at 1500 m above cloud base.

The averaged profiles \(\bar{r}_v(z)\) obtained using the \(\Gamma-K\) model are close to those obtained in measurements and the LES. The influence of droplet concentration on profiles \(\bar{r}_v(z)\) is stronger than that of turbulence intensity and even of cloud width. Indeed, \(\bar{r}_v(z)\) changes by 10\%–20\% as can be caused by changes of \(K\) values from 1 to 20 m\(^2\) s\(^{-1}\) (the possible range of \(K\) values), as well as by changes of cloud width from 400 to 1000 m (the typical range of Cu widths). At the same time, \(\bar{r}_v(z)\) decreases by half when droplet concentration increases from 100 to 1000 cm\(^{-3}\). These results show that the profiles \(\bar{r}_v(z)\) are determined largely by droplet concentration. Again, it appears that the value \(K = 20\) m\(^2\) s\(^{-1}\) provides the best agreement with the observations.

Figure 10 shows that the droplet mean volume radius reaches its maximum in the cloud center and decreases toward cloud edges. At higher vertical velocities, the shapes of the horizontal profiles \(r_v(x)\) are closer to rectangular. In wider clouds, the profiles \(r_v(x)\) are also closer to rectangular (see Fig. 10e). Within the zone \(|x/L| < 0.5\) (the cloud center) the value of \(r_v(x)\) changes by 5\%–10\%.

The decrease of \(r_v(x)\) near cloud edges is overestimated in the \(\Gamma-K\) model as a result of overestimated droplet concentration, as discussed above. Comparison of horizontal profiles of LWC (Figs. 4a,c) and the mean volume radius (Figs. 10a–d) shows that the mean volume radius remains nearly unchanged, while LWC dramatically decreases toward the cloud edge. This result agrees with the recent
study by Yang et al. (2019), where the ARM shortwave spectrometer data were analyzed.

The fact that $r_v(z)$ profiles are smooth and values of $r_v(z)$ averaged over multiple clouds differ insignificantly from the maximum values of $r_v(z)$, indicate a low horizontal variability of the droplet mean volume radius within clouds. At comparatively high vertical velocities, $r_{v,max}(z)$ is close to its adiabatic

Fig. 10. Horizontal profiles of the mean volume radius $r_v(x)$ at the height of 1000 m above cloud base at different values of the turbulent coefficients $K$ and droplet concentrations of (a),(b) $N_1 = 100$ and (c),(d) 1000 cm$^{-3}$ for updraft velocities of (a),(c) 0.5 and (b),(d) 2.0 m s$^{-1}$. The cloud parameters are: $2L = 400$ m, $T = 20^\circ$C, and $S_1 = -40\%$. (e) As in (c), but for $2L = 1000$ m. The figure is based on calculations using only reliable radii, whose values correspond to the liquid water mixing ratio of $q(x, z) > 0.1$ g kg$^{-1}$. 
value (Fig. 8). Accordingly, for clouds containing adiabatic core \( r_c(z) \) is also close to its adiabatic profile and depends solely on droplet concentration. This closeness of \( r_c(z) \) to the adiabatic profile is supported by observations of Cu, especially at those with significant vertical velocity (>2 m s\(^{-1}\)) (Freud and Rosenfeld 2012; Prabha et al. 2011; Khain et al. 2013). In addition, according to observations (Kumar et al. 2017), the shape of \( r_c(x) \) is close to trapezoidal with a quite sharp decrease of \( r_c \) for instance from 10 down to ~5 \( \mu m \), within a few hundred meters near the edge of a developing convective cloud.

### 4. Discussion and conclusions

In the study we propose a simple model of an ascending cloud volume in which only two processes are considered, namely droplet growth due to condensation and the lateral turbulent mixing with dry environment (the \( \Gamma-K \) model). The entrainment and turbulent mixing are described by the turbulent diffusion equation in which the turbulent fluxes are assumed to be proportional to the gradients of the quantities. The coefficient of proportionality (i.e., the turbulent coefficient) is evaluated from the Richardson law using both theoretical and observed data as regards to the values of the external turbulent scale and the dissipation rate. The governing parameters of the \( \Gamma-K \) model are the width of an ascending volume, its updraft velocity, relative humidity and temperature of the environment air, and turbulent intensity (turbulent coefficient). The initial values of the liquid water mixing ratio and droplet concentration in the ascending air volume are prescribed. The cloud volume is initially assumed saturated.

The analytical solution was obtained that allows us to calculate the vertical and horizontal changes of the liquid water mixing ratio, the adiabatic fraction, the droplet concentration and the droplet mean volume radius in an ascending volume. The effect of turbulent mixing on the cloud microstructure at the stage of cloud development is investigated by assuming different values of the turbulent diffusion coefficient, taken from the reasonable range of values. The effect was found to strongly depend on the cloud width and vertical velocity.

In small Cu with width of a few hundred meters and low vertical velocity, the first effects of mixing on the cloud core appear at a few hundred meters above the cloud base, decreasing the liquid water content, the adiabatic fraction and droplet concentration. At vertical velocity equal to or higher than 2 m s\(^{-1}\) and cloud width of \( \approx 1000 \) m, the cloud core remains undiluted at least up to 2 km above cloud base. It seems that the effect of mixing on the cloud core becomes pronounced in such clouds at later stages, when the vertical velocity becomes low and even negative.

Analysis of the solution and its comparison with observed data and the LES results show that the turbulent coefficient values of 10-20 m\(^2\) s\(^{-1}\) provide results that are closest to the experimental data. The value \( K = 1 \) m\(^2\) s\(^{-1}\) seems to be too small to affect the cloud structure anyhow significantly. It means that discussions concerning the existence or absence of adiabatic (undiluted) volumes in a cloud should include the analysis of such parameters as cloud size and vertical velocity, as well as turbulence intensity.

At vertical velocities of about 2 m s\(^{-1}\) and higher, the maximum values of the mean volume radius are close to the adiabatic values even in clouds of 400-m width. At high vertical velocity and large cloud width, there is not enough time for mixing to affect the air ascending in the cloud core. Such clouds are affected by mixing at later stages of cloud evolution. At lower vertical velocities and narrow clouds, high values of the mean volume radius can be attributed to a synchronous decrease in liquid water mixing ratio and droplet concentration. The values of the mean volume radius within cloud body change only slightly, especially at significant vertical velocities (\( \approx 2 \) m s\(^{-1}\)) and cloud widths (\( >400 \) m). The cloud-averaged values calculated for the area where \( q > 0.1 \) g kg\(^{-1}\), are only 1-2 \( \mu m \) less than the maximum values. This relatively small difference is also related to the synchronous changes in both the cloud water mixing ratio and droplet concentration. The strongest decrease takes place near cloud edges. These results agree with in situ observations and the LES results.

By neglecting complete droplet evaporation and droplet coalescence, the \( \Gamma-K \) model overestimates droplet concentration and underestimates the mean volume radii in the vicinity of cloud boundaries. Hence, the horizontal variability of the mean volume radius in reality should be even less than predicted by the \( \Gamma-K \) model.

The results obtained by the \( \Gamma-K \) model support the conclusion that fields of Cu can be characterized by a single robust vertical profile of the mean volume radius (which is close to the adiabatic value) that depends on the aerosol concentration. This conclusion is implicitly assumed in practical applications (Rosenfeld et al. 2014, 2016; Grosvenor et al. 2018) and derived in LES experiments (Khain et al. 2019). This result was also obtained in observations (e.g., Freud et al. 2008). Application of the \( \Gamma-K \) model clearly demonstrates the mechanism of the humid shell formation around clouds due to turbulent mixing between a cloud and its environment.

The condition for a cloud formation in updraft is formulated. It is shown that a cloud can arise when the updraft velocity exceeds a certain critical value. These critical values are of about 0.2 m s\(^{-1}\) for clouds of 1000-m width and of 0.7 m s\(^{-1}\) for plumes of 400-m width. Smaller turbulent plumes should ascend at higher
velocity and within more humid atmosphere, to facilitate cloud emergence despite the existence of lateral turbulent mixing with dry environment.

The good agreement with observations and the LES results, obtained in the study, indicates that the microphysics of developing small nonprecipitating Cu is determined largely by two processes, namely, adiabatic ascent and lateral turbulent mixing. Adiabatic processes determine the maximum values of the droplet mean volume radius in the cloud core. Comparison with in situ measurements and the LES results shows that the \( \Gamma - K \) model based on Eqs. (10) and (19) can be used for evaluation of the effects of mixing on cloud structure in small Cu. It can also be used in different parameterizations in large-scale models. To determine the spatial (horizontal) and temporal (vertical) changes, a model requires prescribing the following parameters: environment humidity and temperature, droplet concentration at cloud base, width and vertical velocity of updraft, and turbulent exchange coefficient values.

**Acknowledgments.** This research was supported by the Israel Science Foundation (Grants 1393/14, 2027/17), the Office of Science (BER). Partial support for this work comes from Grants DE-SC008811, DE-SC0014295, and ASR DE-FOA-1638 from the U.S. Department of Energy Atmospheric System Research program.

**APPENDIX**

**Derivation of Eq. (10)**

The diffusion equation

\[
\frac{\partial \Gamma(x,t)}{\partial t} = K \frac{\partial^2 \Gamma(x,t)}{\partial x^2} + A_{1} w(x) \tag{A1}
\]

with the initial conditions

\[
\Gamma(x,0) = \begin{cases} 
  \Gamma_1 = A_2 q_1 & \text{if } |x| \leq L \\
  \Gamma_2 = S_2 & \text{if } |x| > L
\end{cases} \tag{A2}
\]

is solved for the area \(-\infty < x < \infty\). The horizontal profile of the vertical velocity \( w(x) \) is given by the rectangle function

\[
w(x) = \begin{cases} 
  w_1 & \text{if } |x| \leq L \\
  0 & \text{if } |x| > L
\end{cases} \tag{A3}
\]

The solution can be found in the handbook by Polyanin and Zaitsev (2004). The Green’s function in our case has a form

\[
G(x,\xi,t) = \frac{1}{2\sqrt{\pi Kt}} \exp \left[-\frac{(x-\xi)^2}{4Kt}\right]. \tag{A4}
\]

Using the Green’s function, the solution of Eq. (A1) can be presented in the form

\[
\Gamma(x,t) = \int_{-\infty}^{\infty} \Gamma(\xi,0) G(x,\xi,t) \, d\xi + \int_{0}^{t} \int_{-\infty}^{\infty} A_{1} w(\xi) G(x,\xi,t-\tau) \, d\xi \, d\tau, \tag{A5}
\]

so

\[
\Gamma(x,t) = \int_{-\infty}^{\infty} \Gamma(\xi,0) \frac{1}{2\sqrt{\pi Kt}} \exp \left[-\frac{(x-\xi)^2}{4Kt}\right] \, d\xi + \int_{0}^{t} \int_{-\infty}^{\infty} A_{1} w(\xi) \frac{1}{2\sqrt{\pi K(t-\tau)}} \times \exp \left[-\frac{(x-\xi)^2}{4K(t-\tau)}\right] \, d\xi \, d\tau. \tag{A6}
\]

The first integral and the inner integral in the second term on the right-hand side of Eq. (A6) are represented using the error function (erf) that leads to the formula

\[
\Gamma(x,t) = \frac{1}{2} \left\{ 2\Gamma_2 + (\Gamma_1 - \Gamma_2) \left[ \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) - \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right] \right. \\
+ A_{1} w_1 \left[ \frac{(x + L)^2}{2K} \text{sgn}(x + L) + \frac{x + L}{\sqrt{\pi Kt}} \exp \left[ -\frac{(x + L)^2}{4Kt} \right] + \frac{t + (x + L)^2}{2K} \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) \right] \\
+ \left. \frac{(x - L)^2}{2K} \text{sgn}(x - L) - \frac{(x - L)^2}{\sqrt{\pi Kt}} \exp \left[ -\frac{(x - L)^2}{4Kt} \right] - \frac{t + (x - L)^2}{2K} \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right\}. \tag{A7}
\]

Finally, we come to the equation

\[
\Gamma(x,t) = \frac{1}{2} \left\{ 2\Gamma_2 + (\Gamma_1 - \Gamma_2) \left[ \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) - \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right] \right. \\
+ A_{1} w_1 \left[ \frac{(x + L)^2}{2K} \text{sgn}(x + L) + \frac{x + L}{\sqrt{\pi Kt}} \exp \left[ -\frac{(x + L)^2}{4Kt} \right] + \frac{t + (x + L)^2}{2K} \text{erf} \left( \frac{x + L}{2\sqrt{Kt}} \right) \right] \\
+ \left. \frac{(x - L)^2}{2K} \text{sgn}(x - L) - \frac{(x - L)^2}{\sqrt{\pi Kt}} \exp \left[ -\frac{(x - L)^2}{4Kt} \right] - \frac{t + (x - L)^2}{2K} \text{erf} \left( \frac{x - L}{2\sqrt{Kt}} \right) \right\}. \tag{A8}
\]
REFERENCES


Lehmann, K., H. Siebert, and R. A. Shaw, 2009: Homogeneous and inhomogeneous mixing in cumulus clouds: Dependence on


