A New Time-Dependent Theory of Tropical Cyclone Intensification

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ABSTRACT: In this study, the boundary layer tangential wind budget equation following the radius of maximum wind, together with an assumed thermodynamical quasi-equilibrium boundary layer, is used to derive a new equation for tropical cyclone (TC) intensification rate (IR). A TC is assumed to be axisymmetric in thermal-wind balance, with eyewall convection coming into moist slantwise neutrality in the free atmosphere above the boundary layer as the storm intensifies, as found recently based on idealized numerical simulations. An ad hoc parameter is introduced to measure the degree of congruence of the absolute angular momentum and the entropy surfaces. The new IR equation is evaluated using results from idealized ensemble full-physics axisymmetric numerical simulations. Results show that the new IR equation can reproduce the time evolution of the simulated TC intensity. The new IR equation indicates a strong dependence of IR on both TC intensity and the corresponding maximum potential intensity (MPI). A new finding is the dependence of TC IR on the square of the MPI in terms of the near-surface wind speed for any given relative intensity. Results from some numerical integrations of the new IR equation also suggest the finite-amplitude nature of TC genesis. In addition, the new IR theory is also supported by some preliminary results based on best-track TC data over the North Atlantic Ocean and eastern and western North Pacific Ocean. As compared with the available time-dependent theories of TC intensification, the new IR equation can provide a realistic intensity-dependent IR during weak intensity stage as seen in observations.

KEYWORDS: Tropical cyclones; Hurricanes/typhoons; Nonlinear models

1. Introduction

In the recent literature, two types of time-dependent equations for the rate of tropical cyclone (TC) intensity change [viz., intensification rate (IR)] have been proposed under various assumptions on the basis of different complexities (Emanuel 2012, hereinafter E12; Ozawa and Shimokawa 2015). The time-dependent theory presented in E12 starts with the boundary layer entropy budget equation and assumes an axisymmetric vortex in thermal-wind balance with neutral slantwise convection in the outwardly sloping eyewall in the free atmosphere above the boundary layer. In this IR theory, small-scale turbulent mixing in the upper troposphere determines the spatial distribution of outflow temperature and the vertical stratification. The final IR equation [see Eq. (17) in E12] is given as

\[
\frac{\partial V_m}{\partial \tau} \approx \frac{C_k}{2h} (V_{\text{max}}^2 - V_m^2),
\]

where \(V_m\) is the maximum tangential wind (or simply the TC intensity) at any given time \(\tau\), \(V_{\text{max}}\) is the maximum potential intensity [MPI; defined as Eq. (18) in E12], \(C_k\) is the surface-exchange coefficient of enthalpy at the air–sea interface, and \(h\) is the depth of the well-mixed boundary layer. This theory predicts larger IR for storms with higher MPIs, which implies more rapid intensification of TCs in a warmer climate (Emanuel 2017) and with weaker intensity. A complete evaluation of E12 using numerical simulations can be found in Peng et al. (2018).

The other time-dependent equation for TC IR is the one constructed by Ozawa and Shimokawa (2015), who assumed a TC as a Carnot heat engine with the hypothesis that the TC intensifies, as measured by the increasing rate in the inner-core mechanical energy, when the energy production rate (viz., surface enthalpy flux) is greater than the surface frictional dissipation rate. The resulting TC IR can be given as

\[
\frac{\partial V_m}{\partial \tau} = \frac{C_D}{H} (\mu V_{\text{max}}^2 - V_m^2),
\]

where \(C_D\) is the surface drag coefficient, \(H\) is the height scale of the TC system, \(\mu\) is the percentage of the available energy production rate converted to the rate of mechanical energy in the inner core from the total available energy production rate, which is assumed to be around 70% by Ozawa and Shimokawa (2015), and the other variables have their meanings as in Eq. (1), although the \(V_m\) and MPI here refer to the near-surface total wind speed rather than the tangential wind speed in Eq. (1).

Equations (1) and (2) are very similar but give a different dependence on \(C_D\) and \(C_k\). Equation (1) implies a weak dependence of IR on \(C_D\) but a strong dependence on \(C_k\) (the former is implicitly included in \(V_{\text{max}}\); also see Fig. 1 in the corrigendum for E12). Equation (2) seems to show a strong
dependence of IR on $C_D$. However, since $V_{\text{max}}$ includes $C_D$ as well (see MPI equation in Emanuel 1997), the actual IR implied by Eq. (2) is not very sensitive to $C_D$, consistent with Eq. (1). Note that Eq. (1) is derived under the assumption that the vortex is well developed. Mathematically both time-dependent TC IR equations show the maximum IR when a TC is at its incipient stage (with $V_m = 0$). This has been recently questioned by Montgomery and Smith (2019), who argue that it is unclear how a vortex could intensify from a quiescent state because there would be no surface enthalpy flux as $V_m$ is close to zero and thus no turbulent mixing to initiate intensification.

In addition, the predicted maximum IR at the incipient stage of TCs by Eqs. (1) and (2) is also in contrast with results obtained for real TCs by Xu and Wang (2015, 2018), who analyzed the relationship between the TC intensity and the subsequent 24-h IR based on best-track TC data for the North Atlantic and the western North Pacific. They found that the top 5% IR in observations shows a maximum for TCs at immediate intensities with the maximum sustained 10-m wind speed around 35–40 m s$^{-1}$. The initial IR increase with increasing TC intensity is explained by balanced dynamics because the increasing intensity corresponds to the increasing inner-core inertial stability and thus to the increasing eyewall-heating efficiency (Schubert and Hack 1982; Vigh and Schubert 2009), while the subsequent IR decrease with increasing intensity at high intensity is because as the TC intensifies closer to its MPI, the enhanced heating efficiency is largely offset by the surface frictional dissipation (Xu and Wang 2015, 2018).

To consider the possible control of the inner-core inertial stability on the observed intensity dependence of TC IR, recently Wang et al. (2021, hereinafter W21) introduced the concept of dynamical efficiency into the energetically based theory of TC intensification Eq. (2), namely,

$$\frac{dV_{\text{ir}}}{dt} = \frac{C_D}{H} (EV_{\text{max}}^2 - V_{\text{ir}}^2),$$

where $E$ is the dynamical efficiency of the TC system, which is parameterized as the normalized inner-core inertial stability $I$ in the following form:

$$E = \frac{I}{I_{\text{max}}} = \frac{\sqrt{\left(f + \frac{2V}{r}\right) \left(f + \frac{\partial r V}{r \partial r}\right)}}{r_{\text{ir}}},$$

where $r$ is the radial distance from the TC center; $r_m$ and $r_{\text{max}}$ denote the radius of maximum wind (RMW) at current stage and that in the quasi-steady state, respectively; $V$ is tangential wind speed; and $f$ is the Coriolis parameter at the TC center. With the introduced dynamical efficiency, the modified Eq. (3) can reproduce the observed and numerically simulated intensity dependence of TC IR as demonstrated by W21.

A question arises as to whether a similar modification to the dynamically based TC IR theory Eq. (1) can be achieved so that the two independent theories can converge to give more realistic time dependence of TC IR consistent with the observations. This is a topic that we will discuss in this article. Here, a new time-dependent equation for TC IR is formulated based on the boundary layer tangential wind budget equation. We will show that this alternative derivation uses different assumptions in comparison with that in E12. More importantly, by modifying one of the assumptions, the new IR equation becomes the same as the one empirically obtained recently by W21. The derivation of the new IR equation is introduced in section 2. In section 3, the performance of the new IR equation is evaluated based on results from idealized axisymmetric numerical simulations with atmospheric soundings sorted according to sea surface temperature (SST) over the tropical North Atlantic and western North Pacific. Main results are summarized in the last section, together with a brief discussion of some preliminary results from observations based on best-track TC data.

2. Derivation of the new TC IR equation

We assume the slab boundary layer, within which the tangential wind and entropy budget equations can be written as (cf. E12)

$$\frac{\partial V_{\text{b}}}{\partial t} + u_b \frac{\partial M_b}{\partial r} = \frac{C_D}{H} |V_{10}| V_{10}$$

and

$$\frac{\partial s_{\text{b}}}{\partial t} + u_b \frac{\partial s_{\text{b}}}{\partial r} = C_k \beta |V_{10}| (s_0 - s_b),$$

where $M$ is the absolute angular momentum [$M = rV + (1/2)fr^2$]; $u$ is radial wind speed; $C_D$ and $C_k$ are the surface drag coefficient and surface enthalpy exchange coefficient, respectively; $|V_{10}|$ and $V_{10}$ are the total wind speed and the tangential wind speed at 10-m height above the sea surface, respectively; $s_0$ and $s_b$ are the saturated entropy at the SST $T_b$ and the well-mixed air entropy in the boundary layer, respectively; and $\beta$ is the efficiency parameter describing the effects of subsaturation or downdrafts (Emanuel 1997; Gray and Craig 1998; Frisius 2006). All variables with subscript “b” are their vertical means in the slab boundary layer, which has a depth of $h$. Note that the horizontal diffusions of both tangential wind and entropy are neglected as in previous theoretical studies of TC MPI and intensification although they are important across the eyewall in strong TCs (Emanuel 1997; Rotunno and Bryan 2012; Zhang and Montgomery 2012).

Since our interest is in TC IR, which is assumed to be the rate of change in the near-surface wind speed following the RMW, all terms in Eqs. (5) and (6) should be evaluated following the RMW where $\beta$ is often close to 1.0 (Gray and Craig 1998; Frisius 2006). Following Raymond (1995) and Emanuel (1995), we further consider that the TC boundary layer is in thermodynamic quasi equilibrium and assume $\beta$ to be less than 1.0 during the intensification stage of a TC because part of the surface enthalpy flux would contribute to the increase in entropy near the RMW, and Eq. (6) becomes

$$u_b \frac{\partial s_{\text{b}}}{\partial r} = \frac{C_k}{H} |V_{10}| (s_0 - s_b).$$

In this study, the radial wind speed equation is not considered, and $u_b$ can be eliminated by combining Eqs. (5) and (7) to give

$$\frac{dV_{\text{ir}}}{dt} = \frac{1}{r_m} \frac{\partial M_b}{\partial r} C_D |V_{10}| (s_0 - s_b) \frac{\partial s_{\text{b}}}{\partial r} - C_D |V_{10}| V_{10}.$$
Note that in Eq. (8) we use \( \tau \) instead of \( t \) because the budget and all variables are now evaluated at and following the RMW. Since the total wind speed at the RMW is close to the tangential wind speed, we can assume \( |V_\theta| \approx V_\theta = V_m \). We further assume \( V_m = \alpha V_{h,m} \), where \( \alpha \) is the reduction factor of the depth-averaged boundary layer wind speed to the near-surface wind speed at the RMW, which is roughly between 0.7 and 0.8 based on observations (Vickery et al. 2000; Powell et al. 2005). With these approximations, Eq. (8) can be rewritten as

\[
\frac{\partial V_m}{\partial \tau} = \frac{\alpha}{\alpha h_m} \left[ -\frac{1}{\alpha h_m} \frac{\partial M}{\partial s} \bigg|_{s=r_m} C_k V_m (\sigma_0 - \sigma_s) \right] - C_k V_m^2. \tag{9}
\]

Note that \( \partial V_m/\partial \tau \) is equivalent to \( dV_m/d\tau \), considering \( dV_m/d\tau = \partial V_m/\partial \tau + r_m (\partial V_m/\partial \tau) \) and \( \partial V_m/\partial \tau = 0 \). Equation (9) can be closed if \( (\partial M/\partial s)_{s=r_m} \) could be determined. It is not unrealistic to approximate this term using the value at the top of the boundary layer at the RMW (E12), namely,

\[
(\partial M/\partial s)_{s=r_m} \approx (\partial M/\partial s)_{s=r_m} \bigg|_{s=r_m}. \tag{10}
\]

Following Emanuel (1986), we assume that the TC vortex in the free atmosphere above the boundary layer is axisymmetric and in hydrostatic and gradient wind balance and thus follows the thermal-wind equation:

\[
\frac{1}{r^2} \left( \frac{\partial}{\partial r} \right) \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \right] = -\left( \frac{\partial T}{\partial r} \right)_{s} \frac{\partial s}{\partial r}, \tag{11}
\]

where \( T \) is air temperature, \( p \) is air pressure, and the subscripts "r," "s," and "p" mean the derivatives at given radius, isentropic surface, and pressure surface, respectively. If we assume that both \( s \) and \( M \) are conserved along streamlines, then the \( s \) and \( M \) surfaces are almost congruent in the free atmosphere. This is equivalent to assuming the state of moist-neutral ascent along \( M \) surfaces (Emanuel 1986). In this case, Eq. (11) can be rewritten as

\[
\frac{1}{r^2} \left( \frac{\partial}{\partial r} \right) \left( \frac{1}{r^2} \frac{\partial}{\partial r} \right) = -\frac{1}{2M} \frac{\partial M}{\partial r} \frac{\partial s}{\partial r}. \tag{12}
\]

If we integrate Eq. (12) along a trajectory of moist-neutral ascent (constant \( M \) and \( s \) surfaces) starting from the RMW at the top of the boundary layer \( (r_m, h) \) to the point at a large radius \( r_0 \) in the upper tropospheric outflow where the trajectory becomes essentially horizontal as in Bryan and Rotunno (2009), we can get

\[
\left. \frac{dM}{ds} \right|_{s=r_m} = -\frac{T_h - T_0}{M_{h,m}} \left( \frac{1}{r_m} - \frac{1}{r_0} \right), \tag{13}
\]

where \( T_h \) is air temperature at \( (r_m, h) \), which is often assumed to be the SST without the loss of accuracy (Emanuel 1986; E12; Bryan and Rotunno 2009), and \( T_0 \) is the air temperature at the outer radius \( r_0 \) in the outflow layer.

It seems that Eq. (9) can be closed with Eq. (13). However, although the assumption of moist-neutral convection in the eyewall is a good approximation for a strong TC, this assumption is often not satisfied during the TC-intensification stage. In this case, the \( M \) and \( s \) surfaces would not be congruent but intersect, especially in the boundary layer (Peng et al. 2018). Therefore, we introduce an empirical parameter \( \lambda' \) into Eq. (13) and consider Eq. (10), and we have

\[
\frac{\partial M}{\partial s} \bigg|_{s=r_m} \approx \frac{\partial M}{\partial s} \bigg|_{s=r_m} = -\lambda' \frac{T_h - T_0}{M_{h,m}} \left( \frac{1}{r_m} - \frac{1}{r_0} \right), \tag{14}
\]

where \( \lambda' \) denotes the extent to which the \( M \) surface is parallel to the \( s \) surface, which should satisfy \( 0 \leq \lambda' \leq 1 \) considering \( (\partial M/\partial s)_{s=r_m} \leq 0 \) and is likely to be a function of TC intensity (Peng et al. 2018). This means that \( \lambda' \) should be close to 1.0 when the inner-core convection reaches a nearly moist-neutral state, with the \( M \) and \( s \) surfaces being almost congruent as in Eq. (13), whereas it is close to zero in the early incipient stage when \( V_m \) is close to zero and the \( M \) and \( s \) surfaces are nearly orthogonal (Peng et al. 2018).

Substituting Eq. (14) into Eq. (9) and considering \( r_0 \gg r_m \), Eq. (9) can be rewritten as

\[
\frac{\partial V_m}{\partial \tau} = \frac{\alpha}{\alpha h_m} \left[ -\frac{r_m}{M_{h,m}} \epsilon A' \beta C_b V_m (\sigma_0 - \sigma_s) \frac{\partial s}{\partial r} \right] - C_k V_m^2, \tag{15}
\]

where the approximation \( (\sigma_0 - \sigma_s) \approx T_s (\sigma_0 - \sigma_s) \) is used as in E12, with \( k_0 \) and \( k_b \) being the saturation enthalpy at SST \( T_s \) \((\approx T_h)\) and the air enthalpy of the boundary layer at the RMW, respectively, and \( \epsilon = (T_h - T_0)/T_h \) is the thermodynamic efficiency (Emanuel 1986). Note that \( M_{h,m} = r_m |V_{h,m} + (1/2)f r_m| \), where \( V_{h,m} \) is the maximum gradient wind speed at the top of the boundary layer. Since in the inner core, \((1/2)f r_m \ll V_m\), we can assume \( M_{h,m} \approx r_m V_{h,m} \). If we further assume \( V_m = V_{h,m} \) as in previous studies (e.g., E12) and ignore the weak dependence of \( C_b \epsilon (k_0 - k_b)/C_b \) on TC intensity \( V_m \) (E12; Li et al. 2020), Eq. (15) can be simplified to

\[
\frac{\partial V_m}{\partial \tau} = \frac{\alpha}{\alpha h_m} \left[ A V_m^2 - V_m^2 \right], \tag{16}
\]

where \( A = \beta A' \) and the surface-wind-based MPI is given (Emanuel 1997) by

\[
V_m^2 = C_b \epsilon (k_0 - k_b). \tag{17}
\]

The new theoretical IR Eq. (16) is similar to the time-dependent theories of TC intensification given in Eqs. (1) and (2). If \( A = 1 \) is assumed, which is equivalent to assuming a steady-state TC in which \( \beta \) is close to 1.0 by definition and \( A' \approx 1.0 \) for the congruence between the \( M \) surface and the \( s \) surface, Eq. (16) is reduced to one that is similar to Eq. (1). This is consistent with the assumption of a moist-neutral state through the eyewall in deriving Eq. (1). However, this assumption is not valid for an intensifying TC, particularly in the early intensification stage (Peng et al. 2018). Therefore, if \( A \) is empirically determined, the new IR equation is more appropriate for TC intensification than Eq. (1). On the other hand, if \( A \) is assumed to be a constant as in Eq. (2) and \( \alpha = 1 \),
Eq. (16) is mathematically reduced to Eq. (2). In this sense, $A$ in Eq. (16) can be considered as the constant percentage of the available energy production converted to mechanical energy in the inner core from the total available energy production as in Eq. (2) (Ozawa and Shimokawa 2015). In addition, $A$ in Eq. (16) is mathematically similar to the ad hoc dynamical efficiency $E$ recently introduced by W21 [see Eq. (4)]. However, the parameter $A$ introduced here has a different physical meaning. Since $A$ in Eq. (16) is mathematically similar to the dynamical efficiency $E$ in Eq. (3), we can simply parameterize $A$ using the normalized inertial stability near the RMW as $E$ in W21. However, in this study, $A$ is parameterized in a slightly different form based on results from ensemble numerical simulations as done in W21 and will be discussed in section 3.

Note that in the E12’s theory, the TC intensification is controlled by the parameterized turbulent mixing in the upper-tropospheric outflow layer, which determines indirectly the rate of inward movement of the $M$ surfaces. As argued by Montgomery and Smith (2019), the physics of how the upper-tropospheric turbulent mixing leads to the spinup in and above the top of the frictional boundary layer are unclear. The new IR equation discussed above is obtained based on the boundary layer tangential-wind budget equation following the RMW. The TC intensifies due to the inward movement of $M$ surface driven by surface enthalpy flux under the eyewall. Specifically, the surface enthalpy flux can lead to the increase in the radial entropy gradient and radial inflow under the eyewall. Specifically, the surface enthalpy flux can lead to the increase in the radial entropy gradient and radial inflow under the eyewall as implied by Eq. (6) or (7), which in turn can lead to the inward movement of $M$ surface because $\delta M/\delta s \leq 0$, both in and above the boundary layer. In other words, the first term on the right-hand side in Eq. (9) leads to the increase in local $M$ (and thus the tangential wind) near the RMW, namely, the intensification of the TC. We will show in section 3 that, similar to that in W21, the new IR equation can capture the intensity dependence of TC IR found in the observations.

3. Evaluation using results from idealized full-physics model simulations

3.a. Idealized full-physics model simulations

The new time-dependent IR equation outlined in section 2 is evaluated using results from a series of ensemble numerical experiments with the state-of-the-art axisymmetric Cloud Model 1 (CM1; Bryan and Fritsch 2002) as conducted in Li et al. (2020, 2021) and W21. The model atmosphere has 59 vertical levels with stretched vertical grid spacing below 5.5 and 0.5 km above up to the model top at 25 km. In the radial direction, the grid spacing of 1 km is used within 100-km radius and then the grid spacing is stretched to 12 km at the lateral boundary of 3100 km. The cloud microphysical processes are treated with the microphysics scheme of Thompson et al. (2008). Cumulus convection parameterization is not used in all simulations. As in Rotunno and Emanuel (1987), Newtonian cooling, which is capped at 2 K day$^{-1}$, is used to mimic the longwave radiative cooling. The Smagorinsky diffusion scheme is used for subgrid-scale turbulent mixing with the horizontal and asymptotic vertical mixing lengths being 700 and 70 m, respectively, as in Li et al. (2020). A constant $C_E$ of $1.2 \times 10^{-3}$ is used for surface enthalpy flux calculation. Donelan et al.’s (2004) wind-dependent $C_D$, which increases with the 10-m wind speed when $|V_{10}| \leq 25$ m s$^{-1}$ and is kept constant at $2.4 \times 10^{-3}$ afterward, is used for surface stress calculation, namely,

$$C_D = 10^{-3} \min \{2.4, \max\{1.0, 1.0 + 0.07(V_{10} - 5.0)\}\}.$$ (18)

The experimental design and the simulation results can be found in Li et al. (2020, 2021) and W21, and only some key aspects are briefly described here. Considering the fact that the TC IR varies with MPI, which is primarily a function of SST and the corresponding atmospheric sounding, to ensure the robustness of the results, experiments with different SSTs are performed, including four ensemble simulations with atmospheric soundings with different SSTs for the western North Pacific (labeled as SST28_WNP–SST31_WNP) and four ensemble simulations with atmospheric soundings for the North Atlantic (labeled as SST27_NA–SST30_NA). The SST and atmospheric sounding data are based on the monthly mean European Centre for Medium-Range Weather Forecasts (ECMWF) interim reanalysis data (Dee et al. 2011). Each ensemble simulation has 21 runs with slightly perturbed initial RMW and maximum tangential wind relative to the maximum tangential wind speed of 15 m s$^{-1}$ at the surface at the RMW of 80 km [see Li et al. (2020) and W21 for more details]. All simulations are run on an $f$ plane of 20$^\circ$N in a quiescent environment. As in W21, only the intensity evolution from the ensemble mean of each simulation is used to evaluate the new IR equation described in section 2 to avoid effects of convective and cloud-scale motions, which are not explicitly considered in the new IR theory.

b. Determination of $A$ and some basic features of the new IR equation

As mentioned earlier, $A$ should be close to 1.0 when the TC reaches its MPI while it is small in the early intensification stage. Although $A$ can be taken to be the same as $E$, we prefer to use a different but simpler functional relationship in this study. Since $A$ in Eq. (16) is mathematically similar to the dynamical efficiency $E$ in Eq. (3) in W21, and the inner-core inertial stability in $E$ [Eq. (4)] often increases with increasing TC intensity (W21), it is not unrealistic to assume $A = (V_m/V_{\text{max}})^n$, where $n$ is a constant to be determined. We can simply determine $n$ based on the relationship between $E$ and the TC relative intensity defined as the TC intensity normalized by its MPI ($V_m/V_{\text{max}}$), assuming $E = (V_m/V_{\text{max}})^n$, using results from the ensemble simulations (see Fig. 6 in W21). Note that the MPI ($V_{\text{max}}$) is estimated as the quasi-steady state TC intensity (i.e., the averaged $V_m$ during the quasi-steady state) in model simulations as done in W21, namely, with the superintensity (defined as the normalized excess TC intensity relative to the theoretical MPI) considered (Li et al. 2020). As we can see from Fig. 1a, $n = 3/2$ is the best fit, indicating that we can take $A = (V_m/V_{\text{max}})^{3/2}$ in Eq. (16) as an ad hoc parameterization. This is further confirmed with the diagnostic result based on Eq. (16) assuming $A = (V_m/V_{\text{max}})^n$ using results from the idealized full-physics ensemble model simulations as done in W21 (Fig. 1b), and the result is generally
consistent as $\alpha$ varies between 0.7 and 0.8 (not shown). With this fitted relationship between $A$ and the normalized TC intensity, Eq. (16) becomes

$$\frac{\partial V_m}{\partial \tau} = \frac{\alpha C_D}{h} \left( V_{\text{max}}^{3/2} V_m^{3/2} - V_m^2 \right). \quad (19)$$

Now we will discuss some basic features of the new IR Eq. (19). First, different from previous time-dependent equations of TC intensification Eqs. (1) and (2) reviewed in section 1, the new IR equation implies zero IR at $V_m = 0$, and the maximum intensification rate occurs at the intermediate intensity, similar to the energetically based IR Eq. (3) from W21 and consistent with observations (Xu and Wang 2015, 2018).

Second, from Eq. (19), we know that when

$$V_m = \frac{9}{16} V_{\text{max}} \quad (20)$$

the IR ($\partial V_m/\partial \tau$) reaches its maximum, or the maximum potential IR (MPIR; Xu et al. 2016)

$$\left( \frac{\partial V_m}{\partial \tau} \right)_{\text{max}} = \frac{27 \alpha C_D}{256} h V_{\text{max}}^2. \quad (21)$$

Note that in obtaining Eq. (20), the weak dependence of the thermodynamic disequilibrium or $C_D$ on TC intensity has been ignored. The above analysis indicates that the MPIR occurs at an intermediate TC intensity (roughly 0.5625MPI) and increases with increasing MPI at a rate proportional to the square of the corresponding MPI. For example, for a given $V_{\text{max}}$ of 60 m s$^{-1}$ and taking $\alpha = 0.75$, $C_D = 2.4 \times 10^{-3}$, and $h = 2000$ m as in W21, the MPIR of 29.52 m s$^{-1}$ day$^{-1}$ will occur when $V_m = 33.75$ m s$^{-1}$. This estimate is very close to those estimates from observations based on best-track data for North Atlantic Ocean TCs documented in Xu et al. (2016). Figure 2a shows the distribution of IR in the $(V_m/V_{\text{max}})$ plan calculated using Eq. (19) with the same values for the parameters mentioned above, except for $C_D$, which is calculated using Eq. (18). We can see that the MPIR increases with increasing MPI and occurs at higher intensity for larger MPI [Eqs. (20) and (21)].

In addition, let us introduce a concept of relative intensity defined as the intensity normalized by the corresponding MPI $(V_m/V_{\text{max}})$ and rewrite Eq. (19) as

$$\frac{\partial V_m}{\partial \tau} = \frac{\alpha C_D}{h} \left( V_{\text{max}}^{3/2} \left( \frac{V_m}{V_{\text{max}}} \right)^{3/2} - \left( \frac{V_m}{V_{\text{max}}} \right)^2 \right). \quad (22)$$

This indicates that for a given relative intensity, the IR is proportional to the square of the corresponding MPI. This is an important feature of the new IR equation. Figure 2b shows the distribution of IR in the $(V_m/V_{\text{max}}, V_{\text{max}})$ plan calculated using Eq. (22) with the same parameters as used in Fig. 2a. Now it is more clearly seen that the IR reaches a maximum at an intermediate relative intensity for any given MPI and increases with increasing MPI at a rate proportional to the square of the corresponding MPI as predicted by Eq. (21). Note that the MPIR for a given MPI does not occur exactly at $V_m = (9/16) V_{\text{max}}$ at relatively low MPIs because the dependence of $C_D$ on TC intensity, mainly for weak TC intensity [cf. Eq. (18)], which shifts the MPIR to the higher relative intensity side for weaker MPI (Fig. 2b). Note also that although previous IR Eqs. (1) and (2) also predict an increase of IR with MPI at a rate proportional to the square of the MPI, they cannot be applied at the incipient stage of TCs and mathematically they indicate the maximum IR to occur at $V_m = 0$, which is in contrast with observations. Since many previous studies have projected an increase in MPI in a warmer future climate, Eq. (22) implies that TCs in a warmer climate may intensify more rapidly.
For example, a 5% increase in MPI may lead to a 10.25% increase in TC IR. Since rapid intensification is still a challenge in current operational forecasts, the increasing IR may make the TC intensity to be more difficult to predict in the future under current operational forecasts, the increasing IR may make the TC system to intensify rapidly at the beginning of integration. The result is consistent with the finite-amplitude nature of TC genesis (Emanuel 1989). Note that the finite-amplitude nature of TC genesis in nature is related to many other processes that have not been considered in our simple intensification model. Moreover, some assumptions made in deriving the new IR equation are not relevant to TC genesis in nature, such as the axisymmetric structure with favorable environmental conditions.

c. Comparison with idealized full-physics model simulations

Figure 4 shows the time evolution of the ensemble-mean maximum 10-m wind speed $V_m$ of the model TC for all eight simulations (solid curves). All storms experienced an initial adjustment period, whose duration decreases with increasing SST (Figs. 4a,b). This initial adjustment is common in most of numerical simulations because of the initial spinup of the boundary layer and moistening of the inner core (e.g., Emanuel and Zhang 2017). After the initial adjustment, all storms intensify with the IR depending on the storm intensity as predicted by the new IR equation shown in Fig. 2b. All storms reach their steady-state maximum intensities, which increase with increasing SST as predicted by the theoretical MPI. Note that superintensity exists in most cases as discussed in Wang and Xu (2010) and Li et al. (2020). Therefore, to ensure the consistency and the fair direct comparison of the new IR equation with the full-physics model simulations, we use the ensemble mean steady-state intensity from the full-physics model simulations as the MPI in Eq. (19) or (22) instead of the MPI defined in Eq. (17), as done above and in W21.

Since the parameter $A$ in the new IR equation is closely related to the dynamical efficiency $E$ in W21 [Eq. (3); Fig. 1a], the new IR equation can also reasonably reproduce the TC IR in the full-physics model simulations as in W21. A comparison of the IRs estimated by the new IR Eq. (3) with those in the full-physics model simulations can be found in W21 (see their Figs. 7 and 8). Here, we compare the time evolution of the TC intensity estimated by the time integration of the new IR Eq. (19) or (22) with that from the ensemble simulations described above. To facilitate a direct comparison, we use the same wind-dependent $C_D$ given in Eq. (18) as used in the full-physics model. Again, $\alpha = 0.75$ and $h = 2000$ m are also used as above. In addition, we have already seen from Fig. 3b, the time evolution of the TC intensity based on the new IR equation, in particular the onset time of rapid intensification, is sensitive to the initial TC intensity. Because the dynamics of the intensity evolution before the onset of rapid intensification in the new IR equation are different from that of the initial adjustment in the full-physics model simulations, a fair comparison of the intensity evolution...
in the very early stage should be excluded. Therefore, in our evaluation, we start the time integration of the new IR Eq. (19) or (22) when the 10-m maximum wind speed reached 15 m s$^{-1}$ in all individual simulations.

To allow for a close comparison, we add the intensity evolution obtained from the new IR Eq. (19) or (22) in Fig. 4 in dotted curve with the same color for the corresponding intensity evolution in each of the full-physics model simulations (solid curves). We can see that the new IR equation captures well the intensity evolution in the full-physics model simulations. The agreement is quite promising for all simulations with the SST-sorted atmospheric soundings for both the North Atlantic and the western North Pacific Oceans. However, we can see from the slope of the intensity evolution during the rapid intensification stage, the new IR equation seems to slightly underestimate the IR in the simulations with SST of 29°C for both the North Atlantic and the western North Pacific. Nevertheless, such a small underestimation does not lead to large discrepancies in the intensity evolution obtained from the new IR equation. Therefore, we can conclude that the new IR equation can reproduce the intensity evolution in the full-physics model simulations. Note that we have only shown the evaluation results of intensity evolution from the new IR equation against the control simulations in W21, who also performed several sensitivity experiments by varying model parameters. Indeed, our results (not shown) for all sensitivity experiments are also consistent with those shown for the control simulations because the IR was captured well by the IR Eq. (3) for all sensitivity experiments as shown in W21.

4. Summary and discussion

A new advance in recent years is the development of time-dependent theories of TC intensification. Two such theories have been constructed based on various assumptions and different approaches (E12; Ozawa and Shimokawa 2015). One is based on the boundary layer entropy-budget equation and assumes an axisymmetric vortex in thermal-wind balance with neutral slantwise moist convection in the outwardly sloping eyewall in the free atmosphere (E12). The other is energetically based and interprets a TC as a Carnot heat engine, which intensifies as the energy production rate due to surface enthalpy flux being larger than the surface frictional energy dissipation rate (Ozawa and Shimokawa 2015). The latter has been recently improved by introducing a dynamical efficiency of the TC.
system in W21 so that the modified theory can reproduce the observed intensity dependence of TC IR (Xu and Wang 2015, 2018). In this study, a new time-dependent theory of TC intensification has been developed based on the boundary layer dynamics and evaluated based on results from a series of ensemble axisymmetric full-physics model simulations.

Different from E12, the depth-averaged boundary layer tangential-wind budget equation, instead of the depth-averaged boundary layer entropy budget equation, following the RMW is used to derive a new IR equation. As in previous studies, a TC vortex is assumed to be axisymmetric in thermal-wind balance, and convection in the eyewall in the free atmosphere above the boundary layer is assumed to gradually attain a state of moist slantwise neutrality as the storm intensifies. Considering the fact that the boundary layer entropy is often in thermodynamic quasi equilibrium (Emanuel 1995), namely, the local tendency of entropy near the RMW is very small relative to the dominant counteracting tendencies associated with advection and surface flux forcing in the TC boundary layer, the thermodynamic quasi equilibrium is assumed in the TC boundary layer in our theoretical framework, which is well supported by our full-physics model simulations (not shown). An ad hoc parameter $\beta$ is introduced to guarantee the quasi equilibrium between the advection and surface flux forcing of entropy, which is assumed to explain the effect of convective downdrafts and turbulent entrainment as in previous studies (e.g., Emanuel 1995; Gray and Craig 1998; Frisius 2006). Another ad hoc parameter $A'$, as a measure of the degree of congruence of the absolute angular momentum and the entropy surfaces, is introduced as a closure of the dynamical system motivated by a recent finding based on idealized numerical simulations by Peng et al. (2018). Based on its mathematical similarity to the dynamical efficiency introduced in W21 and also calibrated using results from ensemble simulations, the combined ad hoc parameter $A$ (=$\beta A'$) is formulated as a function of the normalized TC intensity or termed relative intensity, namely, $(V_m/V_{max})^{3/2}$.

The new IR equation indicates a strong dependence of IR on both the TC intensity and the corresponding MPI, which is well supported by observations. A new finding of this study is the dependence of TC IR on the square of the MPI in terms of the near-surface wind speed for any given relative intensity, which is also consistent with observations. In addition, results from some numerical integrations of the new IR equation show that vortices with too weak initial intensity take relatively long time to develop, suggesting the finite-amplitude nature of TC genesis. The new IR equation is also evaluated using results from idealized ensemble numerical simulations.

![Figure 4](image-url)
using the full-physics axisymmetric cloud model CM1 (Bryan and Fritsch 2002). Results show that the new IR equation can reproduce well the time evolution of the numerically simulated TC intensity after the initial adjustment stage. As compared with the available two time-dependent theories of TC intensification (E12; Ozawa and Shimokawa 2015), the new IR equation can provide a realistic intensity-dependent IR of TCs during weak intensity stage that is consistent with observations.

To further demonstrate the validity of the new time-dependent IR theory, here, we briefly introduce some preliminary results for the observed dependence of IR on TC intensity and the corresponding MPI based on best-track data for TCs over the North Atlantic and eastern and western North Pacific to examine. All data were obtained online (https://rammb.cira.colostate.edu/research/tropical_cyclones/ships/developmental_data.asp), which is the dataset used for the Statistical Hurricane Intensity Prediction Scheme (SHIPS) (Knaff et al. 2005). The newest-version SHIPS dataset includes all tropical or subtropical cyclones in both the North Atlantic and Eastern North Pacific during 1982–2019 and those in the western North Pacific for the period 1990–2017. In our analysis, TC IR is defined as 6-hourly intensity change. Only positive IR cases, namely, intensifying cases, are considered. The corresponding MPI is directly adopted from the SHIPS dataset, which was calculated using the Emanuel’s MPI theory as given in Bister and Emanuel (2002). Figure 5 shows the 99th percentile of IRs in the \((V_m, V_{\text{max}})\) plan (Fig. 5a) and the \((V_m/V_{\text{max}}, V_{\text{max}})\) plan (Fig. 5b), respectively. Note that the 99th percentile of IRs is used to represent the IR under favorable environmental conditions for a fair comparison with the theoretical IR developed in this study. Similar distributions of IR as a function of TC intensity (and relative intensity) and MPI in Figs. 5 and 2 demonstrate that the new IR theory is well supported by the observations. A full evaluation/calibration of the new IR theory using best-track TC data is still under way, and the results will be reported separately in due course.

Note that the intensity evolution of a TC in the real atmosphere is also affected by environmental factors, such as the large-scale vertical wind shear, in addition to the storm-scale dynamics as discussed in this study. W21 mentioned that their dynamical efficiency \(E\) in Eq. (3), and thus the TC IR, may be modified by environmental conditions. Similarly, we can consider that both \(A'\) and \(\beta\), and thus \(A\) in Eq. (16), could be also modified by environmental conditions. In this case, we can introduce an ad hoc parameter \(B\) (\(\leq 1.0\)) into Eq. (22) and revise it to

\[
\frac{\partial V_m}{\partial t} = \frac{\alpha C_D}{h} V_{\text{max}}^2 \left[ B \left( \frac{V_m}{V_{\text{max}}} \right)^{3/2} - \left( \frac{V_m}{V_{\text{max}}} \right)^2 \right].
\]

Physically, the parameter \(B\) can be considered as the collective detrimental effect of environmental factors, such as vertical wind shear, that ventilate the eyewall entropy both in and above the boundary layer (Tang and Emanuel 2010; Chavas 2017). As a result, on one hand, part of the surface enthalpy flux under the eyewall would be used to recover the boundary layer ventilation effect due to downdrafts near the RMW, and thus \(\beta\) in Eq. (7) could be reduced. On the other hand, ventilation of the eyewall entropy may destroy the near congruence of the absolute angular momentum surface and the entropy surface, thus reducing the parameter \(A'\) in Eq. (14) or (15). Both processes would cause a reduction of \(A\) in Eq. (16), making the parameter \(B\) in Eq. (23) less than 1.0, and thus leading to a reduced IR or even a weakening of the TC. Therefore, if \(B\) is parameterized using environmental factors, such as the large-scale vertical wind shear and translational speed of the TC, and calibrated based on best-track TC data,
REFERENCES


