Comparing Growth Rates of Simulated Moist and Dry Convective Thermals

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(Manuscript received 8 June 2020, in final form 3 December 2020)

ABSTRACT: The spreading rates of convective thermals are linked to their net entrainment, and previous literature has suggested differences in spreading rates between moist and dry thermals. In this study, growth rates of idealized numerically simulated axisymmetric dry and moist convective thermals are directly compared. In an environment with dry-neutral stratification, the increase of thermal radius with height $dR/dz$ is a larger by a factor of 1.7 for dry thermals relative to moist thermals. The fractional change in thermal volume $\varepsilon$ is also greater for dry thermals within a distance of $-4$ radii from the initial thermal height. Values of $dR/dz$ are nearly constant with height for both moist and dry thermals consistent with classical theory based on dimensional analysis. The simulations are also consistent with theory relating impulse, circulation, and spreading rate for dry thermals proposed in previous papers and extended here to moist thermals, predicting they will spread less than dry thermals. Tests adding heating to dry thermals, either spread uniformly across the thermal volume or concentrated in the inner core, indicate that $dR/dz$ and $\varepsilon$ are smaller for moist thermals because latent heating is confined mostly to their cores. Additional axisymmetric moist simulations using modified lapse rates and large-eddy simulations support this analysis. Overall, these results indicate that slow spreading rates are a fundamental feature of moist thermals caused by latent heating that alters the spatial distribution of buoyancy within them relative to dry thermals.

KEYWORDS: Convection; Entrainment; Small scale processes; Vertical motion; Large eddy simulations

1. Introduction

Entrainment remains a poorly understood aspect of cumulus dynamics [e.g., see the review of de Rooy et al. (2013)]. Limited understanding is reflected by the wide variety of approaches used to formulate entrainment (and detrainment) rates in cumulus parameterization schemes. Such approaches have included separating entrainment and detrainment into types associated with organized convective-scale flow (the so-called dynamic type) from that associated with smaller-scale turbulent mixing (e.g., de Rooy and Siebesma 2010); representing mixing using the buoyancy sorting concept (e.g., Kain and Fritsch 1990; Emanuel 1991; Bretherton et al. 2004); and representing fractional entrainment rates as a function of various convective parameters. These parameters have included inverse updraft radius (e.g., Simpson and Wiggert 1969; Kain and Fritsch 1990), inverse height (e.g., Jakob and Siebesma 2003), vertical velocity (e.g., Neggers et al. 2002), and relative humidity (e.g., Bechtold et al. 2008).

There is evidence that cumulus entrainment is intimately linked to updraft structure (e.g., Moser and Lasher-Trapp 2017; Morrison et al. 2020; Peters et al. 2020a). Many previous studies, and almost all cumulus parameterization schemes, have assumed steady-state plume updrafts. However, recent observational (e.g., Damiani et al. 2006) and numerical modeling studies (e.g., Sherwood et al. 2013, hereinafter S13; Romps and Charn 2015, hereinafter RC15; Hernandez-Deckers and Sherwood 2016; Moser and Lasher-Trapp 2017; Hernandez-Deckers and Sherwood 2018; Peters et al. 2019, 2020a) have indicated that cumulus convection is often composed of a collection of individual, transient thermal-like features. The flow associated with these distinct thermal structures appears to be critical in driving overall entrainment in cumulus updrafts (Moser and Lasher-Trapp 2017; Morrison et al. 2020; Peters et al. 2020a).

Work in the mid twentieth century established the basic dynamics of idealized dry thermals following dimensional analysis (e.g., Morton et al. 1956; Scorer 1957). Specifically, considering an instantaneous point source of buoyancy in an infinite, unstratified fluid, the only independent external parameter is the total buoyancy added. Assuming Boussinesq flow and that the thermal shape does not change over time, dimensional analysis shows that the thermal half-width $R$ must be proportional to height $z$, and thus the rate of spread with height $dR/dz$ must be constant. This scaling was supported by laboratory studies showing that dry thermals grow substantially in width at a nearly constant rate as they rise (e.g., Morton et al. 1956; Scorer 1957). Turner (1957) also developed a theory for thermal spread and growth based on impulse and circulation, showing that dry thermals must grow at a rate proportional to the ratio of buoyant forcing and circulation (for Boussinesq flow). McKim et al. (2020) confirmed this relationship from direct numerical simulation of dry thermals.

The spreading rate of thermals is related to their net entrainment of the surrounding fluid and increase in thermal mass. In addition, entrainment and detrainment can occur by
mixing across the thermal boundary without contributing to a net change in thermal mass. However, Lecanet and Jeenanjee (2019, hereinafter LJ2019) found that entrainment in dry thermals is primarily a laminar process, and is not accompanied by much detrainment or turbulent momentum mixing. This finding contrasts with assumptions made in some earlier studies that thermal entrainment is primarily driven by small-scale (relative to the thermal size) turbulence (e.g., Morton et al. 1956).

The dynamics of moist thermals are comparatively less well understood, though a few recent studies have addressed this subject (e.g., S13; RC15; Hernandez-Deckers and Sherwood 2016, 2018; Peters et al. 2019). From deep convective LES, Hernandez-Deckers and Sherwood (2016, 2018) found that moist thermals changed little in size as they rose through the troposphere. Even though fractional entrainment and detrainment rates individually were large (~2–3 km$^{-1}$), they mostly balanced one another, and the net change in total mass of thermals was generally much smaller. Given the well-documented increase in radius and volume of dry thermals as they rise, these results beg an important question: Are spreading rates for moist thermals indeed small relative to those for dry thermals, and, if so, what are the reasons for this difference?

Shedding light on this question could help to improve understanding of entrainment and cumulus dynamics more generally. This is because understanding the parameters and mechanisms controlling evolution of thermal and updraft size is a critical yet largely uncertain aspect of cumulus dynamics. For example, there is evidence for the importance of updraft width, and by extension the sizes of thermals that the updrafts comprise, in controlling mixing and dilution of updraft properties from observations (Kyle et al. 1976), theory (Morrison 2017; Morrison et al. 2020), and LESs of moist convective thermals (e.g., Rousseau-Rizzi et al. 2017; Hernandez-Deckers and Sherwood 2018; Peters et al. 2020a,b). The idea that relatively large updrafts inhibit dilution from turbulent entrainment has been invoked to explain the transition from shallow to precipitating deep convection in LES studies (Khairoutdinov and Randall 2006). Improving understanding of the factors controlling thermal and updraft size is directly relevant to cumulus parameterizations, which often include an explicit dependence of fractional entrainment rates inversely on updraft radius (e.g., Simpson and Wiggert 1969; Kain and Fritsch 1990). Understanding the evolution of thermal sizes in cumulus clouds also provides context for the possible future development of new thermal-based convection parameterizations.

The basic goal of this study is to address the above question regarding differences in the spreading rate between moist and dry thermals. We analyze the dynamics of dry and moist thermals based on theory and numerical simulations. In section 2, we first review the theoretical work of Turner (1957) and McKim et al. (2020) to understand relationships between impulse, buoyancy, circulation, and spreading rate of dry thermals. Extension of the theory to moist thermals is then discussed. Section 3 describes the numerical model and experimental setup for the simulations. In section 4, we present results from the numerical simulations, showing that indeed moist thermals have smaller $dR/dz$ than dry thermals. We show how results from these simulations are consistent with the theoretical framework discussed earlier in the paper. We then test two hypotheses (described later) explaining the difference in spreading rates of moist and dry thermals using a set of sensitivity experiments. LES results are presented at the end of this section. A summary and conclusions are given in section 5.

2. Theoretical description
a. Dry convective thermals

We first briefly review the classic theory for dry thermals from Morton et al. (1956) and Scorer (1957). This arguably presents the simplest quantitative understanding of the dynamics of dry thermals, but it contains dimensionless constants that must be determined empirically. Assuming Boussinesq flow and that the thermal shape does not change in time, dimensional analysis implies that $dR/dz$ is constant and that the thermal’s volume can be expressed as $V \propto R^3$ (Scorer 1957). Note that this relation applies generally to any thermal shape as long as it is constant, and in this paper we define $R$ generally as the thermal half width at its widest point. It then follows that the fractional change in volume with thermal height $e = d\ln V/dz$ is

$$e = 3 \frac{dR}{R} \frac{1}{dz}.$$  (1)

Because $dR/dz = eR/3$ is a constant, entrainment efficiency $e = eR$ must also be a constant. Values of $dR/dz$ and/or $e$ have been determined empirically from laboratory (Morton et al. 1956; Scorer 1957; Richards 1961; Bond and Johari 2010) and numerical modeling (Lai et al. 2015; LJ2019) studies.

The fractional volume change $e$ is related to the net fractional mass entrainment rate, defined as the net mass inflow into a thermal given by $\epsilon_m = d\ln qV/dz$, where $q$ is air density. For Boussinesq flow, $\epsilon_m = e$. However, in density-stratified flow $\epsilon_m < e$ owing to the decrease of density with height. For tropospheric conditions, $\epsilon_m$ is smaller than $e$ by ~0.1 km$^{-1}$; for context, in the simulations here $e$ is roughly 0.3–0.6 km$^{-1}$. For simplicity, we limit analysis to $e$ herein. Note $\epsilon_m$ for a thermal is not the same as typically defined for a plume. Net mass entrainment for a thermal is the net mass inflow into a volume defined by the thermal boundaries as the thermal rises, whereas in a plume (or plume-based parameterization) it is the change in mass flux with height at a given height, equal to the horizontal mass inflow into the plume at that height by mass continuity.

Thermal radius and volume growth can also be understood from relationships between impulse, circulation, and width as proposed by Turner (1957). These relationships do not invoke the dimensional analysis and self-similarity arguments of Morton et al. (1956) and Scorer (1957). A brief review is given here, and details are in Turner (1957) and McKim et al. (2020).

Fluid impulse $I$ is the total momentum change starting from rest caused by external forcing over a finite region of the space, such as from gravity via a localized buoyancy anomaly. For an idealized infinite, Boussinesq fluid in the
absence of nonconservative forces, \( I \) is related to the vorticity \( \omega \) by (e.g., Batchelor 2000)

\[
I = \frac{P}{2} \int_{\mathcal{D}} x \times \omega \, d^3 x, \tag{2}
\]

where \( \mathcal{D} \) is the entire domain. Impulse is calculated from vorticity (vortical momentum) rather than velocity (linear momentum) because the latter is nonconvergent when integrated over the domain [see Batchelor (2000) for details]. Internal forces arising from pressure gradients do not impact \( I \), so \( I \) is essentially a record of the volume- and time-integrated external (in our case gravitational) forcing on the fluid.

If the region of nonzero vorticity within the fluid is concentrated on a circle with a radius equal to \( R \) (i.e., near the thermal’s boundaries) and this region of vorticity is small relative to \( R \), then (2) can be approximated as (Turner 1957; McKim et al. 2020)

\[
I_c = \pi \rho r^2 \Gamma^2, \tag{3}
\]

where \( \Gamma \) is circulation and subscript \( z \) indicates that the impulse is only in the \( z \) direction, as dictated by gravity and the lack of external forcing in the horizontal plane; \( \Gamma \) can be obtained as the integral of velocity along a circuit \( \delta S \) passing through the thermal center and returning through the ambient fluid (McKim et al. 2020). If we assume axisymmetry, this integral simplifies and can be expressed as the integral of azimuthal vorticity \( \omega_\phi \) over the surface bounded by \( S \):

\[
\Gamma = \int_S u \cdot d\mathbf{A} = \int_S \omega_\phi \, dr \, dz. \tag{4}
\]

The time rate of change of the circulation is given by

\[
\frac{d\Gamma}{dt} = \int_S -\frac{\delta B}{\delta r} \, dr \, dz \tag{5}
\]

for an inviscid fluid, where buoyancy \( B = -g \rho' / \rho \) (\( g \) is gravitationally acceleration and \( \rho' \) is the perturbation density). If the region with nonzero \( B \) is confined within \( S \) such that \( d\Gamma / dt = 0 \), when combined with (3) this implies

\[
\frac{dI_c}{dt} = \pi \rho r^2 \frac{dR^2}{dt} = F_B, \tag{6}
\]

where \( F_B = \int \rho B r \, dr \, dz \) is the buoyant force on a thermal integrated over its volume. Because \( \rho' \) is a tracer with no sources or sinks for inviscid, neutrally stratified, Boussinesq flow, \( F_B \) is constant as long as the region of buoyancy remains confined within the thermal.

In the presence of a positive buoyant force so that \( F_B > 0 \), (6) implies that \( dR^2 / dt > 0 \) if both \( \Gamma \) and \( \rho \) are constant (Turner 1957). In other words, buoyant forcing produces a monotonically increasing impulse that, if \( \Gamma \) and \( \rho \) are constant, requires an increase in thermal size \( R \). Indeed, such behavior is seen in the simulations of LJ2019 and McKim et al. (2020). If \( \Gamma \) and \( \rho \) are constant, it follows from (6) that the change in \( R \) is

\[
\frac{dR}{dt} = (2\pi \rho \Gamma R)^{-1} F_B. \tag{7}
\]

From (7), \( \varepsilon \) is found by relating \( R \) and thermal volume \( V \) using \( V \propto R^3 \), using the chain rule

\[
\frac{d}{dt} = \frac{dz}{dt} \frac{d}{dz} = \frac{w_{th}}{dz} \tag{8}
\]

\( (w_{th} \) is the thermal’s ascent rate), and rearranging terms to yield

\[
\varepsilon = \frac{d \ln V}{dz} = \frac{3F_B}{2\pi \rho R^2 w_{th}}. \tag{8}
\]

Combining the relationship \( w_{th} \propto 1/R^3 \) [McKim et al. 2020, their (9)] with (3) and (8) and assuming \( F_B \) is constant implies that the entrainment efficiency \( \varepsilon = \varepsilon R \) only depends on \( \Gamma \) and thermal shape for an inviscid, Boussinesq, neutrally stable fluid (McKim et al. 2020); if \( \Gamma \) and shape are constant, \( \varepsilon R \) must be constant. This result is consistent with the theory of Morton et al. (1956) and Scorer (1957). From these relationships, McKim et al. (2020) developed a simple analytic model for dry thermals that does not rely on empirically derived entrainment constants. This model provides a complete description of dry thermal behavior provided values of \( w_{th}, B, \) and \( R \) at some time after the thermal’s initial spinup.

Another important result from LJ2019 and McKim et al. (2020) is that the volume growth of dry thermals is primarily accomplished by laminar rather than turbulent flow. Increasing the Reynolds number from 630 to 6300, resulting in a transition from laminar to turbulent flow, led to variation in \( \varepsilon \) of only 20% (LJ2019). This result suggests that the increase in volume of a rising thermal is a fundamental dynamic entrainment process associated with the thermal’s buoyancy and ascent rather than smaller-scale turbulent motion. Motivated by this finding, to simplify the modeling framework as much as possible we compare moist and dry thermal laminar flow simulations in section 4, though we also briefly discuss LES of turbulent moist thermals at the end of this section.

b. Extension of theory to moist convective thermals

A critical feature of moist thermals is condensation and the accompanying latent heating that modifies the buoyancy field. Furthermore, there is typically evaporation along the thermal edge or in regions where dry air is entrained, leading to latent cooling and a decrease in buoyancy. This introduces changes in the buoyancy field that complicate the picture relative to dry thermals.

The basic governing equations are still valid for moist flow, though additional equations are needed to describe moist thermodynamics and microphysics. It follows that the dynamical equations derived directly from the governing equations for impulse \( I_c \) in (3), circulation \( \Gamma \) in (4), and \( d\Gamma / dt \) in (5) remain valid for moist thermals. However, there are three critical differences for moist compared to dry thermals: 1) the buoyant forcing \( F_B \) is no longer constant in time because of latent heating aloft; 2) the buoyancy distribution across moist thermals is altered because of latent heating that is concentrated near the thermal core, where the strongest vertical velocities and condensation rates occur; 3) there is latent cooling near the thermal periphery from cloud evaporation. Critically,
these differences mean that one can no longer assume $\Gamma$ is constant, unlike for dry thermals after an initial spinup period.

To account for nonconstant $\Gamma$, we derive an expression for $dL/dt$ that generalizes (6) simply by taking $d/dt$ of (3) and writing out all of the terms:

\[
\frac{dL}{dt} = \pi \rho \frac{dR^2}{dt} + \pi R^2 \frac{d\Gamma}{dt} + \pi R^2 \frac{dp}{dt} = F_B. \tag{9}
\]

We have written (9) in a general way that also retains the $dp/dt$ term to approximate non-Boussi`nesq effects of density stratification. Note that $dp/dt$ here is the change in mean thermal density following the thermal (i.e., mean density of the region bounded by $R$). Though an exact expression for impulse in a compressible fluid was derived by Shivamoggi (2010), and is given by

\[
I = \frac{1}{2} \int_g x \times (\nabla \times \mathbf{u}) \, dx,
\]

we neglect covariances between local density anomalies and velocity in compressible flow that have a finite but presumably small effect on $I$, for atmospheric conditions.

Although one can derive an expression for $I$ from (9) using $V = mR^3$, this has limited usefulness because it contains terms with $\Gamma$ and $d\Gamma/dz$ that cannot be assumed constant. Indeed, because the buoyancy structure of moist thermals depends on a complicated set of interacting microphysical processes, expressing $\Gamma$ generally in terms of known parameters does not seem possible. Nonetheless, (9) provides qualitative insight into the behavior of moist thermals. Because of latent heating in the core of moist thermals, where the largest vertical velocities and thus condensation rates occur, following (5) this implies $d\Gamma/dt > 0$ and hence $d\Gamma/dz > 0$. From (9), it then follows that $dR^2/dt$ (and $dR^2/dz$ using the chain rule $d/dt = w_{th}d/dz$) must be smaller than it would be if $\Gamma$ were constant, all else being equal. A complication, of course, is that in general all else is not equal for dry and moist thermals. In particular, the latent heating in moist thermals increases overall buoyant forcing $F_B$, which leads an increase in $w_{th}$ with thermal height (for the neutrally stable environmental profile tested here). However, we have carefully designed sensitivity experiments to minimize these differences by adding heating speed uniformly across the thermal volume or only within its core (see section 4b). These simulations show dramatic differences in thermal growth depending on the distribution of this heating even when $w_{th}$ and $F_B$ are similar, supporting the picture given by (9). Specifically, heating confined to the core concentrates the spatial distribution of vorticity and limits thermal spread.

3. Description of model and experimental setup

The Cloud Model, version 1 (CM1), is a nonhydrostatic, compressible model for simulating atmospheric flow (Bryan and Fritsch 2002). A hierarchy of model configurations with varying complexity using CM1 is employed in this study. In the simpler configuration, we use a two-dimensional axisymmetric version of CM1 to model dry and moist thermals. In the other configuration we use a fully three-dimensional (3D) version of CM1 as an LES of turbulent moist thermals within cloud updrafts.

In all configurations the upper and lower boundary conditions are free slip and a rigid lid. A Rayleigh damping layer is applied above 15 km. Radiation and surface fluxes are neglected. In the moist simulations, microphysical processes follow from Morrison et al. (2009), but precipitation generation and ice processes are turned off so that only cloud condensation and evaporation following "saturation adjustment" are included. In both the axisymmetric simulations and LES a first-order Smagorinsky-type subgrid-scale mixing scheme is employed with the mixing length equal to the model grid spacing.

As this study is focused on the behavior of moist and dry thermals, a method for identifying and tracking thermals in the simulations is required. We discuss various methods and in the appendix propose a new method that is a combination of two previous methods. For the analysis, buoyancy was calculated relative to the horizontally homogeneous but height-varying hydrostatic background state. Circulation was calculated simply by summing the azimuthal vorticity calculated at each grid cell times the grid cross-sectional area across the surface defined by a square-shaped circuit through the thermals’ central axes and extending 1 km beyond the thermals’ radii. We separately calculated circulation by numerically integrating the velocity along the circuit $(\mathbf{u} \cdot d\mathbf{l})$ and results were very similar (generally < 1% difference). All quantities that are calculated as derivatives, including vorticity and thermal ascent rates, use central finite differencing. Calculations are based on model data output every 15 s.

a. Description of the axisymmetric simulations

In the axisymmetric configuration, the effects of water vapor and condensate loading on the buoyancy field are neglected in the moist simulations for simplicity, while there is no vapor or cloud in the dry simulations. For the baseline simulations the domain extends 25 km in the horizontal plane and 21.2 km in the vertical direction. The horizontal and vertical grid spacing is 50 m.

Two different sets of simulations using the axisymmetric model with different environmental static stabilities are used. In the first, we use a simple neutrally stable initial sounding through the depth of the model, with uniform potential temperature of 300 K and a surface pressure of 1000 hPa (note that, here and throughout the paper, static stability is defined with regard to dry dynamics). This setup is used for both dry and moist thermal simulations. For dry thermals the water vapor field is zero everywhere. For the moist simulations, the initial relative humidity (RH) profile is taken from the modified Weisman and Klemp (1982, hereinafter WK) profile below 2 km and set to a uniform RH of 42.5% above. Note that a dry-neutral environmental profile extending over a large depth leads to very large convective available potential energy (CAPE) and extreme vertical velocities for the moist simulations. To keep vertical velocities within a reasonable range, we reduced the latent heating rate from cloud condensation by a factor of 4 (note that latent cooling from evaporation was not altered for simplicity). For consistency with the saturation adjustment method used by the microphysics, the vapor field is...
adjusted to maintain exactly saturated conditions inside clouds with the reduced latent heating rate. This reduction of latent heating has minimal impact on \( dR/dz \) and \( e \), although it strongly impacts vertical velocities.

In the second setup using the axisymmetric model, we simulate moist thermals using a more realistic initial temperature profile following the sounding from WK with two different uniform RH profiles above 2 km: 42.5% and 85%. Unlike the first setup, no modification is made to the latent heating rates. This setup is not run for dry thermals.

In all of the axisymmetric simulations, thermals are generated by adding a warm bubble to the initial conditions. The baseline simulations use a 2-K initial potential temperature perturbation \( \theta^\prime \) that is uniform within a spherical bubble for simplicity; LJ2019 showed considerable variability even in their laminar simulations with perturbations to a nonuniform initial \( \theta^\prime \) distribution. Additional dry simulations use different initial \( \theta^\prime \) of up to 5 K. In all simulations the initial water vapor mixing ratio is adjusted within the bubbles to retain the same height-varying RH as the background environment. Unlike the LES, there is no small-scale noise added to the potential temperature or any other quantities and the fields remain smooth and the flow laminar. To improve robustness, small ensembles are run for all moist and dry configurations using slightly different initial \( \theta^\prime \) within the bubbles, modified by \( \pm 0.05 \) K. Because the spread among members of a given ensemble is small, most plots herein only show the “control” member with no modification of the initial \( \theta^\prime \).

The initial bubble radius is 750 m, and bubbles are centered at 750 m height in all axisymmetric simulations except as noted. This is similar to the mean radius of thermals that develop in the LES. Although previous numerical studies (e.g., LJ2019) initialized thermals well away from the lower boundary, here we place thermals initially near the lower boundary to mimic thermals originating from the planetary boundary layer, consistent with the LES. A dry thermal axisymmetric simulation with the thermal center initially at 4 km gives similar results to that initialized at 750 m, with small increases in \( R \) (\(-10\%\)) and \( V \) (\(-30\%\)) for the same relative height (i.e., distance above the initial thermal center height). To explore reasons for differences in behavior between the moist and dry thermals, we also run additional simulations that apply heating to the dry thermals. The configuration for these simulations is described in section 4b.

b. Description of the LES

Fully 3D LES are analyzed to demonstrate that the small thermal spreading rates shown in moist axisymmetric simulations are also characteristic of moist thermals in flow with realistic turbulence. These simulations use a 26.4 km cubic domain with periodic lateral boundary conditions and free-slip top and bottom boundary conditions. The horizontal and vertical grid spacing is 100 m. Surface-to-atmosphere fluxes are turned off. Initial thermodynamic conditions are from the analytic sounding of WK with a boundary layer mixing ratio of 14 g kg\(^{-1}\), except the initial RH above 2 km is set to a uniform value of either 42.5% or 85%. Initial winds are set to zero. Convection is initialized with a 1.5-K Gaussian-shaped warm bubble centered at a height of 0.5 km. Small random perturbations of up to \( \pm 0.1 \) K are applied to the initial \( \theta \) field to spin up asymmetry and help seed turbulent motion. As shown by Peters et al. (2019), this configuration produces reasonable turbulence properties within a few min of model integration, including the existence of a \(-5/3\) slope of the kinetic energy spectra in an inertial subrange. Initial bubble horizontal radii are 1000, 1500, and 2000 m, and vertical radii are 500 m in all simulations. For each of these setups, four ensemble members with different realizations are run by varying the random number seed for the initial \( \theta \) perturbations.

4. Axisymmetric simulation results

a. Neutrally stable sounding

In this section, we first present results from the axisymmetric dry and moist simulations using the neutrally stable environmental profile. This allows for a direct comparison of dry and moist simulations, as otherwise the dry thermals would quickly decay in a stably stratified environment without any additional buoyancy source.

Coherent thermals develop and rise as expected in the moist and dry simulations. This is illustrated by vertical cross sections of the baseline (initial \( \theta^\prime \) of 2 K) dry and moist thermal simulations (Figs. 1 and 2). There is an initial increase of buoyancy over time for the moist thermal from condensation and latent heating, and large positive buoyancies are retained in the central thermal core. In contrast, buoyancy slowly decreases for the dry thermals owing to entrainment and mixing (though the domain-integrated buoyancy is nearly constant), and positive buoyancy becomes confined to the outer ring core near the center of maximum vorticity. This behavior is consistent with recent dry thermal simulations (Lai et al. 2015; LJ2019; McKim et al. 2020).

Evolution of thermal characteristics (\( w_{th} \), \( w_{max} \), thermal half-width \( R \), and thermal volume \( V \)) for the moist simulations and dry simulations with initial \( \theta^\prime \) 2 to 5 K is shown in Fig. 3. Height on the \( y \) axis corresponds to the height of thermal center as it rises (defined by the location of maximum \( w \); see the appendix). Note that LJ2019 leveraged the relation \( w_{th} \sim t^{-1/2} \) to provide smooth estimates of \( w_{th} \) for applying thermal tracking, which improves robustness and reduces noise. While our dry thermals approximately follow this scaling, we do not employ this method because the moist thermals do not follow a \( w_{th} \sim t^{-1/2} \) scaling. As a result, there is some noise in calculating \( w_{th} \), mainly for the dry thermal with a 2-K initial \( \theta^\prime \) because this has the smallest \( w_{th} \). In this case, the finite differencing causes “jumps” in the thermal height every few output times, which also leads to some noise in calculating \( R \) and \( V \). In contrast, for the simulations with larger \( w_{th} \), the thermal height increases one or more vertical levels every output time resulting in smoother estimates. Simply applying a running mean smoother over a fairly short time window (30 s) removes much of this noise as seen in Fig. 3. We anticipate that decreasing the vertical grid spacing would also reduce noise, but such tests are beyond the scope of this study.
The latent heating in moist thermals increases $w_{th}$ and $w_{max}$ over height/time. This latent heating is concentrated within the thermal core where the strongest vertical velocities occur (Fig. 4). As the moist thermals rise the latent heating rates decrease with altitude. This corresponds to the decrease of condensation rate with decreasing temperature (consistent with the moist adiabatic lapse rate approaching the dry lapse rate as temperatures decrease). In contrast to the moist simulations, the dry thermals quickly (with a distance of a few radii) reach maximum values of $w_{th}$ and $w_{max}$, followed by a slow decrease, with the ratio $w_{th}/w_{max}$ near the value of 0.4 consistent with Hill’s analytic spherical vortex.

It is evident from Fig. 3 that net growth rates both in terms of $R$ and $V$ are smaller for the moist thermals than for the dry
ones. For example, once the thermals’ centers reach 8 km, the dry thermals have volumes that are greater than the moist thermals by a factor of ~1.6. Although \( w_{th} \) and \( w_{\text{max}} \) are sensitive to the initial \( \theta' \) for the dry thermals (and scale approximately with the square root of the initial buoyancy perturbation), the spreading rate and volume growth of dry thermals are insensitive to the initial buoyancy. This result is consistent with the theory of Morton et al. (1956) and Scorer (1957) from dimensional analysis.

Figure 5 shows \( dR/dz \), fractional volume change \( e \), and entrainment efficiency \( e \) as a function of thermal center height. \( dR/dz \) is consistently smaller for the moist compared to dry thermals, by an average factor of 1.7 between 3- and 8-km height. The \( e \) is also smaller for the moist thermals, but only below 4 km; above this height it is remarkably similar to that of the dry thermals. This occurs because relative differences in \( e \) and \( R \) between moist and dry thermals above 4 km are similar, compensating one another as \( e \) is equal to the ratio of \( e \) and \( R \). Thus, most of the differences in fractional volume growth between the moist and dry thermals occur below 4 km, with similar rates above (which is also seen in Fig. 3d). The dry thermals have a fairly constant \( dR/dz \) and \( e \) with height above ~3 km and \( e \) scales with \( R^{-1} \) consistent with the theory discussed in section 2. \( e \) from the moist thermal also scales with

![FIG. 2. As in Fig. 1, but for the baseline moist thermal axisymmetric simulation with the dry neutrally stable environment. Simulation times shown here were chosen to give similar thermal heights as the dry thermal shown in Fig. 1.](image-url)
Above 4 km height, though not below this height. Above 4 km the moist and dry thermals have similar values of \( e \) roughly between 0.6 and 0.8. Why, then, is \( \frac{dR}{dz} \) consistently smaller for the moist thermals relative to dry thermals at all heights, while \( e \) and \( e' \) are similar among the moist and dry thermals above 4 km? This is explained by changes in moist thermal shape over time. As seen in the vertical cross section plots (Fig. 2), the moist thermal grows somewhat more vertically than horizontally. This compensates for its relatively slow horizontal spread (the latter defining the growth of \( R \)), producing volume growth rates and hence \( e \) and \( e' \) similar to the dry thermals, but smaller \( dR/dz \).

However, because profiles of thermal volume reflect the integrated change in volume over height and all of the thermals start with almost the same volume, the smaller \( e \) for the moist thermal below 4 km means its volume remains substantially smaller than that of the dry thermals above 4 km.

Fig. 3. Evolution of (a) tracked thermal velocity \( w_{th} \), (b) maximum vertical velocity \( w_{\text{max}} \), (c) radius \( R \), and (d) volume \( V \) as a function of the height of thermal center. Results from the baseline moist (‘‘Moist’’), moist but with evaporation turned off (‘‘Moist–Nevp’’), and dry simulations with initial bubble \( \theta' \) ranging from 2 to 5 K are shown. All simulations use the neutrally stable environmental profile.

For Moist and Moist–Nevp different lines represent individual ensemble members with small perturbations to the initial bubble \( \theta' \). Except for \( w_{\text{max}} \) quantities are calculated as 30-s moving averages to reduce noise.

Fig. 4. Vertical cross sections of latent heating rate (color fill) and vertical velocity (black contour lines) from the baseline moist axisymmetric simulation at three different output times: (a) 6, (b) 9, and (c) 12 min. Vertical velocity contours are every 1 m s\(^{-1}\) between \(-2\) and 2 m s\(^{-1}\) and every 4 m s\(^{-1}\) otherwise. Solid and dashed lines show positive and negative vertical velocities, respectively. The \( X \) axis shows distance from the central axis of symmetry.

Though applying a moving average with a short time window (30 s) removes much of the noise in the \( R \) and \( V \) profiles in Figs. 3c and 3d, there is still some noise using even a longer window (2 min) for the derivatives shown in Fig. 5 (and in the circulation budget discussed below); derivatives of quantities are inherently noisier than the quantities themselves. To estimate uncertainty in the vertically averaged \( dR/dz \) and \( e \) from this noise, we have calculated these averages by randomly sampling the height range used for the calculation 100 times, varying the upper and lower heights by \( \pm 0.5 \) km. This gives about a 10%–15% spread of vertically averaged \( dR/dz \) and
are calculated from differencing used to calculate these quantities, results shown are 2-min moving averages. Theoretical values of fractional entrainment rate for the dry thermal simulations with varying initial perturbation potential temperatures. Results from the baseline moist simulation are shown by the black lines. Theoretical values of fractional entrainment rate from the theory of Morton et al. (1956) and Scorer (1957) are shown by the dashed lines in (b); these calculations use \( e = 0.6 \) for illustration. Because the theoretical \( e \) depends on \( R \), values are calculated from \( R \) for each of the simulations, with the color indicating the corresponding simulation. To reduce noise with the finite differencing used to calculate these quantities, results shown are 2-min moving averages.

The mean \( dR/dz \) and \( e \) calculated from these 100 vertically averaged profiles are given in Table 1, along with values from previous numerical and laboratory studies. Mean values of \( dR/dz \) were 0.25 and \( e = 0.67 \) for the baseline dry thermal (750-m initial radius) here are generally within the range of values from previous studies, though on the upper end of this range. Nonetheless, values of \( dR/dz \) and \( e \) for the baseline dry thermal are somewhat larger than those from recent numerical studies (Lai et al. 2015; LJ2019). This is attributable in part to differences in the thermal tracking method, which is detailed in the appendix. Using the LJ2019 method, which is the same or similar to what was used in these previous studies, leads to reductions of \( dR/dz \) and \( e \) of 12%–16% for the baseline dry thermal compared to the COMB method used here otherwise (Table 1). The remaining differences might be explained by the non-Boussinesq effects of density stratification; Lai et al. (2015) and LJ2019 assumed Boussinesq flow. Density stratification also explains the large sensitivity to the initial thermal size here. Decreasing the initial dry thermal size by one-half to 375 m (with a commensurate reduction in model grid length so the thermals remain well resolved) leads to notable decreases in \( dR/dz \) and \( e \) of \( \sim 30\% \), bringing values much closer to those reported by Bond and Johari (2010), Lai et al. (2015), and LJ2019 (see Table 1). Conversely, an increase in dry thermal initial size by a factor of 2 to 1500 m leads to increases in \( dR/dz \) and \( e \) of \( \sim 70\% \).

One possible mechanism driving sensitivity of \( dR/dz \) and \( e \) to the initial thermal radius is that radius growth rates from horizontal expansion of rising thermals in a density-stratified flow increase with thermal size. Anders et al. (2019) analyzed a similar problem of the effects of density stratification on numerical simulations of sinking cold bubbles. They showed that the change in \( R \) for cold bubbles lies between growth in Boussinesq flow (following the classical scaling of \( R \) increasing with constant \( dR/dz \) and “pure” horizontal compression that scales as \( R \propto \rho^{-1/2} \) (Brandenburg 2016), depending on the ratio

\[
\frac{dR}{dz} = \frac{e}{3}
\]

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\[
\frac{dR}{dz} = \frac{e}{3}
\]

TABLE 1. Summary of entrainment efficiency \( e \) and \( dR/dz \) from the axisymmetric moist and dry simulations in the current study (boldface font) and previous numerical simulations (Lai et al. 2015; LJ2019) and laboratory studies [Morton et al. 1956 (MTT); Scorer 1957 (S57); Richards 1961 (R61); Bond and Johari 2010] of dry thermals. Similar to LJ2019, we give both \( e \) and \( dR/dz \) using the scaling \( dR/dz = e/3 \) for studies that reported one but not the other. Note that Lai et al. (2015) reported simulation results for initial thermal aspect ratios ranging from 0.125 to 2, but we only show values for aspect ratio of unity from their study; “375 m,” “750 m,” and “1500 m” indicate the radius of the initial thermal in the current simulations.

<table>
<thead>
<tr>
<th>Study</th>
<th>( E )</th>
<th>( dR/dz )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dry 375 m</strong></td>
<td>0.46</td>
<td>0.18</td>
</tr>
<tr>
<td><strong>Dry 750 m (base)</strong></td>
<td>0.67</td>
<td>0.25</td>
</tr>
<tr>
<td><strong>Dry 1500 m</strong></td>
<td>1.17</td>
<td>0.41</td>
</tr>
<tr>
<td><strong>Moist 750 m (base)</strong></td>
<td>0.52</td>
<td>0.16</td>
</tr>
<tr>
<td><strong>Dry 750 m (LJ2019)</strong></td>
<td>0.56</td>
<td>0.22</td>
</tr>
<tr>
<td>Lai et al. (2015)</td>
<td>(-0.45–0.51)</td>
<td>(-0.15–0.17)</td>
</tr>
<tr>
<td>L2019</td>
<td>0.36–0.47</td>
<td>0.12–0.16</td>
</tr>
<tr>
<td>MTT56, S57, R61</td>
<td>(-0.75)</td>
<td>(-0.25)</td>
</tr>
<tr>
<td>Bond and Johari (2010)</td>
<td>(-0.5)</td>
<td>(-0.17)</td>
</tr>
</tbody>
</table>

FIG. 5. Evolution of (a) \( dR/dz \), (b) fractional volume change \( e \), and (c) entrainment efficiency \( e = eR \) as a function of the thermal center height for the dry thermal simulations with varying initial perturbation potential temperatures. Results from the baseline moist simulation are shown by the black lines. Theoretical values of fractional entrainment rate \( e = eR \) from the theory of Morton et al. (1956) and Scorer (1957) are shown by the dashed lines in (b); these calculations use \( e = 0.6 \) for illustration. Because the theoretical \( e \) depends on \( R \), values are calculated from \( R \) for each of the simulations, with the color indicating the corresponding simulation. To reduce noise with the finite differencing used to calculate these quantities, results shown are 2-min moving averages.
of bubble size to density scale height. In contrast, the effects of growth from entrainment and pure expansion are in the same direction for rising thermals. We use the \( R \propto \rho^{-1/2} \) scaling to estimate the effect of such expansion on thermal growth between 2 and 8 km height, giving

\[
\frac{dR'}{dz} = \frac{R_0 - R_1}{\Delta z} \approx \frac{R_0}{\Delta z} + \left( \frac{dR}{dz} \right) \left( \frac{\rho_0}{\rho_1} \right)^{1/2} - \frac{R_0}{\Delta z},
\]

where \( R_0 \) and \( R_1 \) are the thermal radii at 2 and 8 km, \( \Delta z = 6 \text{ km} \) is the height difference, \( dR/dz \approx 0.15 \) is the Boussinesq growth rate based on recent modeling studies (Lai et al. 2015; LJ2019; see Table 1), and \( \rho_0 \) and \( \rho_1 \) are the background densities at 2 and 8 km, respectively. Plugging in values for the 750- and 1500-m-initial-radius thermals here gives \( dR'/dz \) of 0.25 and 0.29 as compared with the actual \( dR'/dz \) of 0.25 and 0.41, respectively (see Table 1). Thus, while this estimate of the effect of expansion works in the right direction, it underestimates the magnitude of the differences in \( dR'/dz \) between the 750- and 1500-m thermals.

Another factor related to density stratification is that as the length scale of a flow feature becomes a significant fraction of the density scale height (about 7–8 km in Earth’s atmosphere), the \( \omega \rho \delta \z \) term in the anelastic continuity equation becomes negligible. Because \( \omega \rho \delta \z < 0 \), this implies that the horizontal inflow in the lower part of the thermal’s toroidal circulation is balanced by relatively stronger outflow in the upper part, which may also contribute to greater spread of large thermals than small ones in a density stratified environment. Despite the sensitivity to initial thermal size, overall differences between moist and dry thermals are similar with changes to the initial thermal size; namely, moist thermals have slower growth rates than dry ones, particularly at lower heights (below 4–5 km) for volume growth. A more detailed investigation of sensitivity to thermal size and effects of density stratification is beyond the scope of this study and left to future work.

We next present results for the dry and moist thermal simulations in the context of the theoretical impulse-circulation relation from section 2. Figure 6 shows evolution of the three terms in the impulse budget (9): “entrainment” \( [\pi \rho \Gamma (dR^2/dt)] \), “circulation” \( [\pi \rho R^2 (d\z/dt)] \), and “density” \( [\pi R^2 \Gamma (d\rho/dt)] \). Also plotted is the “total,” which is the sum of the three terms and is approximately equal to \( dL/dt \). The y axis in these plots is the height of the thermal center as it rises. We first focus on discussing results for the baseline dry and moist thermals (Figs. 6a and 6b, respectively).

The circulation term (blue lines) is smaller than the entrainment term (red lines) by a factor of about 4 for the dry thermal, while the circulation and entrainment terms are similar in magnitude for the moist thermal. Note that some studies (Zhao et al. 2013; McKim et al. 2020) have shown that the change in circulation for dry thermals is close to zero, confirming the hypothesis of Turner (1957). This contrasts with our results in Fig. 6a, which show that the

circulation term is relatively small but nonzero. This difference from previous studies does not reflect our analysis method: using a different thermal tracking algorithm (LJ2019 instead of COMB; see the appendix) or changing the region \( S \) over which the circulation is calculated (as long as it encompasses the thermal) has little impact. However, the nonzero circulation change is consistent with an estimate of the circulation tendency calculated directly by integrating \( -\delta B/\delta r \) following (5) over the area defined by the circuit used to calculate the circulation. The small positive circulation tendency is consistent with the buoyancy distribution seen in Fig. 1. In particular, the red line showing the 0.1-K \( \theta' \) contour crosses the central buoyancy axis even when the thermal has risen above 7 km (Fig. 1e). Thus, the circulation increase is caused by small net horizontal buoyancy gradients across the region of integration \( S \), whereas previous studies presumably had near-zero net gradients. This difference in the buoyancy field may result from differences in the numerics between CM1 and previous modeling studies, particularly the subgrid-scale and implicit (numerical) mixing.

Consistent with the buoyancy distribution in Fig. 1, the positive circulation tendency \( dL'/dt \) decreases as the thermal rises. However, the circulation term \( \pi R^2 (d\z/dt) \) for the dry thermal in Fig. 6a does not change much with thermal height (above 2 km) because the decrease in \( dL'/dt \) and \( \rho \) is compensated by the increase in \( R^2 \).

The entrainment term in Fig. 6 is similar to the total (black lines) for the dry thermal, whereas the total is somewhat larger than both the entrainment and circulation terms for the moist thermal. In both simulations the density term is negative, which corresponds with the decrease of \( \rho \) with height. Large differences in the magnitude of all terms between the moist and dry thermals (roughly an order of magnitude) reflect much greater total buoyancy forcing for the moist simulation. Considering all simulations, the total (sum of the circulation, entrainment, and density terms) is strongly correlated with the integrated buoyant forcing \( F_B \) (coefficient \( r_{corr} = 0.96 \)), as anticipated from the discussion in section 2. Overall, these results suggest that the buoyant forcing is compensated mainly by an increase in \( R \) for the dry thermal, and by increases in both \( R \) and circulation for the moist thermal. For dry thermals, this is consistent with Turner (1957) and McKim et al. (2020). This also supports our hypothesis that latent heating in moist thermals drives a stronger increase in circulation but relatively weaker growth rates when compared with dry thermals. The specific linkage between this behavior and latent heating is elucidated with additional tests discussed in the next section.

d. Additional sensitivity tests

In the analysis discussed above, we showed that the radius and volume growth rates of moist thermals are substantially smaller than dry thermals. Here, we explain why this difference occurs. Two competing hypotheses are formulated: 1) weak growth rates of moist thermals result from latent heating regardless of how that heating is distributed; 2) weak growth rates result from latent heating that is concentrated within the cores of moist thermals, as seen in Fig. 4.
To test these two hypotheses, two additional sets of simulations are run. These use the neutrally stable dry thermal configuration with an initial bubble radius of 750 m and initial $u_0$ of 2 K. The first set adds a heating rate that is constant in time and spread uniformly across the approximate thermal volume, referred to as “Dry_spr.” The second adds heating only to the inner core of the thermal, referred to as “Dry_core.” Heating rates tested in both sets of experiments are 0.005, 0.01, and 0.015 K s$^{-1}$. In these simulations the thermal boundaries must be determined during the runs. However, it is not trivial to calculate the thermal boundaries in-line using the thermal tracking method employed here (nor using other tracking methods). To avoid this complication, we approximate the boundary by assuming a spherical thermal shape with $R$ set to 0.9 times the horizontal distance from the location of minimum $w$ to the central axis. This provides a good approximation of the dry thermal boundary (see Fig. 1). In Dry_spr, we apply heating uniformly within the region defined by this boundary. In Dry_core, heating is applied to the inner 500 m core between 1 km height and the thermal top at all times. Thus, heating is applied both within and below the thermal when it rises sufficiently high. This mimics the latent heating within the moist thermals (see Fig. 4). Note that latent heating rates in the moist simulations decrease with altitude, but for simplicity the heating rates in these sensitivity simulations are constant in time and evenly distributed within this inner region.

Evolution of $w_{th}$, $w_{max}$, $R$, and $V$ for the Dry_spr simulations is shown in Fig. 7 as a function of the thermal center height, in a similar format to Fig. 3. Results are compared with the baseline dry and moist thermal simulations. As expected, $w_{th}$ and $w_{max}$ are sensitive to the heating rates,
since they strongly impact the thermal-averaged buoyancy. For the largest heating rate tested, both $w_{th}$ and $w_{max}$ are similar to the baseline moist thermal simulation. Although $w_{th}$ and $w_{max}$ are sensitive to the applied heating rates, there is remarkably little sensitivity of $R$ and $V$. Thus, the Dry_spr simulations produce $dR/dz$ and $\varepsilon$ that are similar to those of the baseline dry simulations and that are consistent with the theory of Morton et al. (1956) and Scorer (1957).

In contrast to Dry_spr, $R$ and $V$ (as well as $w_{th}$ and $w_{max}$) are sensitive to the applied heating rate in the Dry_core simulations, shown in Fig. 8. For the largest core heating rate tested (0.015 K s$^{-1}$), $w_{max}$ and $w_{th}$ are similar to and slightly larger than, respectively, those in the baseline moist thermal simulation, while $R$ and $V$ are smaller than the moist simulation after the thermal centers reach ~5 km. A key result is that $dR/dz$ and $\varepsilon$ decrease with increasing core heating rate, as seen by the profiles of $R$ and $V$. However, this effect appears to saturate as profiles of $R$ and $V$ are similar when applying heating rates of 0.01 or 0.015 K s$^{-1}$.

The Dry_spr and Dry_core experiments are also analyzed in terms of the impulse-circulation relation. Evolution of terms in the impulse “budget” as a function of the thermal center height for representative simulations from the Dry_spr and Dry_core tests is shown in Figs. 6c and 6d. Differences between Dry_spr and Dry_core are somewhat consistent with differences between the baseline dry and moist thermal simulations shown in Figs. 6a and 6b. For Dry_spr the circulation term is smaller than the entrainment term and total, qualitatively similar to the baseline dry thermal simulation but with smaller differences between these terms than in the baseline dry simulation. In contrast, the circulation term is larger than the entrainment term in Dry_core. Because the heating in Dry_core is confined to a relatively small area the total buoyant forcing and hence “total” impulse change (sum of all terms in the impulse budget; black line in Fig. 6d) is smaller than in Dry_spr and the baseline “Moist” simulation, and it decreases somewhat with height above 4 km. This decrease is caused by the decrease in $\rho$ as the thermal rises.

Besides a larger magnitude of all terms in Dry_spr relative to the baseline dry simulation owing to greater buoyant forcing, the other notable difference is that the circulation term increases over time (height). This occurs because the thermal heating increases $\int^t_0(\partial B/\partial R)drdz$ and thus circulation. The key point, however, is that the spreading rate in Dry_spr is almost the same as the baseline dry thermals as seen by comparing profiles of $R$ and $V$ (Fig. 7c). Thus, thermal growth seems to be fundamentally controlled by the spatial distribution of buoyancy, and not by whether the circulation itself is constant or increases with thermal height. That is, circulation increases with an increase in thermal-averaged buoyant forcing from thermal heating, but if the heating is distributed...
uniformly across the thermal (as in Dry_spr) $d\Gamma/dz$ and $\epsilon$ are mostly unaffected. In this case, the circulation term increases with height but remains smaller than the entrainment term. Circulation also increases when heating is confined to the thermal core (as in Dry_core), but in this case the circulation term is larger than the entrainment term. These results are consistent with the theoretical insight gained from (9). Physically, heating confined to the core leads to horizontal buoyancy gradients that generate vorticity inside the thermal. Whereas in dry thermals, or those with heating spread uniformly across the thermal volume, the maximum buoyancy occurs near the thermal edge, which generates vorticity in the outer region but destroys vorticity in the inner region, thus causing the vortex center to move outward (Fig. 9).

Besides latent heating, another difference between the moist and dry simulations is latent cooling from evaporation. To test its impact, an additional set of moist simulations is run with evaporation turned off. Of course, the effects of evaporation and latent cooling depend strongly on the RH of the environment. Even with the fairly dry environment (RH = 42.5% above 2 km) in the baseline moist simulations, the effects of evaporation on profiles of $R$ and $V$ are small; the simulations with no evaporation produce $\sim$10%-larger $R$ and $\sim$25%-larger $V$ than baseline at 8-km height (Fig. 3). Note that the effects of mixing, which occurs only through parameterized subgrid-scale mixing and numerical mixing in these laminar flow simulations, could enhance this effect by increasing mixing along the cloud–environment interface. We leave an investigation of this idea to future work.

c. Weisman–Klemp sounding

To understand moist thermal evolution for more realistic atmospheric conditions, we run axisymmetric simulations using the modified WK thermodynamic sounding (see section 2). These simulations also provide a bridge to the LES setup that uses the same initial sounding. Simulations produce ascending thermals generally similar to those using the neutrally stable profile, but with some important differences (cf. Fig. 10 with Fig. 2). Most notably, static stability in the environment ($\partial \theta/\partial z \approx 3.8 \text{K km}^{-1}$ in the midtroposphere) using the WK sounding leads to the generation and propagation of gravity waves, evident by the patterns of upward and downward motion and weak buoyancy anomalies outside of the main thermal (Fig. 10). Note the WK simulations with a fairly dry free tropospheric RH (42.5%) dissipate rapidly at $\sim$7.5-km height.

Evolution of $w_{\text{wh}}, w_{\text{max}}, R$, and $V$ for the simulations using the WK sounding and free tropospheric RH of 42.5% and 85% are shown in Fig. 11. Also shown are the baseline dry and moist simulations with the neutrally stable environmental profile. The RH = 85% WK simulations produce $R$ and $V$ profiles similar to the baseline moist simulation below 7.5 km, while the RH = 42.5% WK thermals are somewhat smaller, especially above 5 km. The RH = 85% WK thermals spread rapidly above 7.5 km, which occurs after they accumulate substantial negative buoyancy and rapidly decelerate. This is associated with the WK thermodynamic profile, which has its level of maximum buoyancy near this height and thus decreasing buoyancy above this level for a moist pseudoadiabatically lifted parcel. In the presence of mixing with environmental air, this leads to a rapid reduction of thermal buoyancy. A similar deceleration occurs in the RH = 42.5% WK thermals above 6 km but they quickly dissipate instead of undergoing rapid expansion. Overall, these results indicate that increasing environmental static stability has some influence on moist thermal growth rates that appear to be RH dependent, mainly once thermals reach their levels of neutral buoyancy and rapidly decelerate. However, impacts of static stability are small in the lower and mid troposphere. This is consistent with the insensitivity of $d\Gamma/dz$ and $\epsilon$ to a factor-of-4 change in the latent heating rates for the neutrally stable sounding (mentioned in section 3a). Thus, the thermodynamic profile generally seems to have a limited impact on thermal growth rates as long as the buoyancy generated by latent heating aloft is comparable to or greater than the initial thermal buoyancy. Conversely, an additional moist simulation with the baseline neutrally stable sounding but increasing the initial $\theta'$ from 2 to 10 K, giving an initial thermal buoyancy much larger than that generated by latent heating, has substantially increased $d\Gamma/dz$ and $\epsilon$ (not shown). Growth rates for this simulation are comparable to those of the dry thermals.

d. Large-eddy simulation results

The main objective of the LES analysis is to augment the axisymmetric simulation results by assessing radius and volume growth rates of turbulent moist thermals. To accomplish this, we apply the thermal tracking algorithm described in the appendix to the LES output. Although previous studies have already analyzed moist thermals in LES using similar tracking procedures (e.g., S13; RC15; Hernandez-Deckers and Sherwood 2016, 2018), we apply a different LES setup that allows for a more direct comparison with the axisymmetric simulations. We use the COMB thermal identification and tracking method described in the appendix. Only coherent and continuous thermal tracks lasting 4 min or longer are considered in subsequent analysis, resulting in 27 usable thermal tracks among the 24...
simulations (three of the simulations have multiple thermals that meet these criteria). Example vertical cross sections of $w$ and $B$ along with flow vectors and location of a tracked thermal from one of simulations are shown in Fig. 12. This figure highlights the toroidal circulation of the tracked thermal and shows that the largest $B$ and $w$ values are concentrated in its core similar to the axisymmetric thermals.

Profiles of various quantities as a function of the height of all tracked thermal centers are shown in Fig. 13. Ascent rates of thermals in LES are consistent with those in the WK axisymmetric simulations (on the order of 5–15 m s$^{-1}$ (Fig. 13a). A similar picture holds for $w_{\max}$, with values up to $\sim 40–50$ m s$^{-1}$ (Fig. 13b). $R$ ranges from $\sim 0.2$ to 1.5 km, with a mean between 0.5 and 1 km (Fig. 13c). Note also that while some thermals decrease in size with height and others increase in size, the mean $R$ shows a factor of $\sim 2$ increase between 2 and 8 km height that is consistent with the moist axisymmetric WK and neutrally stratified simulations. Because there is considerable variability in $R$ and $dR/dz$ profiles among individual tracked thermals, we assess robustness of the mean profiles using a “jackknife”-type resampling approach, which is similar to bootstrapping. The full dataset including all of the tracked thermals is resampled to create 100 subsets (using more than 100 has little impact on results). Individual thermals have a 50% probability of being included in a subset based on an
to the initial bubble lines represent individual ensemble members with small perturbations.

Moist (‘‘Moist’’) and dry (‘‘Dry’’) simulations using the neutrally stable initial environmental sounding from WK with free tropospheric RH of 42.5% (‘‘WK_lowRH’’) or 85% (‘‘WK_highRH’’) and the baseline independent random number draw for each thermal. The mean $R$ and $dR/dz$ profiles from each subset are then calculated, and the standard deviations of the 100 subset means are evaluated. The full sample mean $R$ and $dR/dz$ profiles $\pm$ these standard deviations are shown by the dotted lines in Figs. 13c and 13d. Variability of $R$ estimated using this approach is small relative to the full sample mean, with standard deviations less than 180 m at all heights. Variability of $dR/dz$ is relatively greater, with standard deviations up to about 0.15. While there is some noise, the full sample mean $dR/dz$ profile shows no apparent trend with height above 3 km, and is equal to 0.15 when averaged from 3 to 8 km. This is very similar to the vertically averaged $dR/dz$ from the WK moist axisymmetric simulations.

LES thermals in the dry environment (RH of 42.5%) reach their maximum ascent rates and dissipate at lower altitudes than those in the moist environment (RH of 85%), as expected. Despite this difference, there are no systematic differences in $R$ between the dry and moist environments, unlike the axisymmetric WK simulations in which $R$ was slightly smaller in the dry environment. However, such small differences may be difficult to distinguish in the LES given much greater variability among individual thermals compared to the axisymmetric simulations, and more data would be needed to detect this possible signal robustly.

5. Discussion and conclusions

This study compared the spreading rates of moist and dry convective thermals, quantified by $dR/dz$ and fractional volume growth rate $\varepsilon$. Several recent studies have investigating growth rates of dry thermals (e.g., Bond and Johari 2010; Lai et al. 2015; LJ2019; McKim et al. 2020), and we have built upon this work to investigate growth rates of moist thermals associated with cumulus clouds.

We first extended, in a simple way, the impulse-circulation theory for spreading rate of dry thermals from Turner (1957) and McKim et al. (2020) to moist thermals. We did this by accounting for circulation changes owing to latent heating in the thermal core associated with condensation. Several sets of numerical simulations were performed with a simple axisymmetric model to directly compare dry and moist thermal behaviors. A simple neutrally stable initial environmental profile (with respect to dry dynamics) was used to simulate moist and dry thermals. It was found that $dR/dz$ was a factor of 1.7 smaller for the moist thermals than the dry ones. The growth rates of our simulated dry thermals were mostly consistent with previous theoretical, numerical and laboratory studies, and $\varepsilon$ approximately scaled with $1/R$ consistent with the classic theory of Morton et al. (1956) and Scorer (1957) from dimensional analysis. Also consistent with this theory, $dR/dz$ and $\varepsilon$ were insensitive to the initial buoyancy of dry thermals.

Additional simulations were analyzed to understand specific mechanisms explaining the reduced $dR/dz$ and $\varepsilon$ for moist thermals relative to dry thermals. It was found that adding heating to the dry thermals as they ascend, spread uniformly across the thermal volume, had little impact on $dR/dz$ and $\varepsilon$ even though it dramatically increased thermal buoyancy and ascent rates. In contrast, when heating was added to the dry thermals but concentrated within their inner cores, consistent with the latent heating rates of moist thermals, $dR/dz$ and $\varepsilon$ decreased substantially. Thus, one of the main conclusions of this study is that the distribution of latent heating within moist thermals is crucial for understanding their reduced spreading rates compared to dry thermals. Physically, latent heating in moist thermals leads to horizontal buoyancy gradients and hence baroclinic vorticity generation in their interior, which limits their spread. In contrast, horizontal buoyancy gradients in dry thermals are concentrated near the thermal edge, which generates vorticity in the outer region, while in the interior buoyancy gradients are opposite in sign, which causes vorticity destruction there. This leads to rapid outward expansion of the center of rotation (Zhao et al. 2013; McKim et al. 2020). These differences are summarized in the conceptual diagram shown in Fig. 9.

Results for the moist and dry thermal simulations were generally consistent with the impulse-circulation theory. The circulation term in the impulse budget for dry thermals was small compared to the entrainment term. In contrast, the moist thermals had similar magnitudes for the circulation and entrainment terms. Thus, changes in impulse associated with the
buoyant forcing led mainly to thermal spreading for the dry thermals and both spreading and circulation increases for the moist thermals. Dry thermals with heating spread uniformly across their volumes had an increase in circulation with height, but the circulation term remained smaller than the entrainment term. Dry thermals with heating confined to the core also had an increase in circulation with height, but the circulation term was larger than the entrainment term. This is consistent with the picture of spreading rates being primarily controlled by how buoyancy is distributed within the thermals. Adding heating uniformly to thermals modified the buoyancy magnitude and increased circulation, but did not fundamentally change the spatial distribution of buoyancy and thus locations of vorticity generation and destruction, in contrast to when heating was only applied to the core.

Additional axisymmetric simulations and 3D LES were analyzed to study moist thermal $R$ and $V$ growth rates under more realistic atmospheric conditions (i.e., statically stable with respect to dry dynamics using the WK sounding). Overall, the axisymmetric and LES results using the WK sounding were consistent with the simple axisymmetric runs using the neutrally stable environment. These results further support the idea that moist thermals grow slower than dry thermals primarily because of how latent heating affects their internal distributions of buoyancy, not because of differences in environmental static stability or how they interact with turbulence.

Although the growth rates of moist thermals were substantially smaller than dry ones, the baseline axisymmetric moist thermals (750-m initial radius) and LES thermals still had a radius increase of about a factor of 2 between 2- and 8-km height. Previous observational studies of ascending cumulus towers based on cloud photographs have suggested $dR/dz$ of 0.2–0.25 (Saunders 1961; Glass and Carlson 1963), somewhat larger than our mean $dR/dz$ for the baseline moist thermals (although cloud boundaries do not strictly coincide with thermal boundaries). These results contrast with the analysis of LES moist thermals by Hernandez-Deckers and Sherwood (2016), who showed almost no increase in $R$ during the thermal

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**Figure 12.** Vertical cross sections of (a),(c) $w$ and (b),(d) $B$ at $t = (top) 8.5$ and (bottom) 11 min from an LES with initial bubble radius of 1500 m and free tropospheric RH of 42.5%. Vertical cross sections in the $X-Z$ plane are along the domain center in the $Y$ direction (i.e., along $Y = 0$). Flow vectors show winds in the 2D plane. Thermal boundaries are shown by the green lines, and blue lines show the thermal track in the 2D plane. Note that there is a small mismatch between the thermal track and location of maximum $w$ seen in the plots because the maximum $w$ considering the full 3D domain (which defines the track) is generally offset from the $Y = 0$ plane.
lifetime overall as indicated by small values of net \( e \) (and sometimes even weakly negative values, see Table 3 therein). Specific reasons for our contrasting results are unclear, though we point out several differences in the model setup and analysis compared to Hernandez-Deckers and Sherwood (2016). First, we focused on relatively large, long-lived thermals that were all well resolved, while the thermals in Hernandez-Deckers and Sherwood (2016) were generally small (\( R < 300 \) m) with short lifetimes (few minutes). Second, we analyzed isolated thermals whereas Hernandez-Deckers and Sherwood (2016) analyzed a field of cloud thermals with multiple thermals sometimes embedded within broader updraft structures (e.g., as seen in their Fig. 4). Third, we initialized thermals from motionless, spherical warm bubbles for both the 3D LES and axisymmetric runs, whereas thermals in Hernandez-Deckers and Sherwood (2016) originated “naturally” from the modeled flow field and hence may have generally had different initial properties than the thermals here. Previous studies have described large sensitivity of (dry) thermal growth rates to the initial conditions including bubble shape (e.g., Lai et al. 2015) and the existence of a pre-existing circulation (Escudier and Maxworthy 1973). Thus, albeit less than in dry thermals, we do find some growth and net entrainment in moist thermals in contrast to some previous LES studies. We suspect that this is due to the differences in the model setup and manner of thermal initialization, but this hypothesis requires further investigation.

Last, we emphasize that this study was focused specifically on radius and volume growth rates of thermals as they ascend. This is only one aspect of entrainment—another component is the entrainment associated with mixing across the thermal boundaries that does not change total thermal mass because it is simultaneously balanced by detrainment (see discussion in Hernandez-Deckers and Sherwood 2016). Indeed, this latter type of entrainment can lead to dilution of important properties for moist thermals, notably buoyancy (Morrison 2017; Morrison et al. 2020; Peters et al. 2020a). There is evidence that entrainment (and detrainment) without a corresponding change in thermal mass is limited for dry thermals (LJ2019), whereas LES has suggested fairly large fractional entrainment and detrainment rates of 2–3 km\(^{-1}\) with little net change in thermal mass for moist thermals (Hernandez-Deckers and Sherwood 2016). Future work should confirm if this difference between moist and dry thermals is indeed real, and if so, explore the underlying mechanisms driving it.

**Acknowledgments.** This material is based on work supported by the National Center of Meteorology, Abu Dhabi, United Arab Emirates (UAE), under the UAE Research Program for Rain Enhancement Science. This work was also supported by the U.S. Department of Energy Atmospheric System Research (Grants DE-SC0016476 and DE-SC0000246356) and the Australian Research Council Grant FL150100035. Author Peters’s efforts were also partially supported by the National Science Foundation Grant AGS-1841674. We thank Dr. Nadir Jeevanjee and an anonymous reviewer for comments and suggestions on the paper that improved it substantially. Author Morrison performed this work while on sabbatical and acknowledges support from ARC Centre of Excellence for Climate Extremes and the University of New South Wales Science Visiting Research Fellowship program. The National Center for Atmospheric Research is sponsored by the National Science Foundation.

**Data availability statement.** Data were generated using the Cloud Model 1 (CM1), version 18. CM1 code is available by download (https://www2.mmm.ucar.edu/people/bryan/cm1/). Data and/or namelists for model runs plus metadata documentation are available at https://doi.org/10.5065/3vqc-qh64.

**APPENDIX**

Thermal Identification and Tracking

Two primary methods for thermal identification and tracking have been employed previously (S13; RC15). In the S13 method, the center of individual thermals are identified by the

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**FIG. 13.** Evolution as a function of thermal center height for (a) tracked thermal velocity \( w_{th} \), (b) maximum vertical velocity \( w_{max} \), (c) thermal radius \( R \), and (d) radius growth rate \( dR/dz \) from the LES. Thin blue and red lines show results for individual thermals in the dry and moist environments, respectively. Thick black lines show mean values considering all tracked thermals. Green lines show mean values from the moist WK axisymmetric simulations, considering both dry (RH of 42.5%) and moist (RH of 85%) environments, up to the height at which the thermals in the dry environment decay. Dotted black lines in (c) and (d) show mean profiles considering all tracked thermals \( \pm 1 \) standard deviation calculated using the resampling technique described in the text.
location maximum vertical velocity, considering a minimum separation distance to filter out smaller local maxima near larger maxima. The change in height of this location over time gives the thermal velocity. The thermal volume is then defined by the volume of a sphere around the thermal center with a mean velocity equal to the thermal velocity. Note that Hernandez-Deckers and Sherwood (2016) introduced a number of refinements to the identification and time-matching of thermals but these are not relevant here since our relatively pristine thermals are unambiguously identifiable using the original approach.

RC15 employed a different method for identifying and tracking thermals. In their approach, cloud tops are identified as a grid point at the local top (within a 1-km² column) of ascending cloud regions. Cloudy updraft points are defined by vertical velocity exceeding 1 m s⁻¹ and cloud liquid plus ice mass fraction exceeding 10⁻³ (a cloud “mask”). These points are connected in time and the change in cloud top position over time is used to calculate the cloud top velocity \( w_{\text{top}} \). The thermal velocity \( w_{\text{th}} \) is assumed to be equal to \( w_{\text{top}} \). A streamfunction \( \psi \) is then calculated in \( r\)-\( z \) coordinates using an azimuthal average of \( \rho (w - w_{\text{top}}) \). The \( \psi = 0 \) streamline defines the thermal boundary. LJ2019 and McKim et al. (2020) extended this approach for dry thermals by replacing the cloud mask with a buoyancy threshold equal to 0.1 times the maximum value of horizontally averaged \( \rho' \) to provide the location of the thermal top. We will refer to this for tracking dry thermals as the “LJ2019” method.

The RC15 method is advantageous in that it does not assume a spherical thermal shape, but it cannot be used to track thermals embedded lower in clouds. On the other hand, the S13 method can track thermals within clouds but is limited by assuming a spherical shape. Here we propose a new method (“COMB”) that essentially combines S13 and RC15. In this method, \( w_{\text{th}} \) is defined by the change in location of maximum vertical velocity following S13. However, instead of assuming a spherical thermal shape, this velocity is used to calculate a streamfunction using the azimuthal average of \( \rho (w - w_{\text{th}}) \) similar to RC15. The thermal boundary is defined by the \( \psi = 0 \) streamline. This can be thought of as a generalization of the S13 method to nonspherical thermal shapes. In the axisymmetric simulations the location of thermals is unambiguous. To apply the COMB method for thermal identification in the LES, at each time we first locate local maxima in \( w \) with a separation distance from other maxima of at least 1 km. We then azimuthally average the flow field around the axes of the locations of these \( w \) maxima to calculate the streamfunction as described above.

We compare the S13, RC15, and COMB thermal tracking methods for the baseline moist axisymmetric simulation (Fig. A1). This gives profiles of thermal velocity \( w_{\text{th}} \), half-width \( R \), and volume \( V \) obtained from applying the methods. The RC15 method gives \( w_{\text{th}} \) up to \( \sim 40\% \) higher than the S13 and COMB methods (Fig. A1a), the latter of which are identical since they use the same approach to calculate \( w_{\text{th}} \). Larger \( w_{\text{th}} \) using RC15 is at least partly attributable to the fact that the thermal expands both vertically and horizontally as it rises, meaning that its top rises faster than its center. It follows that the RC15 method gives higher \( w_{\text{th}} \) because it uses \( w_{\text{top}} \) to define the ascent rate whereas S13 and COMB define \( w_{\text{th}} \) by the location of maximum \( w \) which is approximately at the thermal center.

Since \( w_{\text{th}} \) is greater, the RC15 method gives \( R \) up to 14\% smaller than S13 and 18\% smaller than COMB, with \( V \) smaller by up to 34\%–36\% (Figs. A1b,c). These differences occur because smaller \( w_{\text{th}} \) using the RC15 method means a smaller volume around the thermal core region of locally large \( w \) is needed to produce a volume-averaged \( w \) equal to \( w_{\text{th}} \). However, the particular method chosen has limited impact on the main conclusions of this paper, namely, that radius and volume growth rates are smaller for moist thermals compared to dry ones. As discussed in section 4a, applying the LJ2019 method, which is similar to RC15 but applicable to dry thermals since it does not require a cloud.

**FIG. A1.** Vertical profiles of (a) tracked thermal velocity \( w_{\text{th}} \), (b) thermal \( R \), and \( V \) for the baseline moist thermal axisymmetric simulation with the (dry) neutrally stable environmental profile. Different colors represent results using the different tracking methods discussed in the appendix. Quantities are calculated as 30-s moving averages.
mask, gives similar decreases in $R$ and $V$ relative to S13 and COMB as seen for the moist thermal in Fig. A1. We use the COMB method for the analysis in the rest of the paper, unless otherwise stated, as it uses exactly the same approach to track moist and dry thermals and does not rely on defining a cloud mask for the former.

REFERENCES


