Tropical Background and Wave Spectra: Contribution of Wave–Wave Interactions in a Moderately Nonlinear Turbulent Flow

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ABSTRACT: Variability in the tropical atmosphere is concentrated at wavenumber–frequency combinations where linear theory indicates wave modes can freely propagate, but with substantial power in between. This study demonstrates that such a power spectrum can arise from small-scale convection triggering large-scale waves via wave–wave interactions in a moderately turbulent fluid. Two key pieces of evidence are provided for this interpretation of tropical dynamics using a nonlinear rotating shallow-water model: a parameter sweep experiment in which the amplitude of an external forcing is gradually ramped up, and also an external forcing in which only symmetric or only antisymmetric modes are forced. These experiments do not support a commonly accepted mechanism involving the forcing projecting directly onto the wave modes with a strong response, yet still simulate a power spectrum resembling that observed, though the linear projection mechanism could still complement the mechanism proposed here in observations. Interpreting the observed tropical power spectrum using turbulence offers a simple explanation as to why power should be concentrated at the theoretical wave modes, and also provides a solid footing for the common assumption that the background spectrum is red, even as it clarifies why there is no expectation for a turbulent cascade with a specific, theoretically derived slope such as $-5/3$. However, it does explain why the cascade should be toward lower wavenumbers, that is an inverse energy cascade, similar to the midlatitudes even as compressible wave modes are important for tropical dynamics.

KEYWORDS: Tropics; Turbulence; Waves, atmospheric; Kelvin waves

1. Introduction

In the tropics, the upward motion of air at small scales not only generates clouds and local precipitation, but also impacts the large-scale flow. Specifically, variations in outgoing long-wave radiation (OLR), precipitation, and zonal winds in the tropics peak at wavenumber–frequency combinations given by the theoretical dispersion curves of longitudinally propagating equatorial waves (Matsumo 1966), as compared to the background spectrum in between these dispersion curves. Figure 1 overlays the observed power distribution from satellite brightness temperature, which is a proxy of deep convection, with a reproduction of the dispersion relations of free linear waves, for both antisymmetric (left panel) and symmetric (center panel) waves. Even though the characteristic length scale of individual clouds is tens of kilometers or less, this convection appears to launch global-scale waves with wavelength of order thousands of kilometers, which then propagate away and affect the global atmosphere (Garcia and Salby 1987).

The precise mechanism whereby the small-scale clouds launch these global-scale waves has been discussed by at least

two studies. First, Salby and Garcia (1987) use a linearized primitive equation model to argue that localized, transient, and stochastic tropical heating projects onto larger-scale wave modes, and the modes observed can be interpreted as a by-product of this projection. Specifically, the Fourier transform of a Gaussian is Gaussian, and hence a heating anomaly shaped like a Gaussian in time and space will include lower frequencies and wavenumbers. Second, Yang and Ingersoll (2013) argue that even though individual convective events are local, convection has an intrinsic tendency to self-aggregate on large scales and low frequencies. Hence, convection excites a range of frequencies (including very low frequencies) and spatial scales (including thousands of kilometers), where it can directly impact the large-scale dynamics.

In addition to the power concentrated along the dispersion curves of linear waves, there is substantial power evident in between these curves which is referred to as the background spectrum. This power was postulated by Wheeler and Kiladis (1999) to be red in both wavenumber and frequency, an assumption that was explicitly used in subsequent works to calculate the background spectrum (e.g., Hendon and Wheeler 2008). The underlying theoretical basis for this postulate is still lacking, to the authors' knowledge. Furthermore, Roundy (2020) recently argued that the assumption that the background spectrum is due to red noise is faulty. In particular, Roundy (2020) find that the substantial power lying between the Kelvin mode and the Madden–Julian oscillation (MJO) is

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FIG. 1. Reconstruction of the brightness temperature space–time analysis of Kiladis et al. (2009) for the (left) antisymmetric and (center) symmetric components of the raw spectrum and (right) the background spectrum. The solid (orange) contour marks the level for which the raw/background equals 1.1 and the signal is statistically significant to at least the 95% level (for estimated 100 degrees of freedom, assuming no latitudinal independence). Data were taken from the Cloud Archive User Service (CLAUS) (low resolution) dataset consisting of four-times-daily estimates of brightness temperature from 1 Jul 1983 to 30 Jun 2009 with a spatial resolution of $0.5^\circ \times 0.5^\circ$. The black curves and lines represent the theoretical dispersion relations of equatorial waves as described in section 2.

part of a Kelvin–MJO continuum, and that the whole region of the spectrum from the Kelvin band to the MJO should be considered together as associated with linear waves. They provide evidence for this claim by showing that the power in this region is structurally consistent with a classic Kelvin mode in the associated wind and geopotential height fields. Hence, they reject the assumption that the power in this region was generated by noise.

The goal of this work is to reconsider how small-scale convection triggers large-scale waves and forms the background spectrum using concepts developed in the fields of nonlinear fluid dynamics and incompressible turbulence. We now briefly review a few key concepts from this field. The degree of nonlinearity of a turbulent system governed by the Navier–Stokes equations is characterized by a dimensionless parameter known as the Reynolds number, Re , defined as the ratio between the nonlinear advection term [i.e., the inertial term $(\mathbf{v} \cdot \nabla)\mathbf{v}$] and the dissipation term. The nonlinear inertial term allows for the generation by triad interactions of Fourier harmonics not already present in the system (Vallis 2017). If $Re \ll 1$ then nonlinearity is weak: viscosity dissipates the harmonics that are already present in the flow before nonlinear interactions can produce other harmonics. The flow is laminar and it includes significantly only the harmonics that are present in the initial conditions or the forcing. In contrast, if $Re \gg 1$ then nonlinearity is strong resulting in strong interactions of the harmonics injected by the force or present in the initial conditions. The outcome of these interactions are fluid motions, often referred as “eddies,” that contain harmonics not previously present. The generation of these harmonics can be considered as a cascade process consisting of a sequence of elementary nonlinear interactions, called steps, where at each step an eddy of a given spatial scale produces another eddy of comparable, yet different, scale before any dissipation occurs (Frisch 1995). These eddies in

turn generate, via nonlinear interactions, other eddies, where again the viscous dissipation is negligible. Thus, starting from the eddy injected by the force and having some characteristic scale L , smaller eddies with a wide range of scales, called the inertial range, are produced. The cascade ends at the so-called viscous scale L_v , much smaller than L , where viscosity dominates nonlinear interactions. The wide inertial range of excited scales between L_v and L is characterized by a power-law distribution of energy among the modes, the famous Kolmogorov spectrum (Frisch 1995; Vallis 2017).

The cascade described above tacitly assumes that nonlinear interactions generate scales smaller than the scale of the forcing and not larger. This assumption, valid for three-dimensional flows, is not obvious and, in fact, is invalid for two-dimensional flows that are often relevant in a stratified fluid such as the atmosphere. For two-dimensional flows the unforced ideal flow equations have two quadratic integrals of motion that have density in Fourier space. Besides energy, present as in 3D, there is also enstrophy—space integral of squared vorticity. Turbulent vortices with scale L self-organize into larger-scale vortices, and this process continues until, usually, a vortex of the size of the system is created, the so-called inverse energy cascade.

Finally, the nature of the turbulent cascade is dependent on the degree of nonlinearity (i.e., the value of Re). The case of moderate nonlinearity, which would correspond to turbulence with $Re \sim 1$, is of particular interest for tropical dynamics as we show in this paper. For such an Re the flow still generates harmonics that are not present in the force; however, the spectrum is not a power law and depends on Re (Eswaran and Pope 1988; Mansour and Wray 1994).

A standard practice in studies of the response of turbulent dynamical systems to forcing is to add a stochastic, inhomogeneous, forcing term to the homogeneous nonlinear equations whose linear counterpart (obtained by setting all

nonlinear terms equal to zero) is employed in the derivation of the properties of free waves (Maltrud and Vallis 1991). Such an approach has been followed with a focus on Rossby waves generation in midlatitudes: when a stochastic forcing is added to the barotropic vorticity equation, Rossby waves emerge (Vallis et al. 2004; Barnes et al. 2010). The studies of Vallis et al. (2004) and Barnes et al. (2010) were based on the barotropic vorticity equation, but given the importance of Kelvin and inertia-gravity waves in the tropics, it is clear that the forced-dissipative system associated with the vorticity equation cannot reconstruct all tropical wave modes. An alternate forced-dissipative system that is more relevant to the tropics is that obtained by adding inhomogeneous forcing and dissipation to the rotating shallow-water equations (RSWE) and not the barotropic vorticity equation since only the linearized RSWE include all waves whose dispersion curves are evident in OLR observations (Fig. 1). Such a stochastic and inhomogeneous forcing in the RSWE can be thought of as randomly distributed convection scattered about the tropics. The excitation of all linear waves in a forced-dissipated nonlinear problem with a focus on the tropics has not been demonstrated, though it may have been implicit in the results of Yang and Ingersoll (2013), who introduced nonlinearity through the convective trigger.

Here, we stochastically force the RSWE as described in section 2 in order to understand the processes leading to the wave spectrum. We demonstrate that turbulent transfer of energy can account for both the wave and the background spectra. Two complementary sets of simulations are used to reach this conclusion. The first, presented in section 3, is a parameter sweep where the magnitude of the applied stochastic forcing concentrated in the zonal wavenumber range $30 < k < 40$ is gradually increased from near zero to near-observed values. In these simulations we eventually develop a background and wave spectra that resemble the observed ones, but with these simulations we can use the results of classical turbulence theory to explain why the background spectrum is red and how power becomes concentrated at the frequency-wavenumber combinations where linear theory mandates that waves should be present. The second, presented in section 4, is a set of simulations where we alternately force just symmetric modes or just antisymmetric modes, yet the resulting spectra show that wave modes with no forcing still have substantial power. The implications of these results, and a discussion of our results in the context of previous work, are found in section 5.

2. Methods

To understand the generation of the background spectrum and linear waves due to stochastic forcing we use an idealized model consisting of the forced-dissipated, global-scale, shallow-water equations. Using u and v to denote the zonal and meridional wind components, respectively, and Φ to denote the geopotential height anomaly with respect to a mean geopotential height $\Phi_0 = gH$, where g is Earth's gravitational acceleration and H is the mean layer thickness, the model equations are given by

$$\frac{\partial u}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial u}{\partial \lambda} + \frac{v}{a} \frac{\partial u}{\partial \phi} - \left[2\Omega + \frac{u}{a \cos \phi} \right] v \sin \phi + \frac{1}{a \cos \phi} \frac{\partial \Phi}{\partial \lambda} = D_u, \quad (1a)$$

$$\frac{\partial v}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial v}{\partial \lambda} + \frac{v}{a} \frac{\partial v}{\partial \phi} + \left[2\Omega + \frac{u}{a \cos \phi} \right] u \sin \phi + \frac{1}{a} \frac{\partial \Phi}{\partial \phi} = D_v, \quad (1b)$$

$$\frac{\partial \Phi}{\partial t} + \frac{u}{a \cos \phi} \frac{\partial \Phi}{\partial \lambda} + \frac{v}{a} \frac{\partial \Phi}{\partial \phi} + \frac{\Phi_0 + \Phi}{a \cos \phi} \left[\frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial \phi} (v \cos \phi) \right] = D_\Phi + S, \quad (1c)$$

where t denotes the time; $0 \leq \lambda < 2\pi$ and $-\pi/2 \leq \phi \leq \pi/2$ denote the longitudinal and latitudinal angles; a and Ω denote Earth's mean radius and angular frequency and $D_{u,v,\Phi}$ and S represent the dissipation and forcing terms. Note that the shallow-water equations can be derived from the nonlinear three-dimensional primitive equations if an appropriate vertical average is taken (Zeitlin 2018, chapter 3); however, the shallow-water equations clearly cannot allow for multiple vertical modes to interact. Nevertheless, we consider the shallow-water equations a useful testbed for probing mechanisms as to how the observed tropical spectrum comes about.

To integrate the equations we use the Geophysical Fluid Dynamics Laboratory's (GFDL's) spectral-transformed shallow-water model (SWM), which employs the spherical harmonics as its basis functions and the semi-implicit method with a Robert-Asselin (RA) filter for the time differencing (<https://www.gfdl.noaa.gov/idealized-spectral-models-quickstart/>). The results of the following sections were all obtained using a spectral resolution of T85, i.e., a triangular truncation where the highest retained wavenumber and total wavenumber both equal 85, a depth of the shallow water H of 50 m, a time step of $dt = 600$ sec and an RA coefficient of 0.04.

a. Forcing and dissipation

The model is forced using a stochastic source term to mimic the effect of small-scale convection. Let S_{ml}^i denote the spectral coefficients of the source term in a spherical harmonics decomposition, where m^1 and l are the order (zonal wavenumber) and degree of the spherical harmonics (see, e.g., Boyd 2001), respectively, and i is the time step. Then the applied forcing is described in spectral form by

$$S_{ml}^i = (1 - e^{-2dt/\tau})^{1/2} Q_{ml}^i + e^{-dt/\tau} S_{ml}^{i-1}, \quad (2)$$

where $dt = 600$ sec is the time step, $\tau = 2$ days is the decorrelation time, and Q_{ml}^i is a uniformly sampled array with elements in $(-A, A)$, where A is the forcing amplitude. The physical motivation for such a forcing is described in Vallis et al. (2004), but briefly the first term on the right-hand side of Eq. (2) specifies white noise while the second term specifies the memory of the forcing from the previous time step. The forcing

¹ When talking about the spectral coefficients of the spherical harmonics decomposition, we use m to denote the wavenumber index of the decomposition, as opposed to the actual wavenumber k .

is then masked in physical space by a Gaussian in latitude centered around the equator with an e -folding of 15° latitude, and in spectral space by retaining only those spectral coefficients with order (corresponding to zonal wavenumbers) $30 \leq |m| \leq 40$ and degree $30 \leq l \leq 40$. We intentionally use these values of m and l in order to distance the forcing from the resulting linear-wave response which is concentrated at low wavenumbers (typically ≤ 10). The 2-day time scale is similar to that used in section 6 of [Salby and Garcia \(1987\)](#), and is justified by observations of clouds ([Orlanski and Polinsky 1977](#); [Chen et al. 1996](#)). The discussion section addresses the sensitivity to using a 1-day or a 4-day time scale instead.

A similar forcing was used to represent the stirring of the barotropic flow by baroclinic eddies in midlatitudes by [Vallis et al. \(2004\)](#) and [Barnes et al. \(2010\)](#). Note that [Vallis et al. \(2004\)](#) and [Barnes et al. \(2010\)](#) used the framework of the barotropic vorticity equation to which the source term was added. In the shallow-water framework the source term can be added in any subset of the equations in system (1). The results shown below were obtained by adding the source term to the continuity equation. Physically, one can think of “convection” adding/removing mass stochastically but with some memory. The results remain qualitatively the same when the source term is added to the vorticity equation instead (see online supplemental Fig. 3).

The model is also dissipated using the same forms of dissipation used in [Vallis et al. \(2004\)](#) and [Barnes et al. \(2010\)](#), which consist of Rayleigh friction and ∇^4 hyperdiffusion. The former provides the simplest mechanism for removing momentum from the flow. The latter is included in the model in order to mimic a continuous enstrophy cascade to subgrid scales. In the present work, the Rayleigh dissipation is applied only to the momentum equations, i.e.,

$$D_u = -ru - \kappa \nabla^4 u, \quad D_v = -rv - \kappa \nabla^4 v, \quad D_\Phi = -\kappa \nabla^4 \Phi, \quad (3)$$

where r and κ are the Rayleigh damping coefficient and hyperdiffusion coefficient, respectively. In the present work r was set equal to $1/6 \text{ day}^{-1}$ and κ was set such that the highest resolved Fourier mode has a damping rate of 10 day^{-1} . Supplemental Fig. 4 shows that results are similar if Rayleigh damping is applied to Φ as well.

In section 4, the stochastic forcing in (2) was further masked in spectral space by applying additional weights to differentiate even and odd values of $l - |m|$, which correspond to symmetric and antisymmetric forcing around the equator. The weights were determined so as to preserve the amount of input spectral energy. For $30 \leq |m| \leq 40$ and $30 \leq l \leq 40$, with $l \geq m$, the stochastic forcing consists of a total of 66 modes in spectral space. Out of these 66 modes, 36 correspond to even values of $l - |m|$ and 30 to odd values. Thus, before applying additional weights, the total spectral energy of the applied forcing is $\sum_{m=30}^{40} \sum_{l=m}^{40} l^2 = 66$. The total spectral energy after applying the additional weight is $30w_o^2 + 36w_e^2 = 66$, where w_o and w_e are the odd and even weights, respectively. The resulting weights for an even distribution of the input energy between the symmetric and antisymmetric modes, for example, are $w_o = 1.049$ and $w_e = 0.957$.

b. Analysis

We examine the power spectra of the waves obtained by applying a two-dimensional Fourier analysis in space and time as in [Wheeler and Kiladis \(1999\)](#) and [Kiladis et al. \(2009\)](#). To this end we used the open-source `wkSpaceTime` routine of the NCAR Command Language, which implements the analyses described in the above papers without the tropical depression filter used in the latter paper. In each experiment, the model was integrated for 10 000 days and the analysis was applied for the last 9000 days with a sampling rate of 6 h. The time window was set to 96 days with a 10-day overlap between consecutive windows. The meridional window in all the results shown in following sections is 15°S – 15°N . The raw and background spectra remain qualitatively similar for a 30-day overlap and for meridional windows of 10°S – 10°N and 25°S – 25°N , although the statistical interpretation can change. The background spectrum is computed using a smoothing filter multiple times as done by [Wheeler and Kiladis \(1999\)](#).

The lowest Rossby and Poincaré modes, as well as the Kelvin and mixed Rossby–gravity (MRG) modes, are indicated in Fig. 1 and subsequent figures using solid black lines for layer depths of 12, 25, and 50 m using the relationships derived over 50 years ago by [Matsuno \(1966\)](#) in Cartesian coordinates. Recently, [Garfinkel et al. \(2017\)](#) and [Paldor et al. \(2018\)](#) have shown that while the MRG mode exists also on a sphere the Kelvin mode does not, though on a planet with Earth’s rotation rate and size the first eastward-propagating inertia–gravity mode has essentially the same dispersion relation as that of a Kelvin wave on a plane. Another important finding of wave theory on a sphere is that the planar wave solutions approximate very well the wave solution on a sphere (i.e., the waves amplitudes decay quickly with latitude), so long as the atmospheric layer is sufficiently thin, i.e., at low speed of gravity waves. [Garfinkel et al. \(2017\)](#) and [Paldor et al. \(2018\)](#) developed a new theory for thick layers on a sphere where the planar theory does not accurately approximate the characteristics of horizontally propagating waves. However, the fit between the dispersion relations of planar waves of [Matsuno \(1966\)](#) and satellite observations ([Wheeler and Kiladis 1999](#); [Kiladis et al. 2009](#)) is good because the “equivalent depth” of the atmospheric layer is sufficiently small so the planar wave theory approximates the spherical waves. In contrast, no such match between the satellite observations and planar theory can be expected in a thick layer of large “equivalent depth” where only the spherical theory applies. While for Earth this distinction is unimportant, the atmosphere of Venus lies within this thick layer regime ([Yamamoto 2019](#)) and on Venus the “Kelvin”-like wave can only be described using the spherical theory of [De-Leon and Paldor \(2011\)](#), [Paldor et al. \(2013\)](#), [Paldor \(2015\)](#), and [Garfinkel et al. \(2017\)](#).

c. Fidelity of the resulting spectrum

The net effect of the stochastic forcing on the power spectrum in the SWM is shown in Fig. 2, which compares the power spectrum in zonal wind between the ERA5 reanalysis ([Hersbach et al. 2020](#)) and the SWM. While the model suffers from too-strong power at high frequencies, it captures the

FIG. 2. Space–time analysis of the zonal wind from (a) the ERA5 product at 700 mb, compared to (b) the SWM with a depth of 50 m and a forcing amplitude of $5 \times 10^{-4} \text{ m s}^{-1}$. The ERA5 data consist of four-times-daily estimates between the years 1979 and 2018 taken from ERA5. The SWM data consist of four-times-daily estimates over 9000 days after a 1000-day spinup initialization.

observed increased power for low zonal wavenumbers and spectral peaks are concentrated at the theoretically predicted linear wave modes. The SWM also succeeds in simulating a spectrum of the geopotential similar to that observed and that is red in both frequency and wavenumber (Fig. 3)² The spectrum for the meridional velocity is not red for small wavenumbers (supplemental Figs. 5 and 6), and the SWM does a relatively poorer job of mimicking the observed spectra. The reasons why should be explored for future work.

While the SWM is not perfect, we argue that it is a useful tool for understanding the processes leading to the background and wave spectrum in nature. The rest of this paper demonstrates

that transfer of energy via wave–wave interactions in a moderately nonlinear fluid can account for both the wave and the background spectrum using two complementary sets of experiments.

3. Background and wave spectrum as moderately nonlinear turbulence

a. Parameter sweep of forcing amplitude

We now present a parameter sweep in which the amplitude of the stirring is gradually increased from near-zero to a value that, in the net, leads to a spectrum that compares favorably to that observed. We have performed 31 different integrations in which the maximum permitted amplitude A of Q_{ml}^i in Eq. (2) is gradually increased, but other aspects of the forcing are identical. Figure 4 shows the root-mean-squared value of the meridional wind [$\text{RMS}(v)$] for these integrations as well as the corresponding $\text{RMS}(v)$ from observations. For the experiments with weakest forcing, $\text{RMS}(v)$ is six orders of magnitude below that observed, while for the experiments with strongest forcing, $\text{RMS}(v)$ is similar though somewhat weaker to that

² In addition to this red spectrum and the Matsuno (1966) wave modes, the reanalysis geopotential spectra also contains discrete values with enhanced power for $|k|$ less than 5 and frequencies above 0.2 cpd. This effect is also evident in Fig. 9 of Gehne and Kleeman (2012), and these modes are external modes with much larger equivalent depths (Kasahara 1976; De-Leon et al. 2020). The SWM with a depth H of 50 m is (as expected) incapable of capturing these modes.

FIG. 3. As in Fig. 2, but for the geopotential height in the SWM and geopotential height at 700 mb in ERA5.

observed. Note that additional sources of variability exist in observations that are not represented in the shallow-water model, for example midlatitude disturbances propagating into the tropics, the Madden–Julian oscillation, tropical cyclones, and modes of variability with equivalent depths that differ sharply from the 50 m depth of the shallow-water model. Hence, we should not necessarily expect $\text{RMS}(v)$ in the shallow-water model to reach observed values. However, this parameter sweep can help clarify how the properties of the tropical power spectrum depend on the amplitude of the stochastic “convective” stirring.

The corresponding spectrum for these integrations is shown in Fig. 5. A forcing amplitude of 5×10^{-4} leads to a background spectrum and wave modes that mimic those observed for zonal wavenumbers lower than 30. For wavenumbers higher than 30 the stirring is evident, but we should not expect the spectrum for such high wavenumbers to resemble that observed as convection in the atmosphere is not confined to a narrow range of wavenumbers as we impose here.

A forcing amplitude of 5×10^{-5} leads to a qualitatively similar background spectrum and wave generation for $|k| < 10$, and the forcing is still evident for $|k| > 30$, but the spectrum for intermediate wavenumbers ($10 < |k| < 30$) is weaker than for a

forcing amplitude of 5×10^{-4} . Note that the contour interval and color scale for all rows in Fig. 5 are chosen so that the power associated with the forcing is shown with identical dark green shading, and the power associated with the lightest color is a factor of 10^{-8} weaker. For a forcing amplitude of 5×10^{-6} , the spectrum for $|k| < 10$ is barely detectable, and for a forcing amplitude of 5×10^{-7} , almost all power stays within the wavenumber range at which forcing is applied.

These results are summarized in Fig. 6, which shows the spectral power as a function of the forcing amplitude at wavenumbers (Fig. 6a) $k = 5$, (Fig. 6b) $k = 15$, and (Fig. 6c) $k = 35$ and frequency $\omega = 0.24$ cpd (corresponding to the Kelvin wave at $k = 5$ and equivalent depth of 50 m). The power associated with the Kelvin mode in Fig. 6a drops by four orders of magnitude for every order of magnitude decrease in the forcing, while the power spectrum associated with the forcing at, e.g., $k = 35$ in Fig. 6c decreases by two orders of magnitude for every order of magnitude decrease in the forcing. Figure 6a also shows the spectral power at the same ω value as the $k = 5$ Kelvin wave but for $k = 4$ and $k = 6$, and hence shows how the background spectrum power changes as a function of forcing amplitude. The background spectrum bears the same quantitative dependence on forcing amplitude as the spectrum

FIG. 4. Typical wind speeds in the SWM simulations (blue dots) as a function of the amplitude, compared to the typical wind speeds in ERA5 at 200 (red line), 500 (yellow line) and 900 (purple line) mb. The blue line is the least squares fit. Blue $\times$ markers indicate the particular samples shown by the circles in Fig. 5. The typical winds are estimated as $\text{rms}_{1000 \leq l \leq 10000 \text{ days}} \{ \text{rms}_{0^\circ \leq \lambda \leq 360^\circ} [\text{rms}_{15^\circ \text{S} \leq \phi \leq 15^\circ \text{N}} (v)] \}$.

associated with the wave modes, but the power associated with the wave modes is slightly larger than that associated with the background. Results are similar for the (antisymmetric) eastward inertia-gravity modes (not shown).

Figure 6b is as in Fig. 6a, but for $k = 15$. In contrast to the background spectrum for $k = 4$ or $k = 6$, the background spectrum for $k = 15$ decreases much more sharply, that is it drops by eight orders of magnitude for every order of magnitude decrease in the forcing.

These results motivate two questions: What physical process can explain the rapid decrease in power of the wave modes as the amplitude of the forcing is lowered? Why is the decrease in power with weaker forcing more extreme for $10 < |k| < 30$ than for $|k| < 10$? To answer this, we apply concepts from nonlinear fluid dynamics to the problem at hand.

b. Explanation of the results from the parameter sweep experiment

We first seek to categorize the nature of the nonlinearity in our model experiments, before explaining the results of section 3a. In complete similarity with the Reynolds number of classical turbulence theory (see the introduction), we contrast the inertial term ($\mathbf{v} \cdot \nabla \mathbf{v}$) and the dissipation term ($\mathbf{r}\mathbf{v}$), which

leads to the dissipative dimensionless number of relevance to our simulations $D = U/Lr$, where L is the typical length scale of the forcing and U is the typical velocity at this scale. If $D \ll 1$ then the Rayleigh friction dissipates fluctuations created by the forcing before they can transfer appreciable energy to other modes by triad interactions. In contrast, if $D \gg 1$ an energy cascade would occur. In our simulation there is yet another dissipative process, hyperviscosity used in the numerical scheme. This process however is negligible for the wave modes of interest in this paper.

Using the $\text{RMS}(U)$ in our simulations as the typical velocity, and $L = 10^6 \text{ m}$ as the typical length, corresponding to the typical forcing scale $k = 40$ (which provides an upper bound on the Rayleigh number compared to $k = 30$), yields values of D between 1.5 for the strongest forcing simulations and 1.4×10^{-3} for the weakest forcing simulation. Hence our simulations do not belong to the fully developed turbulent regime of strong nonlinearity (described in, e.g., Frisch 1995), but rather to the regime of moderate nonlinearity.

The resulting moderately nonlinear state depends on the value of the remaining dimensionless parameter(s). The turbulent dynamics of the Navier–Stokes equations is characterized, in addition to the Reynolds number, by the Mach number providing the ratio of U and the speed of sound (Bayly et al. 1992). In our case the role of the speed of sound is played by $\sqrt{gH}$ and the conventionally used parameter is the Froude number, defined as $\text{Fr} \equiv U/\sqrt{gH}$; see the appendix for details. It is well known in the case of gas dynamics that in the limit of small Mach numbers the flow is effectively incompressible and the density is approximately constant (Bayly et al. 1992). Using the $\text{RMS}(U)$ in our simulations as the typical velocity implies that the Froude number is $\text{Fr} = 0.13$. Note that the relevant parameter of the small Fr expansion is Fr^2 (see the appendix), which is even smaller. This implies that our simulations fall in the limit of small Fr and hence the free-surface elevation, which is the counterpart of density, is nearly constant. Therefore, even though the free surface is allowed to move vertically, the fluctuations of the free-surface elevation are weak, and hence the flow acts to a large degree as two-dimensional turbulence (as in midlatitude turbulence of the barotropic vorticity equation; e.g., Maltrud and Vallis 1991). There is a difference between turbulence in our system and a classic two-dimensional moderately nonlinear incompressible turbulent flow—our system rotates. However, the Rossby number associated with the rotation is small and thus it also implies the elevation’s constancy, as in the quasigeostrophic approximation. Thus, one can expect the classical theory of turbulence of a moderately nonlinear incompressible flow to apply in our case with some modifications.

Important results from the classical theory of turbulence of relevance to our simulations include the following:

- 1) The equations without dissipation, rotation, and forcing have two quadratic invariants—enstrophy (space integral of vorticity) and kinetic energy. In this situation an inverse cascade of energy is anticipated at any level of turbulence; see, e.g., Zakharov et al. (2012) and Frisch (1995). Rotation does not change the form of nonlinearity in the system and

FIG. 5. Zonal wind spectra in the forced-dissipated SWM as function of forcing amplitude. (a) The contour intervals and color scale is chosen for visual clarity, and (b)–(d) we use the scaling of the power in the forcing region, which is quadratic in the forcing amplitude, to adjust the color scale and contour interval.

FIG. 6. Spectral power as a function of the forcing amplitude at wavenumbers (a) $k = 5$, (b) $k = 15$, and (c) $k = 35$ and frequency $\omega = 0.24$ cpd (corresponding to the Kelvin wave at $k = 5$ and equivalent depth of 50 m). In all cases the power is also sampled at neighboring wavenumbers with the same value of ω (outside the waves' dispersion curve). The abscissas in all panels change by four orders of magnitude. The ordinates in (a), (b), and (c) change by 16, 32, and 8 orders of magnitude, respectively, so the 1:1 lines correspond to slopes of 4, 8, and 2, respectively. Similar results are obtained for $\omega = 0.1, 0.3, 0.5$ (not shown). The results of the antisymmetric EIG0 wave are also similar to those of the Kelvin wave in (a).

thus the inverse cascade, which is due to the nonlinearity, must still hold. See the [appendix](#) for details and more references.

- 2) Because the forcing amplitude is relatively small and the nonlinearity is weak, the spectrum is maximal in the natural frequencies of the system. Hence, one should expect somewhat more power for the wave modes than in the background spectrum. Importantly, this is not true for fully developed turbulence, realized at larger forcing amplitudes (Frisch 1995).³
- 3) In the regime of weak nonlinearity ($D \ll 1$) a triad interaction between two wavenumbers in the region of forcing will carry more energy than if one of the modes is outside the region of forcing (see the explanation below). This is so only when nonlinearity is weak. Otherwise the power associated with wavenumbers outside the range of the forcing is comparable with that in the range, and so the interaction would carry comparable energy.

Each of these three properties is of crucial importance for understanding the resulting statistical steady state in the shallow-water model. The first explains why the background

spectrum is red both in wavenumber and in frequency,⁴ and hence provides a theoretical justification for the assumption of a red spectrum in observations (e.g., Wheeler and Kiladis 1999; Masunaga et al. 2006; Hendon and Wheeler 2008). The second explains why the linear wave modes derived by Matsuno have slightly more power than the background spectrum.

The last fact can shed light on the slopes shown in Fig. 6, as we now elaborate. Recall that the slope of a log-log plot gives the power of the relationship. Thus, the slope of 2.01 on Fig. 6c indicates that the power at wavenumbers in the forcing region increases with the second power of the forcing amplitude. If two modes in the forcing region with wavenumbers k_1 and k_2 interact via quadratic nonlinearities to launch a wave with $k = |k_1 - k_2|$, the power associated with this low wavenumber wave will be the product of the power of each of the two constituent waves. Recall that the stochastic forcing in the above results was applied in the range of wavenumbers between $30 \leq |k| \leq 40$, so indeed two modes in the forcing region can interact to form waves with $|k| \leq 10$. Hence, we should

³ We also would expect that for wavenumbers within the forcing (between 30 and 40), the spectrum is maximal at natural frequencies of the system if the forcing amplitude is the same for all frequencies (e.g., white noise). This is not so for our forcing where the memory included in the forcing [Eq. (2)] leads preferentially to low frequencies.

⁴ The characteristic time of eddies with size L is proportional to $L^{2/3}$ (Frisch 1995) so generation of small wavenumbers implies generation of small frequencies. Note that power is concentrated at low frequencies both for $k < 10$ and for k in the forcing region; however, this concentration is much more evident for $k < 10$. In the forcing region the dominant process that leads to this effect is the memory in the forcing [Eq. (2)]; in contrast, for $k < 10$ turbulent processes are more important.

expect that the power for $|k| < 10$ should be proportional to the fourth power of the forcing amplitude. Indeed, the slope of the best-fit curves in Fig. 6a are essentially four, and slight changes to, e.g., the Rayleigh friction strength and range of forcing wavenumbers (e.g., $40 \leq |k| \leq 50$) leads to small changes in the slope inferred from the SWM experiments.

A similar explanation can account for the slope of near eight in Fig. 6b. Power in the range $10 < |k| < 20$ can be most effectively generated by a triad interaction between wave modes with $|k| < 10$ ($k = |k_1 + k_2|$). If the power for wave modes with $|k| < 10$ is proportional to the forcing amplitude to the fourth power, and two such modes must interact, then we should expect power for wave modes in the range $10 < |k| < 20$ to be proportional to the eighth power of the forcing. In reality the power is proportional to the 7.98 power, though again changes in the parameters governing the SWM experiments lead to small changes in this slope.

Note that there is no pure power-law spectrum (such as Kolmogorov's $5/3$ slope) in our experiments dictating the ratio of power for two distinct values of k . While it is possible to fit the spectrum in some regions by a power law, such a relationship does not hold globally. This may be due to the following three factors: (i) wavenumber discreteness, which is significant here, (ii) the inertial range in our simulations is too short for the power-law spectrum to develop, (iii) for $\text{Re} \sim 1$ the spectrum is not necessarily a power law and depends on Re (Eswaran and Pope 1988; Mansour and Wray 1994).

Finally, the confinement of the power to the forcing region for simulations with weak forcing (Fig. 5d) can be explained by this theory too. The Reynolds number for this simulation is much less than one, and thus any power generated by a triad interaction is rapidly damped away. That is, this regime is essentially fully linear, and the only power present is that continuously provided by the forcing. In contrast, in the most energetic run (Fig. 5a) the Reynolds number is around 1, and hence a roughly similar amount of power is evident in the forcing region and elsewhere.

Overall, these experiments demonstrate that in the parameter regime of the tropics, nonlinear turbulent processes are crucial and a linearized perspective (as in Salby and Garcia 1987) may miss a key aspect of the dynamics, though we expect that the linear projection argument has merit as well and supplements the mechanism proposed here.

4. Stirring only antisymmetric or symmetric wave modes

We now consider a second set of simulations that further demonstrates the crucial role of turbulence for the background spectrum and wave modes. Here, we only stir combinations of zonal wavenumber and total mode-number that correspond alternately to symmetric or antisymmetric modes, and then consider the subsequent spectrum. In all cases the total stirring is held constant, and is equal to the 5×10^{-4} simulation from section 3. Full details of the forcing for these simulations are included in section 2.

Figure 7a repeats the spectrum from Fig. 5a, while Figs. 7b and 7c show the spectrum when only symmetric and only antisymmetric modes are forced, respectively. The key point is

that spectral power is not limited to those modes which are directly excited, but rather includes, e.g., symmetric modes even when only antisymmetric modes are stirred. The reason the response to an antisymmetric forcing is nearly purely symmetric is that in triad interactions, the meridional structure of the resulting wave is determined by the *product* of the meridional structures of the two interacting waves. As was shown in Fig. 6a, the power in the region $|k| \leq 10$ is consistent with triad interaction in the forcing region $30 \leq |k| \leq 40$. So indeed even if the forcing is purely antisymmetric the response is expected to be purely symmetric. A full discussion of how this turbulent transfer occurs is presented in Shamir et al. (2021).

The results of this section cannot be explained by a linear framework (e.g., of Salby and Garcia 1987), as they would predict that antisymmetric forcing should result in purely antisymmetric waves. Hence turbulent processes must be transferring power from wave modes that receive forcing to wave modes with no direct forcing.

5. Discussion

Variability in the tropical atmosphere occurs on spatial scales that are separated by several orders of magnitude. Specifically, individual convective systems have a characteristic length scale of tens of kilometers or less, but these systems are organized by, and also help launch, global-scale waves with wavelength of order thousands of kilometers. Previous theories on how the small-scale clouds launch these global-scale waves have typically involved a linear perspective—Salby and Garcia (1987) and Yang and Ingersoll (2013) argue that localized, transient, and stochastic tropical heating projects directly onto larger-scale wave modes, and thereby help launch these larger-scale modes.

In addition to the power concentrated along the dispersion curves of linear waves, there is substantial power evident in between where no dry linear wave should exist, and the fundamental processes leading to this so-called background spectrum are still unknown. While the tropical atmosphere hosts variability that is unrelated to dry linear waves, it is unclear why this variability should appear red in both wavenumber and frequency. While some studies posit that this background spectrum is red so as to then remove it when focusing on the wave modes (Wheeler and Kiladis 1999; Masunaga et al. 2006; Hendon and Wheeler 2008), the underlying theoretical basis for this assumption has not been established. Furthermore, the recent paper by Roundy (2020) argues that this background spectrum, at least in the region between the Kelvin mode and the MJO, is part of a Kelvin–MJO wave mode continuum associated with linear waves.

This work offers a different perspective than the previous works mentioned above. It demonstrates that small-scale convection may trigger large-scale waves and form the background spectrum in exactly the same way that any small-scale forcing in a nonlinear, turbulent fluid generates a background spectrum and wave modes. We demonstrate this using a simplified model that, in its most realistic configuration, simulates wave modes and a background spectrum similar to that observed. We then provided two key pieces of evidence for this

FIG. 7. Zonal wind spectra in the forced-dissipated SWM as a function of the forcing symmetry. (a) The input energy is evenly distributed between the symmetric and antisymmetric component of the forcing. (b) All the input energy is in the form of symmetric modes. (c) All the input energy is in the form of antisymmetric modes. The input energy was distributed between the symmetric and antisymmetric modes by further masking the applied forcing in spectral space as detailed in [section 2](#).

revised interpretation of tropical dynamics: a parameter sweep experiment in which the amplitude of an external forcing is gradually ramped up, and also an external forcing in which only symmetric or only antisymmetric modes are forced. For both classes of numerical experiments, we simulate wave modes and a background spectrum well outside of the forcing region. The mechanisms of both [Salby and Garcia \(1987\)](#) and [Yang and Ingersoll \(2013\)](#) are not present in our simulations: the forcing does not directly project onto the wave modes at the wavenumbers where a strong response is seen, nor does our

model support any sort of convective organization. Hence this mechanism is not necessary in order to reconstruct realistic tropical waves, though these mechanisms cannot be ruled out as important in the actual atmosphere, and models with only these mechanisms can reconstruct realistic tropical waves. Note that the shallow-water model can be modified to provide evidence for the linear projection mechanism too. If we do not apply the wavenumber-30–40 mask, and instead stir stochastically with the amplitude of the forcing set to a Gaussian shape in both longitude and latitude, we can also recover a red

FIG. 8. Zonal wind spectra in the forced-dissipated SWM as function of the decorrelation time τ of the stochastic forcing in Eq. (2).

spectrum that resembles that observed (see supplemental Figs. 7 and 8). Finally, our simulations also provide a solid footing for the commonly used assumption that the background spectrum of tropical winds or brightness temperature is red both in frequency and wavenumber.

The results of this study may allow for a reinterpretation of results shown in two previous works. First, Andersen and Kuang (2012) find that the red spectrum in their model is eliminated if moist static energy is damped toward its time- and zonal-mean values with a 12-h time scale, and the zonal average of all prognostic variables but temperature are nudged to their climatological values (see their Fig. 2c). We speculate that this damping and nudging could be effectively removing any

turbulent interactions in the flow. Second, the model of Yang and Ingersoll (2013) is capable of simulating a background and wave spectrum similar to that observed (see their Fig. 3). While they note that the degree of nonlinearity in their model is small, we find that even with $\text{RMS}(U)$ of order 0.1 m s^{-1} or even 0.02 m s^{-1} , the system is already weakly nonlinear and triad interactions can take place and meaningfully transfer energy to large scales even as most of the forcing is dissipated before it can be cascaded (Figs. 5b,c). Hence, we speculate that the spectrum shown in Yang and Ingersoll (2013) could be due, at least in part, to the mechanism presented here.

It is well established that the midlatitude atmosphere is characterized by turbulence (Rhines 1975; Maltrud and Vallis

1991; Vallis et al. 2004; Barnes et al. 2010). In contrast to midlatitude dynamics where turbulence is fully developed, in the tropics the flow is only moderately turbulent. This distinction has implications for the properties of the turbulence. First, a moderately turbulent fluid should have its spectrum maximal at the natural frequencies of the system so one should expect somewhat more power for the wave modes than in the background spectrum. This is not true for fully developed turbulence. Second, a triad interaction between two wavenumbers in the region of forcing will carry more energy than if one of the modes is outside the region of forcing. This leads to a time of energy transfer from a given k mode to another k mode that is dependent on k , and hence there is no scale invariance and also no expectation for a turbulent cascade with a specific, theoretically derived slope such as $-5/3$ (Mansour and Wray 1994). Furthermore, we do not have a large inertial range away from the wavenumbers k_f of the forcing. That is, we lack at least a decade of wavenumbers smaller than k_f that would allow us to see the power-law spectrum associated with the inverse cascade. Future work should consider the situations where a larger range of wavenumbers is present both for $k > k_f$ and $k < k_f$ and study the formation of power-law spectra.

In the turbulence community, the moderately nonlinear regime is often known as wave turbulence (Zakharov et al. 2012). The fluid in our simulations demonstrates some of the properties of wave turbulence (e.g., the spectrum maximizes at the natural frequencies of the system) and also some of the properties of incompressible two-dimensional turbulent fluid (the nascent inverse cascade of energy; Frisch 1995). However, we do not observe the famous power-law spectra that are associated with turbulence (Zakharov et al. 2012; Frisch 1995) for at least two reasons: the discreteness of wavenumbers and the shortness of the inertial range.

While our model is able to represent the observed tropical spectrum well, it certainly does not include all of the physics present in the actual atmosphere. Specifically there is no moisture anywhere (and consistent with this simplification no MJO). Furthermore, “convection” is parameterized in a very simple manner as a stochastic process with memory, and hence the large-scale waves are unable to affect the strength and location of the convection (no two-way coupling). We leave for possible future work an exploration of the effect that moisture or two-way coupling may have on the turbulence, though note that Yang and Ingersoll (2013) did allow for two-way coupling and found a very similar tropical spectrum to what we find. Furthermore, the memory of the stochastic convection mimics to some degree an external process that helps organize the convection, and as shown in Fig. 8 the main effect of this memory on our results is to shift power toward lower frequencies but without affecting the distribution of power among different wavenumbers. It is beyond the scope of this work to consider the regime change as the forcing amplitude is further strengthened and the system transitions to fully developed turbulence, but this transition should also be a focus for future work. Finally, future work should consider repeating the experiments performed here but with a multilevel primitive equation model, which would then allow for vertical modes to

interact and also for a more quantitative comparison of the importance of the turbulence mechanism proposed here to the linear projection mechanism. These limitations notwithstanding, the results of this paper provide a theoretical basis for the background spectrum evident in the tropical atmosphere, and also may explain why the power associated with wave modes should be stronger than that of the background spectrum.

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APPENDIX

Theory for Moderately Turbulent Fluids for the Rotating Shallow-Water System

As discussed in the main body, the properties of a moderately turbulent fluid depend on three dimensionless parameters: the Reynolds number, characterizing the nonlinearity strength, the Froude number Fr , characterizing the strength of the force (gravity) restoring constant height, and the Rossby number Ro , characterizing the rotation speed. The presence of Ro and spherical geometry in our simulations differs from conventional derivations of turbulence (Frisch 1995), which consider nonrotating fluids on the plane. In a strongly nonlinear nonrotating shallow-water plane flow with Reynolds number larger than 1, the smallness of Fr is enough to establish the inverse energy cascade. Here the reversal of the cascade direction, in comparison with the three-dimensional case, is due to the existence in two dimensions of one more quadratic integral of motion besides the energy—the enstrophy (space integral of squared vorticity). Similarly, the existence of two quadratic integrals of motion implies an inverse cascade in the weakly nonlinear turbulent regime, holding at small Re and known as the wave turbulence (Zakharov et al. 2012). Asymptotic continuation indicates then that preferential energy transfer to small wavenumbers must hold at $Fr \ll 1$ at any Re . Here we confirm that the addition of rotation does not alter this conclusion by reducing the evolution to incompressible flow on a rotating sphere which is known to produce an inverse cascade (see, e.g., Maltrud and Vallis 1991; Obuse et al. 2010).

We start by writing the nonlinear shallow-water equations on the sphere in a form that allows us to make analogies with plane geometry. It is preferable to use a coordinate free approach with covariant differentiation (e.g., Falkovich and Gawędzki 2014). We however use a mathematically less advanced approach for greater readability. We observe that a solution of Eqs. (1) generates three-dimensional flow $\mathbf{v} = v(\phi, \lambda)\hat{\phi} + u(\phi, \lambda)\hat{\lambda}$, which solves the three-dimensional equations

$$\begin{aligned} \partial_t h + \tilde{\nabla} \cdot (h\mathbf{v}) &= q, \\ \partial_t \mathbf{v} + (\mathbf{v} \cdot \tilde{\nabla})\mathbf{v} + 2\Omega \sin\phi \hat{R} \times \mathbf{v} + \frac{(v^2 + u^2)\hat{R}}{a} &= -g\tilde{\nabla} h - r\mathbf{v}, \end{aligned} \quad (\text{A1})$$

where $\eta = (\Phi_0 + \Phi)/g$ is the height and we introduced the operator $\tilde{\nabla} \equiv R\nabla/a$. For instance using the ∇ operator in spherical coordinates we have

$$\tilde{\nabla} \cdot (h\mathbf{v}) = \frac{R}{a} \nabla \cdot \mathbf{v} = \frac{1}{a \cos\phi} \left[\frac{\partial(hv \cos\phi)}{\partial\phi} + \frac{\partial(hu)}{\partial\lambda} \right]. \quad (\text{A2})$$

We use above capital R for the radial coordinate to distinguish from the Rayleigh friction coefficient r . In accord with the main text the forcing is included in Eqs. (A1) only in the height equation, $q \equiv S/g$, and the Rayleigh friction is included only on the winds (see section 2). Hyperdiffusion is not included as it is small relative to the Rayleigh damping for the wavenumbers of interest.

Conversely, an R -independent solution of Eqs. (A1) with $v_R = 0$ produces a solution of Eqs. (1) so the formulations are equivalent. The above formulation shows explicitly that qualitatively we have the usual hydrodynamic evolution. The only differences are in the presence of the centrifugal force, needed to prevent generation of radial flow, and the presence of $\tilde{\nabla}$ instead of ∇ to which it is equivalent as far as tangential derivatives on the sphere are considered. Equations (A1) with Ω set to zero coincide with the equations of two-dimensional compressible gas flow on the sphere. Here h plays the role of the density and $gh^2/2$ is the “pressure” p as seen by checking that $h^{-1}\nabla p$ equals the first term in the left-hand side of the momentum equation. The corresponding square of the speed of sound $c^2 = \partial p / \partial \rho$ is gh . It will be seen below that in the case of our interest the deviations of h from the mean height H are small so that the speed of sound is approximately constant and given by $\sqrt{gH}$.

We provide an example of usage of Eqs. (A1) by taking the curl of the momentum equation, remembering that derivatives act also on R in the definition of $\tilde{\nabla}$ operator. The radial component gives

$$\partial_t w + (\mathbf{v} \cdot \tilde{\nabla})w + w\tilde{\nabla} \cdot \mathbf{v} + 2\Omega \sin\phi \tilde{\nabla} \cdot \mathbf{v} + \frac{2\Omega v \cos\phi}{a} = -rw, \quad (\text{A3})$$

where we designate the radial component of $\nabla \times \mathbf{v}$ by aw/R with

$$\begin{aligned} w &\equiv \frac{1}{a \cos\phi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos\phi} \frac{\partial}{\partial \phi} (u \cos\phi); \\ \mathbf{w} &\equiv \nabla \times \mathbf{v} = \frac{aw\hat{R} + u\hat{\phi} - v\hat{\lambda}}{R}. \end{aligned} \quad (\text{A4})$$

We refer to w below as vorticity. This definition appears most reasonable since circulation around any contour belonging to the sphere can be written via surface integral of w . In other words, w is the normalized circulation density. We used in the calculation

$$\begin{aligned} (\mathbf{v} \cdot \nabla)\mathbf{v} &= \nabla \frac{v^2}{2} + \mathbf{w} \times \mathbf{v}, \\ \nabla \times [(\mathbf{v} \cdot \nabla)\mathbf{v}] &= \mathbf{w}(\nabla \cdot \mathbf{v}) - (\mathbf{w} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{w}, \\ [\nabla \times (\sin\phi \hat{R} \times \mathbf{v})]_R &= \{\nabla \times [\sin\phi(u\hat{\phi} - v\hat{\lambda})]\}_R \\ &= \sin\phi \nabla \cdot \mathbf{v} + \frac{v \cos\phi}{R}, \\ [(\mathbf{v} \cdot \nabla)\mathbf{w}]_R &= (\mathbf{v} \cdot \nabla) \frac{aw}{R}, \quad [(\mathbf{w} \cdot \nabla)\mathbf{v}]_R = 0. \end{aligned} \quad (\text{A5})$$

The vorticity equation, given by Eq. (A3), can also be derived directly from Eqs. (1).

We assume that the forcing acts at a characteristic spatial scale L and produces flow field with characteristic velocity U at this scale. The relation between U and the force depends on the qualitative nature of the solutions that is determined by the values of the dimensionless parameters that characterize the equations. We observe that the fluid mechanics of a gas with the polytropic index 2 and no gravity coincides with the shallow-water equations up to relabeling of the constants in the equations, and this analogy helps to clarify the nature of the resulting solutions. The first dimensionless parameter of the SWE is the Froude number Fr defined as $\text{Fr} \equiv \sqrt{gH}/U$. In the gas dynamic parallel, upon relabeling, this number becomes the Mach number, which is well known to describe the strength of density variations in space. Since gas density corresponds to the elevation, we conclude that Fr provides a measure of strength of fluctuations of the surface elevation. In our simulations, the maximal value of Fr is 0.13, and we estimate U from the RMS of the velocity as outputted from the simulations. Figure A1 shows that the velocity at the forcing scale $[\sum_{30 < k < 40} \langle |\mathbf{u}(k)|^2 \rangle]$ is of the same order as the total velocity $[\sum_k \langle |\mathbf{u}(k)|^2 \rangle]$. Thus there is appreciable energy contained in $30 < k < 40$ of the forcing scale and also at $k < 30$. Hence this figure justifies the use of $\text{RMS}(U)$ as the typical velocity scale when estimating the Froude number.

The role of the Froude number and also of the Rossby number in setting the properties of the turbulent regime can be seen by passing to dimensionless variables

$$\mathbf{v} = U\mathbf{u} \left(\frac{\mathbf{x}}{L}, \frac{Ut}{L} \right), \quad h = H\eta \left(\frac{\mathbf{x}}{L}, \frac{Ut}{L} \right), \quad (\text{A6})$$

where η is of order one;

$$\begin{aligned} \partial_t \eta + \tilde{\nabla} \cdot (\eta \mathbf{u}) &= \tilde{q}, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \tilde{\nabla})\mathbf{u} + \frac{2 \sin\phi \hat{R} \times \mathbf{u}}{\text{Ro}} &= -\frac{\tilde{\nabla} \eta}{\text{Fr}^2} - \frac{\mathbf{u}}{D}, \end{aligned} \quad (\text{A7})$$

where $\tilde{q}$ is q multiplied with $L/(HU)$ and taken at rescaled arguments. Besides Fr the system in dimensionless form involves the dimensionless Rossby number defined by

$$\text{Ro} = \frac{U}{\Omega L}. \quad (\text{A8})$$

If we use the spatial scale corresponding to $k = 40$ for L (which yields an upper bound on the Rossby number), we find

FIG. A1. The sum of squares of magnitudes of Fourier components of velocity with wavenumber smaller than (blue) or larger than (red) a cutoff k_0 as a function of the cutoff. Vertical dashed lines indicate the forcing region in the range $30 \leq |k| \leq 40$. All modes with $k < 70$ have significant energy (seen from the blue line), which is a characteristic property of the inverse cascade in the case of nonlarge inertial range (for a large range the energy is predominantly contained in the smallest wavenumbers).

that the maximum Rossby number in our simulations is 0.05 for the largest forcing amplitude, and hence is small.

In the limit of $\text{Fr} \rightarrow 0$, Eq. (A7) demands that $\tilde{\nabla} \cdot \mathbf{v} \propto \text{Fr}^2$ in order for accelerations to remain finite, implying that any correction to $h = H$ is small (Bayly et al. 1992). A similar argument is true also of the Rossby number, just as in the usual quasigeostrophic approximation. Hence, the near constancy of the elevation is caused by the demand that the pressure and rotational terms in the momentum equation do not get infinitely large in the limits of Fr or Ro going to zero. We conclude that both Fr^2 and $\text{Ro} \ll 1$ imply $H - h \ll 1$ (i.e., fluctuations of the elevation are weak). Hence, the turbulence of the system should be expected to behave as if there was a two-dimensional incompressible flow.

The incompressibility however needs additional inspection in our case since the presence of the source in the continuity equation destroys the usual implication that constant density (height) gives $\nabla \cdot \mathbf{v} = 0$. Returning to dimensional variables, and using $h = H$ in the continuity equation implies that

$$\tilde{\nabla} \cdot \mathbf{v} = \frac{1}{a \cos \phi} \left[\frac{\partial(v \cos \phi)}{\partial \phi} + \frac{\partial u}{\partial \lambda} \right] \approx \frac{q}{H}. \quad (\text{A9})$$

Thus, in contrast with the usual small Mach number limit of the Navier–Stokes equations, in our case the velocity divergence is finite because it is directly injected by the source q . This source is absent in conventional nonrotating theories of fluid mechanics which are stirred in vorticity. Here, for our application of interest, we elect to stir height to maintain a stronger connection to convection which most directly affects the height field.

We now demonstrate that this direct injection of velocity divergence does not affect the direction of the turbulent cascade, and specifically that finiteness of $\nabla \cdot \mathbf{v}$ by itself does not imply that the flow is compressible. The divergence must be compared with the vorticity w . We find by using Eq. (A9) in Eq. (A3) that

$$\partial_t w + (\mathbf{u} \cdot \tilde{\nabla})w + \frac{2\Omega v \cos \phi}{a} = -\frac{qw}{H} - \frac{2\Omega \sin \phi q}{H} - rw. \quad (\text{A10})$$

We see from the right-hand side of Eq. (A10) that q is the source of the vorticity which is injected both additively and multiplicatively. The vorticity is injected also directly by the rotation via the Ωv term, which in our simulations never exceeds the Rayleigh friction rw term.⁵ Moreover, in our simulations q/H is smaller or of order r . Thus, for estimating the vorticity we can discard both qw/H and Ωv terms in Eq. (A10). Finally, we observe that the decorrelation time of the force (2 days) is comparable with the characteristic decay time of the fluctuations (6 days). This allows us to estimate the magnitude of the vorticity by equating the last two terms in Eq. (A10), which could be wrong if the correlation and decay time were vastly different. We find that vorticity can be estimated as $w \sim 2\Omega \sin \phi q / (Hr)$. Therefore, the so-called compressibility ratio, defined as the ratio of the flow's divergence and curl, obeys

$$\frac{\nabla \cdot \mathbf{v}}{w} \sim \frac{r}{2\Omega \sin \phi} \ll 1, \quad (\text{A11})$$

where we use the smallness of $r/(2\Omega \sin \phi)$ in the simulations (the smallness holds for $\phi > 1^\circ$ degrees and thus applies at relevant $\phi \sim 15^\circ$). Thus the flow is weakly compressible and the nonlinear term in Eq. (A10) is approximately as in the usual incompressible two-dimensional flow.

⁵ The smallness of the rotation term here differs from the regime studied in section 4 of Maltrud and Vallis (1991). Specifically, Maltrud and Vallis (1991) study a regime where the rotation term is of the same order as the inertial term for small but nonzero k , at which point the turbulent transfer of energy ends and a wavelike regime begins for even smaller k , with the transition occurring at the Rhines scale (Rhines 1975). Our simulations never enter such a regime, and rather the rotation term is always smaller than the inertial term. Future work should consider how tropical wave dynamics changes if the rotational term was made more important (say, by increasing the rotation rate while keeping other parameters fixed).

Finally, we consider the resulting approximation for the flow. The condition of vanishing divergence allows to introduce the streamfunction ψ giving

$$v = \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda}, \quad u = -\frac{1}{a} \frac{\partial \psi}{\partial \phi}, \quad \frac{\partial(v \cos \phi)}{\partial \phi} + \frac{\partial u}{\partial \lambda} = 0. \quad (\text{A12})$$

The vorticity definition becomes

$$w = \frac{1}{a \cos \phi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \phi} \frac{\partial}{\partial \phi} (u \cos \phi) = \frac{1}{a^2 \cos^2 \phi} \frac{\partial^2 \psi}{\partial \lambda^2} + \frac{1}{a^2 \cos \phi} \frac{\partial}{\partial \phi} \left(\cos \phi \frac{\partial \psi}{\partial \phi} \right) = \frac{\zeta}{a^2}, \quad (\text{A13})$$

where we introduced tangential Laplacian $\zeta \equiv a^2 \nabla^2 \psi$. The vorticity equation, written in terms of $\zeta \equiv a^2 \nabla^2 \psi$ becomes

$$\partial_t \zeta + \frac{J(\psi, \zeta)}{a^2 \cos \phi} + 2\Omega \frac{\partial \psi}{\partial \lambda} = -\frac{q\zeta}{H} - \frac{2\Omega a^2 \sin \phi q}{H} - r\zeta, \quad (\text{A14})$$

where we introduced the Jacobian operator

$$J(\psi, \zeta) = \frac{\partial \psi}{\partial \lambda} \frac{\partial w}{\partial \phi} - \frac{\partial \psi}{\partial \phi} \frac{\partial w}{\partial \lambda}. \quad (\text{A15})$$

This resembles the standard formulation of incompressible flow on a rotating sphere [the barotropic vorticity equation studied by, e.g., Vallis et al. (2004), Obuse et al. (2010), and Barnes et al. (2010)]. The left-hand side of Eq. (A14) describes the inertial evolution of the flow, and the RHS describes Rayleigh friction and the forcing by q . The inertial evolution implies inverse cascade of energy as in the similar case of two-dimensional incompressible flow on the plane (see, e.g., Obuse et al. 2010). Our consideration here demonstrates how forcing the height variable results in production of vorticity and an inverse energy cascade. However, our use of the shallow-water equations allows for the inertia-gravity modes, i.e., the counterpart of sound waves in classic turbulence, which are missing in the barotropic vorticity equation where divergence is identically zero.

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