The Efficiency of Upward Wave Propagation near the Tropopause: Importance of the Form of the Refractive Index

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ABSTRACT: The connection between the polar stratospheric vortex and the vertical component of the Eliassen–Palm flux in the lower stratosphere and upper troposphere is examined in model level data from ERA5. The particular focus of this work is on the conditions that lead to upward wave propagation between the tropopause and the bottom of the vortex near 100 hPa. The ability of four different versions of the index of refraction to capture this wave propagation is evaluated. The original Charney and Drazin index of refraction includes terms ignored by Matsuno that are shown to be critical for understanding upward wave propagation just above the tropopause both in the climatology and during extreme heat flux events. By adding these terms to the Matsuno index of refraction, it is possible to construct a useful tool that describes wave flux immediately above the tropopause and at the same time also describes the role of meridional variations within the stratosphere. It is shown that a stronger tropopause inversion layer tends to restrict upward wave propagation. It is also shown that while only 38% of extreme wave-1 Eliassen–Palm flux vertical component (F_z) at 100 hPa events are preceded by extreme F_z at 300 hPa, there are almost no extreme events at 100 hPa in which the anomaly at 300 hPa is of opposite sign or very weak. Overall, wave propagation near the tropopause is sensitive to vertical gradients in buoyancy frequency, and these vertical gradients may not be accurately captured in models or reanalysis products with lower vertical resolutions.

KEYWORDS: Atmospheric circulation; Rossby waves; Stratospheric circulation

1. Introduction

The variability of the Northern Hemisphere wintertime stratospheric polar vortex influences tropospheric climate and weather (Limpasuvan et al. 2004; Baldwin and Dunkerton 1999; Polvani and Kushner 2002). An anomalously weak vortex is often associated with the negative phase of the Arctic Oscillation (the surface signature of the northern annular mode) in the following weeks or months (Baldwin and Dunkerton 2001; Limpasuvan et al. 2004; Polvani and Waugh 2004; Kidston et al. 2015). While much polar vortex variability is quasi random and deterministic predictability of the vortex is generally limited to 2 weeks (Karpechko 2018; Tripathi et al. 2016, 2015) probabilistic predictability may extend for longer if the upward flux of anomalous wave activity that drives the vortex anomaly could be predicted (Garfinkel and Schwartz 2017; Domeisen et al. 2020a,b; Rao et al. 2019). The goal of this study is to better understand upward wave propagation from the troposphere to the lower stratosphere.

It has long been recognized that upward wave flux in the stratosphere may not always be dictated by upward wave flux

in the troposphere (Shiotani 1986). Birner and Albers (2017) found that less than 15% of the climatological lower-tropospheric (at 700 hPa) wave 1 + 2 vertical Eliassen–Palm (EP) flux propagates up to 100 hPa. They conclude that anomalous upward fluxes of lower-tropospheric wave activity are not a necessary or sufficient precursor to sudden stratospheric warmings (SSW) events. White et al. (2019) also examined wave flux in the troposphere and stratosphere preceding SSWs, and found that the tropospheric vertical EP flux is around 0.75 standard deviations above the mean while in the stratosphere the anomaly reaches about two standard deviations (their Fig. 2). Most sudden decelerations of the stratospheric polar vortex are not preceded by extreme wave activity in the lower troposphere (de la Cámara et al. 2019). The relationship between tropospheric vertical EP flux and vertical EP flux at 100 hPa both in the time mean and associated with extreme events, and the factors that govern the efficiency of the vertical propagation, still warrants further investigation.

The index of refraction of Charney and Drazin (1961) and especially of Matsuno (1970) have been used by numerous studies to offer guidance on the direction of wave propagation within the stratosphere. Specifically, waves are preferentially refracted toward regions with a more positive index of refraction and ducted away from regions in which the index of refraction is more negative (Andrews et al. 1987). The derivation of Matsuno assumes that the buoyancy frequency (N) is constant in height, i.e., he ignores height-dependent buoyancy frequency terms (see the appendix for a self-contained derivation). This assumption is clearly violated just above the tropopause, where the buoyancy frequency varies substantially within a few kilometers, and as shown by Chen and Robinson (1992), the tropopause strongly influences wave propagation.

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The goal of this work is to better understand how the tropopause region modifies upward wave propagation. Specifically, we explore the impact of the rapid vertical variations in the buoyancy frequency near the tropopause on wave propagation by allowing for a more complicated height dependence of the background state. These rapid vertical variations are in turn associated with the tropopause inversion layer, a narrow region with a pronounced local maximum in N immediately above the troposphere (Birner 2006).

Section 2 introduces four different forms of the index of refraction and considers which can successfully account for the rapid decay of climatological wave activity just above the tropopause. Section 3 shows that most of the time, standardized anomalies of upward zonal wavenumber-1 (hereafter wave-1) flux at 300 and 100 hPa are quantitatively similar. Section 4 examines the exceptions to the usual tendency for there to be similar standardized anomalies at 300 and 100 hPa, i.e., events in which there is a large difference in the normalized upward propagating wave activity at 300 and 100 hPa, and shows that a term not included in Matsuno (1970) but is included in Charney and Drazin (1961) is crucial for understanding these events. Section 5 considers synoptic wave-numbers, and section 6 presents a discussion of our results.

2. Four different forms of the refractive index

The refractive index arises from the more general statement that for linear stationary Rossby waves, where the linearization is with reference to a zonally symmetric basic state, theoretical insight as to the nature of the solutions to the equation for the perturbation potential vorticity is attainable only if certain terms are approximated or treated as negligible. Matsuno (1970) and Charney and Drazin (1961) make distinct approximations in order to arrive at their respective expressions for the perturbation potential vorticity, and specifically they approximate either the meridional gradient of the basic state potential vorticity, the perturbation potential vorticity, or both (see the appendix for details). The net effect of these approximations are the first (for Matsuno 1970) and fourth (for Charney and Drazin 1961) indices of refraction below.

The second and the third are generalizations of Matsuno's original index that are not rigorously justifiable but include physical effects included in the definition of Charney and Drazin (1961) (which in turn neglected other physical effects) as described below.

- (i) The original form by Matsuno (1970) is given here as written in Andrews et al. [1987, Eq. (5.3.7)]:

$$M70n_{\text{const}} = \frac{\bar{q}_\phi}{a\bar{u}} - \frac{s^2}{a^2 \cos^2 \phi} - \frac{f^2}{4N^2 H^2}, \quad (1)$$

where

$$\bar{q}_\phi = 2\Omega \cos \phi - \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} (\bar{u} \cos \phi) - \frac{af^2}{\rho_0} \frac{\partial}{\partial z} \left(\frac{\rho_0}{N^2} \frac{\partial \bar{u}}{\partial z} \right) \quad (2)$$

is the zonal-mean quasigeostrophic potential vorticity meridional derivative [Andrews et al. 1987, Eq. (5.3.4)]. The notation used is as follows:

- a is the Earth radius,
 - $\bar{u}$ is the zonal mean zonal wind, and varies both with ϕ and z ,
 - s is the zonal wavenumber,
 - $f = 2\Omega \sin(\phi)$ is the Coriolis parameter, where Ω is the rotation rate of Earth, and ϕ is the latitude,
 - $H = 7000$ m is the height scale, and
 - $\rho_0 = \rho_s \exp(-z/H)$ is the density and $\rho_s = 1.2 \text{ kg m}^{-3}$.
- The third term in Eq. (2) can be expanded as

$$-\frac{af^2}{\rho_0} \frac{\partial}{\partial z} \left(\frac{\rho_0}{N^2} \frac{\partial \bar{u}}{\partial z} \right) = af^2 \left[\frac{1}{H} \frac{1}{N^2} \frac{\partial \bar{u}}{\partial z} + \left(\frac{2}{N^3} \right) \frac{\partial N}{\partial z} \frac{\partial \bar{u}}{\partial z} - \frac{1}{N^2} \frac{\partial^2 \bar{u}}{\partial z^2} \right]. \quad (3)$$

The form in Eq. (1) assumes a constant buoyancy frequency for the entire atmosphere following the derivation of Matsuno (1970), so that the second term on the right-hand side in Eq. (3) is identically zero. Figures later in this paper using this form of the index of refraction use the average of N from 100 to 10 hPa and from 40° to 80°N. In this paper we assume a zero phase-speed when computing the index of refraction, though results are essentially the same if we allow for a nonzero phase speed in the calculation of the index of refraction (e.g., $c = 5 \text{ m s}^{-1}$ in supplemental Fig. 5).

- (ii) In order for Matsuno (1970) to derive his index of refraction, he had to assume that N has no z dependence (see the appendix for details). Nevertheless, some papers have plotted a Matsuno (1970) index of refraction with N varying spatially (e.g., Chen and Robinson 1992; Li et al. 2007). Our second form of the refractive index is therefore identical to the first, but with N allowed to vary in the vertical and horizontal:

$$M70n_{\text{CR92}} = \frac{\bar{q}_\phi}{a\bar{u}} - \frac{s^2}{a^2 \cos^2 \phi} - \frac{f^2}{4N^2(\phi, z)H^2}; \quad (4)$$

N is also allowed to vary in the expression for the mean-state potential vorticity gradient [Eq. (2)], so that the second term on the right-hand side in Eq. (3) is retained (as in Li et al. 2007). Note that latitudinal structure in N is generally inconsistent with the assumptions underlying theories of linearized wave dynamics, including quasi-geostrophy (Andrews et al. 1987), which typically assume that N is a function of z only, but for this form we allow for latitudinal dependence.

- (iii) In contrast to Matsuno (1970), Charney and Drazin (1961) do not assume that N is fixed spatially, and are able to derive an index of refraction by instead assuming that the zonal wind and also N have no meridional variation (see the appendix for a self-contained derivation). It can be shown that the net effect of allowing vertical variations in N in Charney and Drazin's (1961) derivation is that the $f^2/(4N^2H^2)$ term in Eq. (1) (which arises from the perturbation potential vorticity) must be replaced by an expression that includes vertical derivatives of N (see the appendix). The third form of the index of refraction is based on Matsuno's (1970) definition but is broadened to include these vertical derivatives:

$$M70n_{N_{\phi,z}} = \frac{\bar{q}_\phi}{a\bar{u}} - \frac{s^2}{a^2 \cos^2 \phi} - \frac{f^2}{N(\phi, z)} \sqrt{\frac{1}{\rho_0} \frac{\partial^2}{\partial z^2}} \sqrt{\frac{\rho_0}{N^2(\phi, z)}}. \quad (5)$$

While the rightmost term in Eq. (5) was derived by Charney and Drazin (1961) under the assumption that N is only a function of z , here the full spatial variations in N are considered when calculating the refractive index. As in form 2, N is also allowed to vary in the expression for the mean-state potential vorticity gradient [Eqs. (2) and (3)]. The net effect is that form (iii) includes vertical derivatives of N that arise both from the mean-state potential vorticity gradient (discussed by Li et al. 2007) already included in the second form and also from the perturbation potential vorticity.

We acknowledge that both the second and third indices of refraction [Eqs. (4) and (5)] cannot be rigorously justified as the indices are derived with N held constant but in computing the index of refraction we allow N to vary.

- (iv) Finally, our fourth form is that derived by Charney and Drazin (1961) on the β plane, allowing for $N(z)$ and $\bar{u}(z)$ to vary with height, and therefore, vertical derivatives of $N(z)$ and $\bar{u}(z)$ are retained, but with no meridional dependence for either N or $\bar{u}$. The resulting index of refraction is

$$\begin{aligned} CD61 = & \frac{\beta}{\bar{u}} - \frac{1}{\bar{u}} \frac{f_0^2}{\rho_0} \frac{d}{dz} \left[\frac{\rho_0}{N^2(z)} \frac{\partial \bar{u}}{\partial z} \right] - (k^2 + l^2) \\ & - \frac{f_0^2}{N(z)} \sqrt{\frac{1}{\rho_0} \frac{\partial^2}{\partial z^2}} \sqrt{\frac{\rho_0}{N^2(z)}}. \end{aligned} \quad (6)$$

Our focus is on the EP flux at 60°N in sections 3 and 4, and hence we set the constant Coriolis parameter to $f_0 = 2\Omega \sin(60^\circ)$. The meridional wavenumber is set $l = 0$ and the zonal wavenumber is $k = s/a \cos(60^\circ)$.

Equation (6) is equivalent to Charney and Drazin (1961) Eq. (3.2) if multiplied by $f_0^2/N(z)^2$ and where the phase speed is set $c = 0$. Note that both $\bar{u}(z)$ and $N^2(z)$ are functions of height only and independent of latitude. We set the values for $\bar{u}(z)$ and $N^2(z)$ by their mean values from 45° to 75°N.

We show in the appendix that the last term in (5) and (6) if expanded out gives

$$\begin{aligned} -\frac{f^2}{N} \sqrt{\frac{1}{\rho_0} \frac{\partial^2}{\partial z^2}} \sqrt{\frac{\rho_0}{N^2}} = & -\frac{f^2}{4N^2 H^2} + \frac{f^2}{N^3} \frac{\partial^2 N}{\partial z^2} - \frac{f^2}{HN^3} \frac{\partial N}{\partial z} \\ & - \frac{2f^2}{N^4} \left(\frac{\partial N}{\partial z} \right)^2. \end{aligned} \quad (7)$$

The first term in Eq. (7) is identical to that in Eq. (1), but the last three terms are new and their effects on upward propagation have not been explored in previous work, though they are implicitly included in the linearized models of Chen and Robinson (1992) and Harnik and Lindzen (2001). The relative importance of these terms will be discussed later.

We then examine the squared refractive index climatology of wave 1 (other wavenumbers will be discussed in section 5)

The refractive index climatology, wave-1, 5% trimmed

FIG. 1. The refractive index multiplied by a^2 climatology (dimensionless) in DJF for (a) $M70n_{Nconst}$, (b) $M70n_{CR92}$, (c) $M70n_{N_{\phi,z}}$, and (d) CD61. We trimmed out 2.5% of the largest and 2.5% of the smallest values before computing the mean. The zero line is shown in magenta.

for the four definitions using model level data from the European Centre for Medium-Range Weather Forecasts reanalysis version 5.1 (ERA5.1) dataset over the period 1979 to 2018 (Hersbach et al. 2020). The native vertical resolution of ERA5.1 is approximately 10 hPa, or around 300 m, near the tropopause and lowermost stratosphere. We interpolate the data on the native hybrid vertical coordinate to pressure levels with a resolution of 10 hPa between the levels of 70 to 400 hPa for a more accurate analysis at and above the tropopause as compared to earlier reanalyses (<https://www.ecmwf.int/en/forecasts/documentation-and-support/137-model-levels>). The data were downloaded at a horizontal resolution of 1.25° latitude $\times$ 1.25° longitude. The indices of refraction are computed for each day, and averages are computed with a trimmed mean that excludes the top and bottom 2.5% of the values. Supplemental Figs. 1 and 2 show that results are similar but noisier if we do not apply a trimmed mean. For the analysis presented below we focus on northern winter [December through February (DJF)].

The climatological squared refractive index for all four forms considered in this paper is shown in Fig. 1. Within the stratosphere, $M70n_{Nconst}$, $M70n_{CR92}$, and $M70n_{N_{\phi,z}}$ are qualitatively similar and the assumption by Matsuno (1970) of constant N is reasonable for that region; however, just above the tropopause this is not the case. The region of negative index of refraction in the subtropical lower stratosphere (at 30°N and from 100 to 50 hPa) is due to the strong vertical shear of the zonal wind above the subtropical jet which leads to meridional shear in the mean-state potential vorticity in Eq. (2) (Li et al. 2007). This effect is apparent only in Matsuno's form (Figs. 1a–c) which allow for meridional gradients in the mean-state potential

The index of refraction terms, climatology, wave-1

FIG. 2. (top) The refractive index profile from Figs. 1b and 1c is repeated for (a) $M70n_{CR92}$ and (b) $M70n_{N_{\phi,z}}$. (bottom) Each of the z dependent buoyancy frequency derivative terms: (c) $a^2(f_0^2/N^3)\partial_{zz}N$, (d) $-a^2(f_0^2/HN^3)\partial_zN$, and (e) $-2a^2(f_0^2/N^4)(\partial_zN)^2$ (section 3 and the appendix). We trimmed out 2.5% of the largest and 2.5% of the smallest values. The zero line is shown in magenta.

vorticity. In contrast, the meridional averaging required for CD61 smears out any meridional structure of the zonal wind.

The importance of the additional terms included in $M70n_{N_{\phi,z}}$ and CD61 can be deduced by comparing Figs. 1a and 1b with Figs. 1c and 1d. These panels clearly differ in the lowermost stratosphere. The squared index of refraction becomes negative (blue region in Figs. 1c,d just above the tropopause from 300 to 200 hPa poleward of 50°N) and leads to a partial barrier for upward propagation just above the tropopause in the climatology.¹ This partial barrier is consistent with the rapid reduction in wave activity between 300 and 100 hPa climatologically (Birner and Albers 2017).

The latter two definitions, $M70n_{N_{\phi,z}}$ and CD61, include the additional terms in Eq. (7), and we now seek to understand which one of these terms makes the largest contribution immediately above the tropopause. The climatology for these terms is shown in Fig. 2. To make the comparison easier, the $M70n_{CR92}$ and $M70n_{N_{\phi,z}}$ climatology from Figs. 1b and 1c is repeated in Fig. 2a and 2b. At 200 hPa it is clear that the largest contributor for the negative values is the second derivative

term $(f^2/N^3)\partial^2N/\partial z^2$. It is therefore the second derivative term that has a large contribution for the barrier just above the tropopause while in the original Matsuno index of refraction it is neglected. Physical processes that affect this term will be discussed later in the paper, but briefly a vertically confined region in which N has a pronounced local maximum will lead to a large contribution from this term and hence more of a barrier to upward wave propagation. The next two sections will consider the ability of these different forms of the index of refraction to explain upward planetary wave propagation above the tropopause not only in the time mean, but also in periods with enhanced or reduced upward propagation.

3. Categorizing events with enhanced and reduced upward wave propagation

We now describe how we form composites of events with more efficient upward propagation and less efficient upward propagation between 300 and 100 hPa. Our first step is to calculate the normalized anomaly of the EP flux vertical component at 300 hPa ($F_{z,300}$) and 100 hPa ($F_{z,100}$) using the ERA5.1 data as described in section 2. The quasigeostrophic form of the EP flux is used here defined using Eq. (12.131) in Vallis (2006):

$$F_z = f p_0 \cos \phi \frac{\overline{v'\theta'}}{\partial_z \overline{\theta}}, \quad (8)$$

where v' and θ' are deviations from the zonal mean for the meridional wind and potential temperature, respectively. We

¹ Note that the local maximum just above the tropopause in $M70n_{CR92}$ is located at a lower elevation than the corresponding local maximum in Fig. 6 of Chen and Robinson (1992). This is likely because the tropopause occurs at a lower elevation in ERA5 than it does in the data used by Chen and Robinson (1992), where the transition to stratospheric values of N^2 occurs above 10 km (see their Fig. 7).

TABLE 1. The distribution of winter (December–February) days based on the normalized F_z anomaly (section 3) at 60°N. For example, days in which the $\hat{F}_{z,100}$ anomaly is smaller than 3 STD and greater than 2 STD are placed in the rightmost column. Days in which the $\hat{F}_{z,100}$ is smaller than 2 STD and greater than 1 STD are placed in the second column from the right, and so on. Similarly, the row in which an event is placed depends on $\hat{F}_{z,300}$. Events are defined allowing for a 1 day lag between $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$. Larger (smaller) number of events than would be expected if the same random process led to both $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ (as described in section 3) are denoted by a plus (minus) sign.

F_z anomaly events						
	$-3 < \hat{F}_{z,100} < -2$	$-2 < \hat{F}_{z,100} < -1$	$-1 < \hat{F}_{z,100} < 0$	$0 < \hat{F}_{z,100} < 1$	$1 < \hat{F}_{z,100} < 2$	$2 < \hat{F}_{z,100} < 3$
$-3 < \hat{F}_{z,300} < -2$	3	19+	16+	1	0	0
$-2 < \hat{F}_{z,300} < -1$	11	110 –	224	34	2–	0
$-1 < \hat{F}_{z,300} < 0$	3–	274+	1008	310+	29	2–
$0 < \hat{F}_{z,300} < 1$	0	28	317	428 –	168	32+
$1 < \hat{F}_{z,300} < 2$	0	3+	35–	164	159	59
$2 < \hat{F}_{z,300} < 3$	0	0	5+	21	47	45

first apply a 1–2–1 running-mean smoothing to $F_{z,100}$ and $F_{z,300}$ wave 1 for each set of 3 days (e.g., weights of 1/4 are applied to days $n + 1$ and $n - 1$ and a weight of 1/2 is applied to day n). A daily climatology over the dataset is computed, and is then subtracted from the raw fields to generate daily anomalies. We then define the normalized anomalous $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$ as the ratio of the daily anomaly and the climatological standard deviation (STD) for each calendar day at each level. We allow for a 1 day lag between $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$, as such a 1 day lag maximizes the correlation between these levels at 60°N when considering the entire ERA5.1 DJF record (supplemental Fig. 4). We have repeated all analysis in this paper for $\hat{F}_z$ at 400 hPa rather than 300 hPa, and results are similar though the maximum lagged correlation drops from 0.69 to 0.57 and the lag increases from 1 day to 3 days.

Then we composite each winter (December through February) day into one of 36 categories based on the relative values of $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$ (see Table 1) at 60°N (results are essentially identical at 50°N or at 70°N, not shown). For example, all days in which the $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ anomaly are less than 3 STD and greater than 2 STD are composited together, and similarly all days in which the $\hat{F}_{z,300}$ anomaly is smaller than 3 STD and greater than 2 STD, while the $\hat{F}_{z,100}$ smaller than 2 STD and greater than 1 STD are included in a second composite, and so on. The number of days that fall into each category is shown in Table 1. Here we focus on wave-1 F_z ; see section 5 for sensitivity tests of this choice. The focus of our discussion will be on the rightmost and leftmost columns, and top and bottom rows, as these constitute the extreme events.

It is immediately apparent that the majority of extreme $\hat{F}_{z,100}$ events are preceded by extreme $\hat{F}_{z,300}$ while there are almost no extreme events at 100 hPa preceded by close to climatology or negative events at 300 hPa (first and last columns in Table 1). Specifically, the bottom two elements on the rightmost column include 75% of the events in which the anomaly at 100 hPa is greater than 2 STD. Similarly, the top two elements on the first column include 82% of the events in which the anomaly at 100 hPa is smaller than -2 STD.

These results lead us to form the following hypothesis: the same quasi-random process leads to variability in both $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$. To assess this hypothesis, we perform the following test. First, we compute a maximum likelihood estimator for the

parameters of a Weibull distribution² that can describe the first three moments of the PDF of $\hat{F}_{z,100}$ in DJF in ERA5.1, and then perform a similar estimation for $\hat{F}_{z,300}$. We then use these parameters to produce a fictitious time series of equal length to and with the same statistical properties as the observed $\hat{F}_{z,300}$, and a corresponding time series for $\hat{F}_{z,100}$, with the same sequence of random numbers used both for the fictitious $\hat{F}_{z,300}$ and the fictitious $\hat{F}_{z,100}$. Note that the fictitious time series using the Weibull parameters capture the skewness and variance of the observed time series. We then create a table corresponding to Table 1 for these fictitious $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ time series. This process is then repeated 10 000 times. If an element in Table 1 is larger (smaller) than 97.5% (2.5%) of the corresponding fictitious F_z tables, then the observed relationship between $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ is not merely due to the same stochastic noise forcing both, and the corresponding element in Table 1 is marked with a plus (minus) sign. Both of the last two elements on the rightmost column are not significant, neither are the rightmost two elements of the last row. If anything, the $-1 < \hat{F}_{z,300} < 0$ and $2 < \hat{F}_{z,100} < 3$ composite is underpopulated as compared to what we might expect if random noise led to variability at 100 and 300 hPa, while the $0 < \hat{F}_{z,300} < 1$ and $2 < \hat{F}_{z,100} < 3$ is overpopulated. Overall, a day with extreme $\hat{F}_{z,300}$ is likely to be associated with anomalous $\hat{F}_{z,100}$ due to the same quasi-random process forcing both.

Extreme negative $\hat{F}_{z,100}$ ($\hat{F}_{z,100} < -2$ STD, the leftmost column in Table 1), is never preceded by positive $\hat{F}_{z,300}$, and a null hypothesis that the same stochastic noise forces both cannot be rejected either. That being said, wave reflection can lead to extremely negative $\hat{F}_{z,100}$ independent of $\hat{F}_{z,300}$ (Shaw et al. 2010), and this possibility should be explored for future work.

4. What leads to events with large differences in $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$?

We now seek to understand the factors distinguishing events well off the main diagonal of Table 1. Namely, why on some days is $\hat{F}_{z,100}$ much larger than $\hat{F}_{z,300}$, and vice versa? The

² A Weibull distribution can capture first three moments of the PDF of $\hat{F}_z$, whereas a normal distribution cannot.

FIG. 3. A histogram of the difference between the standardized $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$. The bar to the right (left) of zero presents the sum of events in which $\hat{F}_{z,100}$ is equal to $\hat{F}_{z,300}$ up to 0.5 STD greater (–0.5 STD smaller). One more column to the right (left) presents the sum of events in which $\hat{F}_{z,100}$ is greater (smaller) than $\hat{F}_{z,300}$ by 0.5 STD (–0.5 STD) up to 1 STD (–1 STD) and so on. Days included in the blue bars form the transmitting composite while the days included in the red bars on the left form the decaying composite (section 4).

results in Table 1 are summarized in Fig. 3, which shows a histogram of the difference between $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$. As discussed in section 3, the majority of the time the standardized anomalies at 300 and 100 hPa are similar, but our focus now is on the tails of this distribution.

For simplicity, we composite each DJF day into one of three categories. A day in which the standardized $\hat{F}_{z,100}$ is greater

than the standardized $\hat{F}_{z,300}$ by at least 1.5 STD after 1 day lag is composited in the “transmitting” composite (blue bars in Fig. 3). Results below are similar if a threshold of 1.25 or 1.75 STD, or if wave flux at 50° or 70°N, is used (not shown). Note that in absolute terms, there is substantially more F_z at 300 hPa than at 100 hPa (consistent with the region of negative squared refractive index evident in Figs. 1c,d), and hence, this composite does not necessarily represent an actual increase in F_z but rather includes days in which the layer between 300 and 100 hPa leads to less decay than usual.

A similar procedure is used to define the “decaying” composite: a day in which $\hat{F}_{z,100}$ is smaller than $\hat{F}_{z,300}$ by at least 1.5 STD (red bars in Fig. 3). Note that in absolute terms, there is nearly always decay of F_z between 300 and 100 hPa, and hence this composite includes days when this decay is more severe than usual, as well as days with reflection. The third category includes all other events in which the difference between $\hat{F}_{z,100}$ and $\hat{F}_{z,300}$ is smaller than 1.5 STD in absolute value (gray bars in Fig. 3).

We now examine the profile of the refractive index for these decaying and transmitting composites in Fig. 4. The top row presents the transmitting composite and the bottom row presents the difference between the transmitting and decaying composites. Regions with a statistically significant difference using a Student’s t test are denoted with gray dots. The barrier for wave propagation near 200 hPa for $M70n_{N_{\phi,z}}$ and CD61 previously shown for the climatology (Figs. 1b,c) is apparent for all four definitions (top row). Matsuno’s linear theory derivation concluded that a more negative squared refractive index leads to stronger wave decay. More importantly, for all four definitions, the index of refraction is more positive in the transmitting as compared with the decaying composite at 200 hPa, 60°N (Fig. 4). However, this region of positive index

FIG. 4. (a),(e) $M70N_{\text{const}}$, (b),(f) $M70N_{\text{CR92}}$, (c),(g) $M70N_{N_{\phi,z}}$, and (d),(h) CD61 are shown for (top) the transmitting composite and (bottom) the difference of the transmitting minus decaying composite mean. Black dots indicate grid points in which the difference Student’s t -test p values are smaller than 5%. We trimmed out 2.5% of the largest and 2.5% of the smallest values.

FIG. 5. The second derivative term of Eq. (7) in the transmitting minus decaying composites. (a) Using the high-vertical-resolution data used in most of this paper (grid spacing of 10 hPa just above the tropopause, section 2). (b) Using ERA5.1 standard vertical resolution with a grid spacing of 25 hPa between the levels of 250 and 100 hPa (i.e., 300, 250, 225, 200, 175, 150, 125, and 100 hPa).

of refraction is more extensive just above the tropopause for the definitions that include the terms in Eq. (7). Hence the neglect of these terms by Matsuno (1970) leads to a misestimation of wave decay immediately above the tropopause not just in the climatology, but also when considering individual extreme events. Supplemental Fig. 3 shows that the difference between the composites starts at least 2 days before a event while the time it takes for waves to propagate upward from 300 to 100 hPa is 1 day (supplemental Fig. 4). Thus, the refractive index may be considered as representing the basic state for wave propagation even though it may generally coevolve with the wave field on daily time scales.

Which term in the refractive index is most important for discriminating between the transmitting and decaying composites? Figure 5 presents the difference between the transmitting and decaying composite for the second derivative term $(f^2/N^3)\partial^2 N/\partial z^2$. The second derivative term is more positive in the transmitting composite. The difference of the other terms in Eq. 7 between the decaying and transmitting composites are relatively negligible just above the tropopause (not shown). Recall, that the second derivative term is the largest contributor to the barrier at 200 hPa in the climatological index of refraction (section 2 and Fig. 2).

To better understand why the second derivative term differs so strongly between the transmitting and decaying composite, we present the buoyancy frequency profile in the transmitting and decaying composites in Figs. 6a and 6b. For both composites, N^2 is maximized near 200 hPa and its second derivative is negative in that region. However, the local maximum is stronger in the decaying composite (Fig. 6b as compared with Fig. 6a) leading to a more negative second derivative in the decaying composite and hence a stronger barrier to wave propagation.

The transmitting and decaying composites differ not only in the distribution of N^2 , but also in the zonal wind (Figs. 6c,d). In particular, the stratospheric vortex is stronger in the decaying composite, which could lead to weakened upward propagation of waves (Chen and Robinson 1992). We now ask the question, Which is more important, $\bar{u}$ or N^2 , for the negative region of index of refraction? Figure 7 attempts to quantify the relative

FIG. 6. The buoyancy frequency is shown for the (a) transmitting and (b) decaying composites. (c),(d) Similarly, the zonal mean zonal wind profiles are shown. The n value indicates the number of composited events. Units of N^2 are s^{-2} , and units of $\bar{u}$ are $m s^{-1}$.

importance of $\bar{u}$ and N^2 . Specifically, Fig. 7a shows the third form of the refractive index $M70n_{N_{\phi,z}}$ as in Fig. 4c but where we first compute the composite average of the buoyancy frequency and the zonal mean zonal wind and then compute the index of refraction, rather than computing the index of refraction using daily $\bar{u}$ and N^2 and then averaging the index of refraction. The main features are essentially the same regardless of the order of the operations. Next, we present $M70n_{N_{\phi,z}}$ but using either N^2 or u from the transmitting composite and the other from the decaying composite (Figs. 7b,c). The overall pattern of the index of refraction is again similar. Finally, we calculate the difference between each of Figs. 7a–c and $M70n_{N_{\phi,z}}$ for the decaying composite calculated using the average N^2 and u (bottom row of Fig. 7). The difference shown in Fig. 7d is consistent with that shown in Fig. 4g when the time averaging is performed after the index of refraction is already calculated. Figures 7e and 7f allow us to answer the question of whether $\bar{u}$ or N^2 is more important. In the lowermost stratosphere, the dominant contributor is the buoyancy frequency, as Fig. 7e exhibits a stronger and more widespread increase in the index of refraction in this region as compared to Fig. 7f. Higher in the stratosphere both in the subtropics and subpolar latitudes, however, $\bar{u}$ is more important. It is beyond the scope of this work to quantify whether these changes higher in the stratosphere associated with $\bar{u}$ could lead to wave reflection near 60°N and thereby to changes in $\bar{F}_z$ between 100 and 300 hPa; however, the local changes near the tropopause are clearly driven by N .

5. Upward propagation of synoptic waves immediately above the tropopause

Thus far we have considered planetary wave-1 upward wave flux and in particular we have shown that the index of refraction can account for the difference between transmitting and

FIG. 7. (a) As in Fig. 4c, but first averaging the buoyancy frequency and the zonal-mean zonal wind over all days included in the transmitting composite before evaluating $M70n_{N\phi,z}$. (b) As in (a), but using the mean N^2 from the transmitting composite and the mean u from the decaying composite. (c) As in (a), but using the mean N^2 from the decaying composite and the mean u from the transmitting composite. (d)–(f) the difference between (a)–(c) and $M70n_{N\phi,z}$, calculated using both N^2 and u averaged over the decaying composite.

decaying wave-1 events. Now we assess sensitivity to analyzing zonal wavenumber-6 F_z as opposed to wave 1. It has long been known that wave 6 decays with height (Charney and Drazin 1961) and Fig. 8 shows that the barrier for wave 6 is even more severe for the indices of refraction that include the extra terms. While the DJF climatological squared refractive index is negative in a broader region of the stratosphere for wave 6 as compared to wave 1 for all four forms, the magnitude of this barrier differs. Crucially, the four forms differ just above the tropopause, and the forms of the index of refraction that include the terms in Eq. (7) capture a stronger barrier at the tropopause (Figs. 8c,d).

It is interesting to note that for wave 6, the connection between $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ is much weaker than for wave 1. Table 2 is as in Table 1 except for wave 6, and it is clear that wave-6 anomalies at 100 hPa are not strongly connected to wave-6 anomalies at 300 hPa. Specifically, a null hypothesis that the same random process leads to variability of both $\hat{F}_{z,300}$ and $\hat{F}_{z,100}$ can be rejected for 34 of the 36 elements of Table 2. This implies that the tropopause region leads to a near total decay of wave 6 that enters from below, and that any wave 6 at 100 hPa is unrelated to upward propagating waves in the upper troposphere. This is in contrast to wave 1, where extreme $\hat{F}_{z,300}$ events generally precede $\hat{F}_{z,100}$ events.

6. Discussion

Probabilistic weather forecasts can be enhanced by better prediction of stratospheric variability (Sigmond et al. 2013; Domeisen et al. 2020b), and stratospheric variability is in turn dictated in part by the ability of waves of tropospheric origin to

reach the stratosphere. The focus of this work is to better understand upward propagation of wave activity just above the tropopause. A commonly used diagnostic of Rossby wave propagation is the index of refraction of Matsuno (1970), with a more negative squared refractive index leading to stronger wave decay. Matsuno's derivation does not permit height

The refractive index climatology, wave-6, 5% trimmed

FIG. 8. As in Fig. 1, but for the wave-6 component. The spherical wavenumber is $s = 6$ and $k = 6/a \cos(60^\circ)$ in the β -plane version (section 3). The zonal phase speed is set equal to zero.

TABLE 2. As in Table 1, but using the wave-6 component of the F_z anomaly.

F_z anomaly events						
	$-3 < \hat{F}_{z,100} < -2$	$-2 < \hat{F}_{z,100} < -1$	$-1 < \hat{F}_{z,100} < 0$	$0 < \hat{F}_{z,100} < 1$	$1 < \hat{F}_{z,100} < 2$	$2 < \hat{F}_{z,100} < 3$
$-3 < \hat{F}_{z,300} < -2$	1+	13+	26–	23	8+	1+
$-2 < \hat{F}_{z,300} < -1$	11+	49+	162–	79–	36+	6+
$-1 < \hat{F}_{z,300} < 0$	27+	120+	636+	491–	114+	23+
$0 < \hat{F}_{z,300} < 1$	29+	90+	517+	453–	140+	24+
$1 < \hat{F}_{z,300} < 2$	5+	34+	119+	108–	51+	15+
$2 < \hat{F}_{z,300} < 3$	4+	6+	24+	36–	10	4+

dependence of the buoyancy frequency, but this restriction is clearly violated at and above the tropopause. We explore the impact of the ignored terms above the tropopause in the ERA5.1 dataset with 137 vertical levels and ~ 300 m resolution at the tropopause and in the lowermost stratosphere using four forms of the refractive index: the original refractive index of Matsuno (1970, $M70N_{\text{const}}$), two generalizations of the original index ($M70n_{\text{CR92}}$ and $M70n_{N_{\theta,z}}$), and the index of Charney and Drazin (1961). While these generalizations cannot be rigorously justified, they are shown here to offer guidance on regions of wave decay not evident upon using the original Matsuno (1970) index, and this additional insight is also evident in the index of refraction of Charney and Drazin (1961) that explicitly allows for height dependence of the buoyancy frequency.

Within the stratosphere, $M70N_{\text{const}}$, $M70n_{\text{CR92}}$, and $M70n_{N_{\theta,z}}$ are qualitatively similar and the assumption of constant N is reasonable for that region (Fig. 1). However, immediately above the tropopause we conclude that the additional terms in $M70n_{N_{\theta,z}}$ and CD61 are not negligible. The squared index of refraction becomes negative and leads to a barrier for upward propagation just above the tropopause in the mean state (Figs. 1a,b compared with Figs. 1c,d). Among the additional terms included in $M70n_{N_{\theta,z}}$ but not in $M70n_{\text{CR92}}$, the largest contributor for the negative values just above the tropopause is the second derivative term $(f^2/N^3)\partial^2 N/\partial z^2$ (Fig. 2) associated with a pronounced local maximum of N in this region.

We then consider the intraseasonal variability in the connection between wave flux at 300 and 100 hPa. We use Table 1 to answer two questions: 1) How many extreme events at 100 hPa are preceded by extreme events at 300 hPa? 2) To what extent can anomalous $\hat{F}_{z,300}$ events be used as a precursor to extreme $\hat{F}_{z,100}$ events? We find that the majority of extreme $\hat{F}_{z,100}$ events are preceded by anomalous $\hat{F}_{z,300}$. Specifically, 75% (82%) of the days in which $\hat{F}_{z,100}$ is greater than 2 STD (smaller than -2 STD) are preceded by greater than 1 (smaller than -1) STD anomaly at 300 hPa (first and last columns on Table 1). There are almost no extreme events at 100 hPa in which the anomaly at 300 hPa is of opposite sign or very weak. These findings likely can be used to enhance predictability of F_z at 100 hPa: given near-climatological F_z at 300 hPa one should not expect an extreme event at 100 hPa, and given an extreme event at 300 hPa $F_{z,100}$ will almost certainly have the same sign (95% of the days in the bottom row of Table 1). However, of the days in which F_z at 300 hPa is between 2 and 3 standard deviation, only 38% are followed by a similarly

extreme F_z at 100 hPa [consistent with Birner and Albers (2017), de la Cámara et al. (2019), and White et al. (2019)]. Extreme $\hat{F}_{z,300}$ events may be used as a precursor to anomalously positive $\hat{F}_{z,100}$ events as 78% of the days in which $\hat{F}_{z,300}$ is greater than 2 STD are followed by $\hat{F}_{z,100}$ greater than 1 STD (upper row on Table 1). The relationship for negatively anomalous $\hat{F}_{z,300}$ events is much weaker as only 56% of the days in which $F_{z,300}$ is smaller than -2 STD are followed by $F_{z,100}$ smaller than -1 STD.

To understand why on some days the standardized $F_{z,100}$ is much larger than the standardized $F_{z,300}$, and vice versa, we have formed “transmitting” and “decaying” composites in which the standardized $F_{z,100}$ is greater than the standardized $F_{z,300}$ by at least 1.5 STD, and smaller than $\hat{F}_{z,300}$ by at least 1.5 STD, respectively (blue and red bars in Fig. 3, respectively). We then considered which of the four indices of refraction can account for the difference between the transmitting and decaying composites. While both composites feature a barrier near 200 hPa, for $M70n_{N_{\theta,z}}$ and CD61 (Fig. 4) the index of refraction is more negative in the decaying as compared to the transmitting composite at 200 hPa (Fig. 4). In contrast, the other two indices are relatively unsuccessful in explaining the difference between the transmitting and decaying composites. The second derivative term in the refractive index is the most important term for discriminating between the transmitting and decaying composites (Figs. 5 and 6a,b), as was concluded in the climatology.

In both the mean state and in the difference between the transmitting and decaying composites, the second derivative term $(f^2/N^3)\partial^2 N/\partial z^2$ near 200 hPa was found to play a crucial role in upward wave propagation from the troposphere to the stratosphere (Fig. 5a). This term is associated with a local maximum of N just above the tropopause, otherwise known as the tropopause inversion layer (Birner 2006). Lower-resolution reanalyses struggle to capture this local maximum (Birner et al. 2006), and while a tropopause inversion layer (TIL) is present (albeit with biases) in some more recent reanalysis (Gettelman and Wang 2015; Pilch Kedzierski et al. 2016), Bell and Geller (2008) estimate that a vertical resolution of 240 m or coarser will miss aspects of this local maximum (see their section 4). Here we use ERA5, which has 137 vertical levels and a nominal vertical resolution of 300 m above and at the tropopause, which is significantly better than previous reanalyses.

To better understand the importance of fine vertical resolution for correctly representing this feature, we have repeated

the analysis of Fig. 5a created with model level data but instead using the ERA5 standard output with 37 vertical levels and a vertical resolution of 25 hPa instead of 10 hPa in the lowermost stratosphere (Fig. 5b). While the general sense of a maximum near 200 hPa is still present, it is much weaker. Similarly, Fig. 8 of Shaw et al. (2010) shows a region of near zero vertical wavenumber (i.e., limited wave propagation and near-reflection) just above the tropopause in ERA-40 data, and a similar effect is evident in NCEP–NCAR reanalysis data (Figs. 4 and 7 of Perlwitz and Harnik 2003), but in both reanalyses this effect is weak. It is likely that 137 vertical levels and a nominal vertical resolution of 300 m are still not sufficient to capture the full magnitude of this effect (see section 4 of Bell and Geller 2008), and hence it is worth revisiting this in future even higher-resolution reanalysis.

It is important to note that a region of limited vertical extent with negative index of refraction does not lead to total wave absorption, as some wave activity is able to tunnel through such a region. The amount of tunneling is dependent on the vertical extent of the negative region and the degree to which the values are negative (Charney and Drazin 1961; Harnik and Lindzen 2001; Harnik 2002; Perlwitz and Harnik 2003), and therefore, a narrow region of sharply negative index of refraction can have the same effect as a broader region of weakly negative index of refraction as the wave decay is proportional to the product of the index of refraction and the depth (Charney and Drazin 1961, see the discussion near the top of p. 89). Ongoing work will quantify the role of the TIL for the transmission versus reflection of an upward directed wave using an analytic perspective.

The additional terms in $M70n_{N_{\phi,z}}$ were found to be important for governing upward wave flux for wave 1 and wave 6 both in the mean state and also associated with extreme events. These extra terms were less successful in explaining extreme events of wave 2, however (not shown), and the reasons why should be explored for future work.

Chen and Robinson (1992) perform experiments with a primitive equation model linearized about a winter-like background state to show that the tropopause is a partial barrier for wave propagation. Specifically, they compare upward wave flux in an experiment with a jump in N at the tropopause to one with a constant value of N . There is substantially less convergence of F_z near the tropopause in their experiment with constant N (their Fig. 11 versus Fig. 10, and also their Fig. 12). This effect is consistent with the $M70n_{N_{\phi,z}}$ definition used here. Chen and Robinson (1992) argue that this effect is also consistent with the $M70n_{CR92}$ definition, but as discussed above the tendency of the tropopause to act as a barrier is much clearer if the three additional terms in Eq. (7) are included.

Chen and Robinson (1992) also find in their linearized primitive equation model that weaker vertical shear in the lower stratosphere leads to enhanced upward wave flux near the tropopause (their section 4c). We find a similar effect in reanalysis: Figs. 6c and 6d show the mean zonal wind for the decaying and transmitting composite, and it is clear that the vertical shear in subpolar latitudes is larger for the decaying composite. This is consistent with the refractive index in which a larger first vertical derivative of the zonal wind in the

term $-(af^2/\rho_0)\partial[(\rho_0/N^2)(\partial\bar{u}/\partial z)]/\partial z$ is expected to lead a more negative refractive index. This effect should be captured by all four forms, including that of CD61 (consistent with Chen and Robinson 1992), and likely is responsible for the relatively weak region of positive index of refraction in Figs. 4e and 4f just above the tropopause. However, this region of positive index of refraction is far weaker than that in Figs. 4g and 4f which include the three additional terms in Eq. (7).

Overall, the results in this work highlight that the index of refraction of Charney and Drazin (1961) is better suited to capturing wave propagation immediately above the tropopause than that of Matsuno (1970). However, it is possible to construct an index of refraction [given in Eq. (5) in this paper] that combines the best properties of each: it captures both the role of meridional gradients in e.g., zonal wind as in Matsuno (1970), and also the importance of rapid changes in buoyancy frequency just above the tropopause. While this blended index of refraction cannot be theoretically justified, we argue that it is nonetheless a useful tool for understanding the propagation of waves above the troposphere.

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APPENDIX

Comparison of the Derivations of Matsuno versus Charney and Drazin

The goal of the appendix is to highlight the source of the differences in the index of refraction derived by Charney and Drazin (1961) versus that derived by Matsuno (1970). As the β plane is the simplest geometry for elucidating the differences between the assumptions of Charney and Drazin (1961) versus Matsuno (1970), we use this geometry even though the Matsuno (1970) index of refraction is typically examined in spherical coordinates as is done in the main text. We begin with the equation for perturbation potential vorticity linearized about a basic state with zonal wind that changes both meridionally and vertically but not zonally. The quasigeostrophic linearized potential vorticity equation, using the notation of Andrews et al. (1987) in their Eqs. (4.5.1), (4.5.2), and (4.5.4), is

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)q' + v'\bar{q}_y = H', \quad (A1)$$

where H' denotes nonconservative terms, which we take as zero,

$$q' = \psi'_{xx} + \psi'_y + \rho_0^{-1} \left(\rho_0 \frac{f^2}{N^2} \psi'_z \right)_z \quad (A2)$$

is the perturbation quasigeostrophic potential vorticity, $v' = \psi'_x$ and

$$\bar{q}_y = \beta - \bar{u}_{yy} - \rho_0^{-1} \left(\rho_0 \frac{f^2}{N^2} \bar{u}_z \right)_z \quad (\text{A3})$$

is the basic-state northward quasigeostrophic potential vorticity gradient. All notation is identical to that in [Andrews et al. \(1987\)](#).

The term $[\rho_0(f^2/N^2)\psi'_z]_z$ in Eq. (A2) naturally arises when the quasigeostrophic (QG) PV equation is derived from the thermodynamic and vorticity equations. Specifically, the buoyancy frequency multiplies the vertical wind in the thermodynamic equation, and when the thermodynamic and vorticity equations are combined to eliminate vertical wind, the net effect is buoyancy frequency appears inside a vertical derivative. The buoyancy frequency appears inside the vertical derivative as it corresponds to the stretching of the vertical grid that is needed to “align” it with the horizontal grid in a dynamical sense. That is, the factor f/N inside the vertical derivative corresponds to the ratio of depth to length scales under QG scaling. So if you want to transform the system so that horizontal and vertical scales become equal in scales you need to scale height by N (as well as density to incorporate differences in mass). Note that the amount of stretching required naturally differs in the troposphere versus in the stratosphere. The net effect is to transform the operator in Eq. (A2) to a 3D Laplacian.

For the second and third forms of the index of refraction in the text, we allow for latitudinal structure in N . Note, however, that the derivations below are generally not possible if such structure is allowed. More generally, latitudinally varying N is inconsistent with the assumptions underlying theories of linearized wave dynamics, including quasigeostrophy ([Andrews et al. 1987](#)), as N describes the reference state which is a function of z only.

a. Matsuno

The key assumption of [Matsuno \(1970\)](#) is that the buoyancy frequency is assumed to be constant, which greatly simplifies the vertical derivatives in Eqs. (A2) and (A3). Setting

$$\psi' = e^{z/2H} \text{Re}[\Psi(y, z) e^{ik(x-ct)}] \quad (\text{A4})$$

leads to Eq. (4.5.27) of [Andrews et al. \(1987\)](#) repeated here:

$$\Psi_{yy} + \frac{f^2}{N^2} \Psi_{zz} + n_k^2 \Psi = 0, \quad (\text{A5})$$

which specifies the wave equation for the complex amplitude Ψ . The index of refraction n_k^2 (i.e., it determines the propagating/evanescent regions) is defined as

$$n_k^2(y, z) = \frac{\bar{q}_y}{\bar{u} - c} - k^2 - \frac{f_0^2}{N^2} \frac{1}{4H^2}. \quad (\text{A6})$$

Equation (A6) is the beta-plane equivalent of Eq. (1).

Equations (A3) and (A6) can be combined to give

$$n_k^2(y, z) = \frac{1}{\bar{u} - c} \left[\beta - \bar{u}_{yy} - \rho_0^{-1} \left(\rho_0 \frac{f^2}{N^2} \bar{u}_z \right)_z \right] - k^2 - \frac{f_0^2}{N^2} \frac{1}{4H^2}. \quad (\text{A7})$$

The key assumption of Matsuno, namely, that the buoyancy frequency is constant, allows him to neglect vertical variations of N both in the equation for perturbation potential vorticity and mean-state potential vorticity gradient [Eqs. (A2) and (A3)]. Note that it is inconsistent to retain vertical variations of N in only one of these equations.

b. Charney and Drazin

[Charney and Drazin \(1961\)](#) do not make this limiting assumption on N ; rather, N is allowed to vary in the vertical. Instead, their key assumption concerns the meridional structure: $\bar{u}$ is fixed in the meridional direction ($\bar{u}_y = 0$ but $\bar{u}$ is allowed to vary in height) and the perturbation potential vorticity varies sinusoidally in y . We can then define the perturbation streamfunction [analogously to Eq. (A4)] as

$$\psi' = e^{z/2H} \text{Re}[\Psi(z) e^{i(kx+ly-kct)}], \quad (\text{A8})$$

and the $\bar{u}_{yy}$ term in Eq. (A3) is identically zero. Note that Ψ of [Charney and Drazin \(1961\)](#) contains vertical variations only, while that of [Matsuno \(1970\)](#) also includes meridional variations. For simplicity we use the same symbol for both.

Substitution of Eq. (A8) in Eq. (A2) leads to CD61's perturbation quasigeostrophic potential vorticity,

$$q' = \text{Re} \left\{ (-k^2 - l^2) e^{z/2H} \Psi + \rho_0^{-1} \left[\rho_0 \frac{f^2}{N^2} (e^{z/2H} \Psi)_z \right]_z \right\} e^{i(kx+ly-kct)}, \quad (\text{A9})$$

and the mean-state quasigeostrophic potential vorticity is

$$\bar{q}_y = \beta - \rho_0^{-1} \left(\rho_0 \frac{f^2}{N^2} \bar{u}_z \right)_z. \quad (\text{A10})$$

Substitution of Eqs. (A9) and (A10) in Eq. (A1) leads to (after some algebra)

$$0 = (\bar{u} - c) \left[\frac{\rho_0}{N^2} (e^{z/2H} \Psi)_z \right]_z \quad (\text{A11})$$

$$- \left\{ \left(\frac{\rho_0}{N^2} \bar{u}_z \right)_z + \frac{\rho_0 \beta}{f^2} \left[(\bar{u} - c) \frac{(k^2 + l^2)}{\beta} - 1 \right] \right\} e^{z/2H} \Psi. \quad (\text{A12})$$

Equation (A12) is identical to the equation of [Charney and Drazin \(1961\)](#) at the very beginning of their section 3, if V is replaced by $e^{z/2H} \Psi$. [Charney and Drazin \(1961\)](#) show that Eq. (A12) can be transformed into the canonical form $\Xi_{zz} + n^2 \Xi = 0$ if $\Xi \equiv \sqrt{\rho_0/N^2} e^{z/2H} \Psi$, where

$$n^2 = \frac{N^2}{f^2} \left\{ \frac{\beta}{\bar{u} - c} - \frac{1}{\bar{u} - c} \frac{f_0^2}{\rho_0} \left[\frac{\rho_0}{N^2(z)} \bar{u}_z \right]_z - (k^2 + l^2) \right\} - \frac{N^2}{f^2} \frac{f_0^2}{N(z)} \sqrt{\frac{1}{\rho_0} \left[\frac{\rho_0}{N^2(z)} \right]_{zz}}. \quad (\text{A13})$$

We can now compare the index of refraction of [Charney and Drazin \(1961\)](#) in Eq. (A13) (if multiplied by f_0^2/N^2) with the beta-plane version of [Matsuno \(1970\)](#) in Eq. (A7) to illustrate how the differing assumptions affect the index of refraction. These differences include the following:

- 1) The second derivative of the basic-state zonal wind ($u_{yy} = 0$) which Charney and Drazin (1961) assume to be zero
- 2) The f^2 term only appears in Eq. (A13) because Charney and Drazin (1961) assume a sinusoidal dependence in the meridional direction
- 3) The last term on the right-hand side of Eq. (A13), which is the term we include in our generalized refractive index $M70n_{N_{\phi,z}}$ and which we focus on now

We expand the last term in Eq. (A13) [also the last term in Eqs. (5) and (6)] in order to explicitly show the terms rendered zero by the assumption of constant N in Matsuno (1970) but which are retained by Charney and Drazin (1961). This term becomes

$$-\frac{f_0^2}{\sqrt{\rho_0}N} \frac{\partial^2}{\partial z^2} (N^{-1} \sqrt{\rho_0}) = -\frac{f_0^2}{\sqrt{\rho_0}N} \left[\frac{\partial}{\partial z} \left(-N^{-2} \frac{\partial N}{\partial z} \sqrt{\rho_0} \right) + \frac{\partial}{\partial z} \left(N^{-1} \frac{\partial \sqrt{\rho_0}}{\partial z} \right) \right] \quad (\text{A14})$$

$$= \frac{f_0^2}{\sqrt{\rho_0}N} \left[-\frac{1}{N} \frac{\partial^2 \sqrt{\rho_0}}{\partial z^2} + \frac{2}{N^2} \frac{\partial N}{\partial z} \frac{\partial \sqrt{\rho_0}}{\partial z} + \frac{1}{N^2} \frac{\partial^2 N}{\partial z^2} \sqrt{\rho_0} - \frac{2}{N^3} \left(\frac{\partial N}{\partial z} \right)^2 \sqrt{\rho_0} \right] \quad (\text{A15})$$

$$= \frac{f_0^2}{N^2} \left[-\frac{1}{\sqrt{\rho_0}} \frac{\partial^2 \sqrt{\rho_0}}{\partial z^2} + \frac{2}{N} \frac{\partial N}{\partial z} \frac{1}{\sqrt{\rho_0}} \frac{\partial \sqrt{\rho_0}}{\partial z} + \frac{1}{N} \frac{\partial^2 N}{\partial z^2} - \frac{2}{N^2} \left(\frac{\partial N}{\partial z} \right)^2 \right]. \quad (\text{A16})$$

We can then substitute $\rho_0 = \rho_s e^{-z/H}$ to the right-hand side of Eq. (A16), and the net result is

$$-\frac{f_0^2}{\sqrt{\rho_0}N} \frac{\partial^2}{\partial z^2} (N^{-1} \sqrt{\rho_0}) = \frac{f_0^2}{N^2} \left[-\frac{1}{4H^2} - \frac{1}{HN} \frac{\partial N}{\partial z} + \frac{1}{N} \frac{\partial^2 N}{\partial z^2} - \frac{2}{N^2} \left(\frac{\partial N}{\partial z} \right)^2 \right] \quad (\text{A17})$$

The first term on the right-hand side of (A17) is identical to that in Eqs. (1) and (A6), but the remaining three terms include the extra physics that we show to be important in the lower stratosphere and near the tropopause.

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