ON PERMANENT PERTURBATIONS OF ZONAL ATMOSPHERIC MOTION, WITH APPLICATIONS TO LOW LATITUDES

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ABSTRACT

If the pressure is eliminated from the vorticity equation, this vector equation and the continuity equation constitute a system of four, independent scalar equations in the four dependent variables, consisting of the momentum components and the density. In this paper, these basic equations are used for study of those zonally propagated permanent-type flows which have small relative velocities. At first, relationships are deduced directly from the differential equations. Perhaps chief of these is a positive correlation in the lower layers of the atmosphere between the vertical and poleward momentum-components. Finally, the equations are integrated in terms of arbitrarily assignable fields, and a numerical example is presented.

1. Introduction

It is common experience that many disturbances move across the weather map with little or no change in shape. A disturbance which moves at a constant velocity of propagation, and with no change in shape, is either said to be of permanent type [6] or to be relatively steady [7]. The case of relatively steady disturbances which are zonally propagated is particularly simple, but nevertheless is of meteorological interest. It is aspects of this case to which we address ourselves in this paper.

Previous investigations of this problem include the line of attack developed by Rossby [15], Haurwitz [4; 5], Craig [2] and Neamtan [8]. By 1946, this had produced a theory of permanent-type waves for horizontal motion at a level of non-divergence on a spherical earth. Another attack on the problem is exemplified by the work of Queney [14], who treated steady-state flow past mountain barriers of infinite length. The stationary wave is, of course, a special case of relatively steady flow. In 1947, Charney [1] obtained the structure of a relatively steady long wave in a specific model of baroclinic westerlies. He assumed the motion to be independent of latitude. All these writers have used the device of treating certain parameters (for example, the Coriolis parameter and the planetary vorticity) as constants, save for the process of differentiation. In particular, this has been done in the choice of eigenvalues, satisfying integrability equations, and thus justifying the building of general solutions from elementary ones of exponential form.

The present paper differs in two respects: Attention is not restricted to problems which are two-dimensional, either in the sense that the streamlines are confined to a coordinate surface or in the sense that the velocity field is independent of one of the space coordinates. Also, neither the Coriolis parameter nor the planetary vorticity are treated as constants. However, we do not obtain results concerning changes of the systems, as did Charney. Nor do we consider frictional effects, as Queney did. Finally, the work does not go beyond the use of perturbation methods, as did that of Rossby, Haurwitz, Craig and Neamtan.

2. The governing equations

It is possible to write the vorticity equation in terms of velocity and density alone. With this done, one may regard the vorticity and continuity equations as a complete set, in the sense that they are four scalar equations in four dependent variables. Together with appropriate boundary conditions, they may be used as a basis for the solution of a large class of boundary-value problems.

In particular, one may use them for study of relatively steady motion, zonally propagated. This case may be treated particularly simply, for study of it may be reduced to the solution of steady-state equations by referring the fields to a coordinate system moving with the fluid disturbance. When this is done, the angular velocity of the coordinate system and the gravity remain constant (although altered from the values observed from a point fixed to the earth), and the velocity field is the field of relative motion. This device will be adopted in deriving the governing equations below.

The equation of motion relative to the earth, referred to unit mass, is

$$\dot{\mathbf{v}} + 2\mathbf{\Omega} \times \mathbf{v} + \alpha \nabla p + \nabla \Phi = \alpha \mathbf{F}, \qquad (1)$$

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where $\boldsymbol{v}$ is velocity, $\boldsymbol{\Omega}$ the angular velocity of the earth, α specific volume, p pressure, Φ geopotential, and $\boldsymbol{F}$ the viscous force per unit volume. If we multiply (1) by the density ρ, we obtain the same equation referred to unit volume:

$$\rho\dot{\boldsymbol{v}} + 2\boldsymbol{\Omega} \times \rho\boldsymbol{v} + \nabla p + \rho\,\nabla\Phi = \boldsymbol{F}. \quad (2)$$

The identity

$$\dot{\boldsymbol{v}} = \partial\boldsymbol{v}/\partial t + \boldsymbol{q} \times \boldsymbol{v} + \nabla(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}) \quad (3)$$

permits (2) to be rewritten as

$$\rho\,\partial\boldsymbol{v}/\partial t + (\boldsymbol{q}+2\boldsymbol{\Omega})\times\rho\boldsymbol{v} + \nabla p + \rho\,\nabla(\Phi+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}) = \boldsymbol{F}, \quad (4)$$

where $\boldsymbol{q}$ is the vorticity. Taking the curl of each member of (4), we obtain a form of the vorticity equation:

$$\begin{aligned}\rho\,\partial\boldsymbol{q}/\partial t + \nabla\rho\times\partial\boldsymbol{v}/\partial t + (\nabla\cdot\rho\boldsymbol{v})(\boldsymbol{q}+2\boldsymbol{\Omega}) + (\rho\boldsymbol{v}\cdot\nabla)\boldsymbol{q}\\ -[(\boldsymbol{q}+2\boldsymbol{\Omega})\cdot\nabla]\rho\boldsymbol{v} + \nabla\rho\times\nabla(\Phi+\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}) = \nabla\times\boldsymbol{F}.\end{aligned} \quad (5)$$

The terms of (5) may be grouped as:

$$2\boldsymbol{\Omega}\cdot\nabla\rho\boldsymbol{v} + \nabla\Phi \times \nabla\rho = \boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C}, \quad (6)$$

where

$$\begin{aligned}\boldsymbol{A} &= (\rho\boldsymbol{v}\cdot\nabla)\boldsymbol{q} - (\boldsymbol{q}\cdot\nabla)\rho\boldsymbol{v} + \nabla\rho \times \nabla(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}),\\ \boldsymbol{B} &= \nabla\rho \times \partial\boldsymbol{v}/\partial t + \rho\,\partial\boldsymbol{q}/\partial t - (\boldsymbol{q} + 2\boldsymbol{\Omega})\,\partial\rho/\partial t,\\ \boldsymbol{C} &= -\nabla \times \boldsymbol{F}.\end{aligned} \quad (7)$$

In the expression for $\boldsymbol{B}$, use has been made of the equation of continuity,

$$\partial\rho/\partial t + \nabla\cdot\rho\boldsymbol{v} = 0. \quad (8)$$

If we now confine our attention to zonally propagated, relatively steady motion and make the change of coordinates described above, *i.e.*, if $\boldsymbol{\Omega}$, Φ and $\boldsymbol{v}$ are redefined to be the values relative to coordinates moving with the system, the term $\boldsymbol{B}$ vanishes identically. Further, if only the non-viscous case is considered, $\boldsymbol{C}$ vanishes as well. We are then left with equations whose left members are linear in density and momentum:

$$\begin{aligned}(2\boldsymbol{\Omega}\cdot\nabla)\rho\boldsymbol{v} + \nabla\Phi \times \Delta\rho &= \boldsymbol{A},\\ \nabla\cdot\rho\boldsymbol{v} &= 0.\end{aligned} \quad (9)$$

These equations govern the motion relative to the moving system.

In the particular case of steady zonal motion with a constant angular speed, the relative velocity field $\boldsymbol{v}$ in (9) vanishes identically. Accordingly, for such motion there is the restriction on the density that

$$\nabla\Phi \times \nabla\rho = 0. \quad (10)$$

We shall consider permanent type motions which differ *but little* from zonal motion at a constant angular speed. In the relative coordinate system used for (9), the motion will be a perturbation on the state of relative rest. Such motions, to a good approximation, satisfy

$$\begin{aligned}(2\boldsymbol{\Omega}\cdot\nabla)\rho\boldsymbol{v} + \nabla\Phi \times \nabla\rho &= 0,\\ \nabla\cdot\rho\boldsymbol{v} &= 0.\end{aligned} \quad (11)$$

Relation (10) implies that we are essentially considering motion of a perturbed, barotropic basic current; the motion, however, is not auto-barotropic. The justification for ignoring the term $\boldsymbol{A}$ is the quadratic nature in $\boldsymbol{v}$ and its space derivatives. For small relative velocities, such product terms may be ignored compared to the linear terms retained in (11). This is analogous to the treatment of waves in the ocean as perturbations on the state of rest [6; 17].

We introduce the spherical coordinates ψ = longitude, ϕ = latitude, and r = radial distance, and the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ directed, respectively, toward the east, north and zenith. Accordingly, $Z = r\sin\phi$ and ∇Z are a length coordinate and a unit vector parallel to the axis of the earth.[2] The following notation for differentiation parallel to the axis of the earth is introduced:

$$\nabla Z\cdot\nabla \equiv \frac{\partial}{\partial Z} \equiv \frac{\cos\phi}{r}\left(\frac{\partial}{\partial\phi} + r\tan\phi\,\frac{\partial}{\partial r}\right) \equiv \frac{\cos\phi}{r}L.$$

In the light of these definitions (11) becomes

$$\begin{aligned}r^{-1}\cos\phi\,L(\rho\boldsymbol{v}) + \tfrac{1}{2}\Omega^{-1}g\,\boldsymbol{k}\times\nabla\rho &= 0,\\ \nabla\cdot\rho\boldsymbol{v} &= 0.\end{aligned} \quad (12)$$

If the relative momentum $\rho\boldsymbol{v}$ is written as

$$\rho\boldsymbol{v} = \rho(u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k}) = U\boldsymbol{i} + V\boldsymbol{j} + W\boldsymbol{k},$$

the system (12) is equivalent to

$$\begin{aligned}&\cos\phi\,L(U) - \tfrac{1}{2}\Omega^{-1}g\,\partial\rho/\partial\phi = 0,\\ &\cos^2\phi\,[L(V) + W] + \tfrac{1}{2}\Omega^{-1}g\,\partial\rho/\partial\psi = 0,\\ &L(W) - V = 0,\\ &\frac{\partial U}{\partial\psi} + \cos\phi\,\frac{\partial V}{\partial\phi} + r\cos\phi\,\frac{\partial W}{\partial r} + 2W\cos\phi - V\sin\phi = 0.\end{aligned} \quad (13)$$

Notice that the relative-velocity components have been designated by small u, v and w, and the relative-momentum components by capital U, V, and W. The system (12) [or its equivalent (13)] is the set of equations governing the motions to be studied.

3. Vertical component of the vorticity equation

Before going on, it may be well to relate the third equation in (13) to the more familiar form of the

[2] We treat the earth as a sphere; the error in so doing is small and will not be elaborated.

vertical component of the vorticity equation,

$$d\zeta/dt + \beta v + (\zeta + f)\nabla_H \cdot \boldsymbol{v} - (\boldsymbol{q}_H + \beta r \boldsymbol{j})\cdot\nabla w + \boldsymbol{k}\cdot\nabla\alpha \times \nabla p = C, \quad (14)$$

where $\boldsymbol{q}_H$ is the horizontal component of $\boldsymbol{q}$ and $\beta = 2r^{-1}\Omega\cos\phi$. $\boldsymbol{k}\cdot\nabla\alpha \times \nabla p$ is the solenoidal term and C the viscous term. The vertical component of (5) is a regrouping of the terms of (14); it takes the form[3]

$$\beta[L(\rho w) - \rho v] = A + B + C, \quad (15)$$

where

$$A = \rho \boldsymbol{v}\cdot\nabla\zeta - (\boldsymbol{q}\cdot\nabla)\rho w + \boldsymbol{k}\cdot\nabla\rho \times \nabla(\tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}),$$

$$B = \rho\frac{\partial\zeta}{\partial t} - (\zeta + f)\frac{\partial\rho}{\partial t} + \boldsymbol{k}\cdot\nabla\rho \times \frac{\partial \boldsymbol{v}}{\partial t}.$$

If (15) is written in relative coordinates and only the non-viscous case is considered, the terms B and C vanish identically, and A is to be computed from the relative velocities. In these relative coordinates, (15) becomes

$$\beta[L(\rho w) - \rho v] = A. \quad (16)$$

To see what is involved in our neglect of A, let us compare our assumptions to those of Rossby [15]. He retained the term $\rho\beta v$ and part of the term $\rho \boldsymbol{v}\cdot\nabla\zeta$ of A, namely, $\rho u\, \partial\zeta/\partial x$. Since he did not use relative coordinates, our u is to be compared to his $U - c$. We have ignored $\rho u\, \partial\zeta/\partial x$ just as Rossby ignored $\rho v\, \partial\zeta/\partial y$; this is permissible in our work, since attention is confined to cases of small relative velocities—for such cases these terms are of the same order of magnitude. In common with Rossby, we retain the $\rho\beta v$ term; we also retain the $\beta L(\rho w)$ term. The reader may readily verify, by substituting reasonable orders of magnitude, that the vertical velocity terms of (14) and, more particularly, the term $L(\rho w)$ which combines these with the horizontal divergence term, are of the same order of magnitude as ρv (see section 5).

4. Solution of the equations of section 2

The plan for the solution of (13) is the following: we shall initially assume the W field known and compute the U, V and ρ fields; we shall utilize but three equations in doing this. Accordingly, it will then be possible to enter these results in the remaining equation to obtain conditions on W. For the moment, then, consider W known. An immediate consequence is that V is determined by the third equation of (13):

$$V(r, \phi, \psi) = L(W). \quad (17)$$

With both V and W known, U may be obtained, by means of a single integration, from the fourth equation of (13):

$$U(r, \phi, \psi) = A(r, \phi) - \int_{\psi_0}^{\psi}\left\{\frac{\partial}{\partial\phi}[V(r, \phi, \xi)\cos\phi] + \frac{\cos\phi}{r}\frac{\partial}{\partial r}[r^2 W(r, \phi, \xi)]\right\} d\xi, \quad (18)$$

where, clearly, $A(r, \phi) = U(r, \phi, \psi_0)$. We are also enabled to determine, from the second equation of (13),

$$\rho(r, \phi, \psi) = B(r, \phi) - 2\Omega g^{-1}\cos^2\phi \times \int_{\psi_0}^{\psi}\{L[V(r, \phi, \xi)] + W(r, \phi, \xi)\}\, d\xi, \quad (19)$$

where $B(r, \phi) = \rho(r, \phi, \psi_0)$. The fields U and ρ determined from (18) and (19) may be introduced into the first equation of (13); this will yield a condition on the fields $A(r, \phi)$, $B(r, \phi)$ and $W(r, \phi, \psi)$. We obtain

$$\cos\phi\, L(A) - \tfrac{1}{2}\Omega^{-1}g\, \partial B/\partial\phi = \int_{\psi_0}^{\psi} I(r, \phi, \xi)\, d\xi, \quad (20)$$

where

$$I(r, \phi, \psi) = \cos\phi\, L\left[\frac{\partial}{\partial\phi}(V\cos\phi) + \frac{\cos\phi}{r}\frac{\partial W r^2}{\partial r}\right] - \frac{\partial}{\partial\phi}[\cos^2\phi\,\{L(V) + W\}]. \quad (21)$$

Since the left member of (20) is independent of ψ, we obtain, by differentiating (20) with respect to ψ, the condition on W:

$$I(r, \phi, \psi) = 0. \quad (22)$$

Accordingly, (18) becomes

$$\cos\phi\, L(A) = \tfrac{1}{2}\Omega^{-1}g\, \partial B/\partial\phi. \quad (23)$$

Thus, (20) is equivalent to (22) and (23)—conditions, respectively, upon W and upon A and B.

Consider first the implications of (23); it yields

$$B(r, \phi) = B(r, \phi_0) + 2g^{-1}\Omega\int_{\phi_0}^{\phi}\cos\xi\, L[A(r, \xi)]\, d\xi, \quad (24)$$

where $B(r, \phi_0) = \rho(r, \phi_0, \psi_0)$. Thus, we have in (17), (18), (19) and (24) a solution for U, V and ρ in terms of $U(r, \phi, \psi_0)$, a single density sounding $\rho(r, \phi_0, \psi_0)$ and the field $W(r, \phi, \psi)$, arbitrary save that it must satisfy (22), the boundary conditions and any other conditions we choose to specify.

If the restriction upon W imposed by (22) is now considered, we find it to be only an apparent one, since $I(r, \phi, \psi)$ vanishes identically. To see this, it is sufficient to replace V in (21) by means of (17) and expand.

It is natural to assume, from the arbitrary nature of the W field, that perhaps the independence of the vorticity and continuity equations has been lost as a result of the simplifications. To see that this is not the case, it is sufficient to find four sets of functions such that each set satisfies three, but not all four, equations in (13). One such check of the independence is the

[3] The reader more familiar with (14) than with the full, vector vorticity equation may wish to derive (15) directly from (14). This is readily accomplished by this procedure: multiply (14) by ρ; substitute for $\rho\nabla_H\cdot\boldsymbol{v}$ from the continuity equation; substitute for $\alpha\nabla p$ in the solenoidal term from (4); regroup terms.

following:

$\rho = \phi,\ U = V = W = 0,$ satisfies all of (13) except the first equation;

$\rho = \psi,\ U = V = W = 0,$ satisfies all of (13) except the second equation;

$V = r^{-1} \ln r,\ W = -L(V),\ U = \rho = 0,$ satisfies all of (13) except the third equation;

$U = \psi,\ V = W = \rho = 0,$ satisfies all of (13) except the fourth equation.

The reader can readily supply similar sets of his own. Clearly the equations comprising (13) are independent.

From physical grounds, we should not expect to be able to decide whether purely zonal flow at the boundary of a region is accompanied by purely zonal flow throughout, or whether the boundary is simply a nodal surface for a system of waves, since either solution is consistent with the boundary conditions. This is in accord with general mathematical experience with steady-state problems. Thus, we cannot expect to achieve unique solutions.

The degree of freedom found in the motion, expressed above as the arbitrary nature of W, may appear in terms of similar arbitrariness of any one of the fields U, V, W, ρ. Just as above we began by assuming the field W, solving for consistent fields U, V and ρ, and then considering the conditions on W (which turned out to be trivial), we may similarly begin with any of the other fields and obtain similar results.

For example, if it is assumed that the field $V(r, \phi, \psi)$ is known, the third equation of (13),

$$\partial W/\partial\phi + r \tan\phi\, \partial W/\partial r = V,$$

may be solved by standard methods. We consider the auxiliary system

$$\frac{d\phi}{1} = \frac{dr}{r \tan\phi} = \frac{dW}{V(r, \phi, \psi)}.$$

This has the solutions

$$r \cos\phi = c_1,$$

$$W - \int_a^r c_1 V(\xi, \cos^{-1} c_1\xi^{-1}, \psi)\xi^{-\frac{1}{2}}[\xi^2 - c_1]^{-\frac{1}{2}}\, d\xi = c_2.$$

Accordingly, the third equation of (13) has the solution

$$W(r, \phi, \psi) = F(r\cos\phi, \psi) + \int_a^r V[\xi, \cos^{-1}(r\xi^{-1}\cos\phi), \psi] r\xi^{-\frac{1}{2}} \cos\phi \times [\xi^2 - r^2\cos^2\phi]^{-\frac{1}{2}}\, d\xi, \quad (25)$$

where the arbitrary function $F(r\cos\phi, \psi)$ has the interpretation

$$F(a\cos\phi, \psi) = W(a, \phi, \psi),$$

so that

$$F(r\cos\phi, \psi) = W[a, \cos^{-1}(ra^{-1}\cos\phi), \psi].$$

Hence (25) becomes

$$W(r, \phi, \psi) = W[a, \cos^{-1}(ra^{-1}\cos\phi), \psi] + \int_a^r V[\xi, \cos^{-1}(r\xi^{-1}\cos\phi), \psi] r\xi^{-\frac{1}{2}}\cos\phi \times [\xi^2 - r^2\cos^2\phi]^{-\frac{1}{2}}\, d\xi. \quad (26)$$

Once W has been computed, U may be obtained in terms of V, W and $U(r, \phi, \psi_0)$ by means of (18), and ρ may be determined in terms of $\rho(r, \phi_0, \psi_0)$ and the previously computed fields from (19) and (25). A condition on V, analogous to (22), turns out to be satisfied identically.

Summarizing, we have found that

given	consistent fields are
$W(r, \phi, \psi)$	$V = L(W)$
$U(r, \phi, \psi_0)$	$U = U(r, \phi_0, \psi_0) - \int_{\psi_0}^{\psi} \left\{ \frac{\partial}{\partial\phi}[\cos\phi\, V(r, \phi, \xi)] + \frac{\cos\phi}{r}\frac{\partial}{\partial r}[r^2 W(r, \phi, \xi)] \right\} d\xi$ (27)
$\rho(r, \phi_0, \psi_0)$	$\rho = \rho(r, \phi_0, \psi_0) + 2g^{-1}\Omega \int_{\phi_0}^{\phi} \cos\xi\, L[U(r, \xi, \psi_0)]\, d\xi - 2g^{-1}\Omega\cos^2\phi \int_{\psi_0}^{\psi} \{L[V(r, \phi, \xi) + W(r, \phi, \xi)]\}\, d\xi,$

and

given	consistent fields are
$V(r, \phi, \psi)$	$W = W[a, \cos^{-1}(ra^{-1}\cos\phi), \psi] + \int_a^r r\xi^{-\frac{1}{2}}\cos\phi\, V[\xi, \cos^{-1}(r\xi^{-1}\cos\phi), \psi][\xi^2 - r^2\cos^2\phi]^{-\frac{1}{2}} d\xi$ (28)
$U(r, \phi, \psi_0)$	U as in (27).
$W(a, \phi, \psi)$	ρ as in (27).
$\rho(r, \phi_0, \psi_0)$	

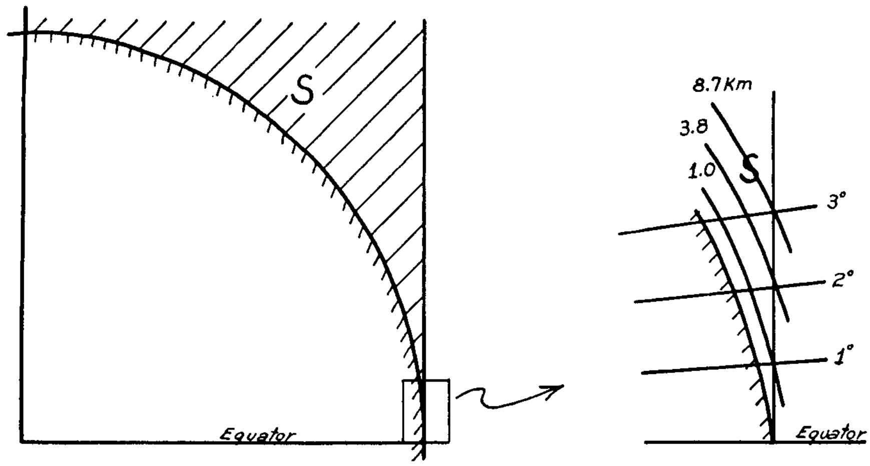

FIG. 1. Meridional section of the region S.

The solution (28) has meaning only for the region S: ($r \cos \phi \leq a$). However (see fig. 1), S encompasses the meteorologically significant portions of the atmosphere, save in the immediate neighborhood of the equator. Further, if we annex to the list of given functions a specification of W on some surface cutting Z-lines (such as the surface $\phi = 0$), we will be able to compute W from the third equation of (13) by an integration along $d\psi = d(r \cos \phi) = 0$. Thus, for $r \cos \phi > a$,

$$W(r, \phi, \psi) = W(r, 0, \psi) + \int_0^\phi \{ V(r, \xi, \psi)\, r^{-1} \cos \xi + V(r \cos \phi/\cos \xi, \phi, \psi) \cos \phi \tan \xi \}\, d\xi. \quad (28')$$

5. Comments upon the solutions

In the preceding section, two solutions to the problem have been written: one primarily in terms of W, and one primarily in terms of V. The first is most useful for model-making; the second may be used, in selected synoptic situations, for the computation of the W field (and, hence, for an inference concerning the weather). Before discussing synoptic applications, we shall find it convenient to review the assumptions implicit in the work and to consider the consequent restriction of the class of problems to which the results may be applied. The principal simplifications are:

1. Non-viscous flow is considered;
2. Disturbances moving zonally, at a constant angular speed and without change in shape, are considered;
3. Applicability of perturbation methods is assumed; (29)
4. The angular speed of propagation is considered to be very nearly the speed of the basic current.

The first of these assumptions may restrict applicability to regions above a friction layer. This is not too serious a restriction, either for model-making or for the study of actual synoptic examples.

The second assumption restricts applicability in several ways. The restriction to zonal motion is not overly serious. Not only do a great number of disturbances move nearly zonally, but some which appear to have a non-zonal component in their velocity of propagation may well be essentially zonally propagated; this will be illustrated in section 6, below. The restriction to a constant speed of propagation and to unchanging shape has obvious implications for both model-making and synoptic analysis.

The third and fourth assumptions of (29) require a bit more detailed discussion than has been given for the first two.

We have derived solutions (27) and (28) in terms of various sets of functions. In the work it was implied that these functions could be arbitrarily chosen. This is true within the framework of problems to which we shall apply our solutions; however, these functions must be chosen to be consistent with the use of perturbation methods. It must be remembered that the motions studied are to have the momentum-components relative to the earth

$$\rho\omega r \cos \phi + U,\ V,\ W, \quad (30)$$

where ω is a constant, and U, V and W are perturbation quantities. Hence, so must the corresponding boundary values, $U(r, \phi, \psi_0)$, $W(r, 0, \psi)$, *et al.*, be perturbation quantities.

We comment further upon the zonal component of the momentum. It is of the form

$$\rho\omega r \cos \phi + U = \rho\omega r \cos \phi + U(r, \phi, \psi_0) + \text{a function of } V, W, etc. \quad (31)$$

In a sense, $\rho\omega r \cos \phi + U(r, \phi, \psi_0)$ rather than $\rho\omega r \cos \phi$ may be regarded as the basic current. The derivations required approximate equality of ω and the angular speed of propagation of the systems. With $\rho\omega r \cos \phi + U(r, \phi, \psi_0)$ regarded as a basic

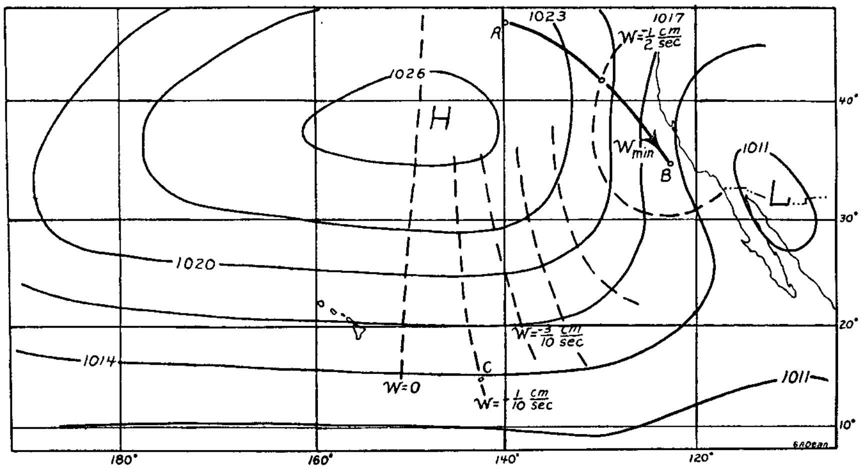

FIG. 2. North Pacific high-pressure cell in July. Vertical-velocity field (dashed lines) is schematic.

current, the fourth assumption of (29) amounts to assuming that the relative zonal momentum of the basic current, $U(r, \phi, \psi_0)$, is a perturbation quantity.

Finally, we make the obvious remark that, in practice, we may well find our solutions applicable to only a limited region. With these restrictions in mind, we shall turn in the next sections to consideration of some of the ways in which the dynamic theory presented above may be applied to synoptic problems. This may be done in two general ways, which will be exemplified (by no means completely) in the succeeding sections. In the first place, deductions may be drawn from the governing differential equations themselves. In the second place, the integrated solutions, derived in terms of the sets of boundary conditions, may be utilized.

Before going to the next sections for illustration of the theory, we pause to consider some orders of magnitude.

First, consider large-scale atmospheric motions; in particular, consider the equatorward portion of the North Pacific anticyclone. Using kinematic computations based upon Werenskiolde's map [18], Neiburger [9] estimated a contraction from 1000 to 70 m for an air column moving from point A of fig. 2 to point B. The movement is at an average speed of about 4 m/sec; thus it takes about 60 hr for the air column to go from A to B. Hence, the mean vertical velocity for the top of the column is about $-\frac{1}{2}$ cm/sec. This gives an order of magnitude for the value of the w field, at the 500-m level, near the midpoint of AB. With this value as a guide and a general picture (all we require is general subsidence in the eastern portion of the high—the vertical-velocity pattern of fig. 2 is not to be taken to imply more than that) of the vertical-velocity pattern, we can assign the order of magnitude $w = -0.1$ cm/sec to a point C in that equatorward part of the anticyclone were v might reasonably be regarded as a perturbation quantity. At point C, $\partial w/\partial\phi < 0$; so we underestimate the value of the magnitude of v if we neglect this term of $L(w)$. Thus, neglecting ρ, at the point C, for the 500-m level, we have a v component of -1.7 m/sec or less. That is, we have a v component greater than 1.7 m/sec (or greater than 3 knots) from the north. Clearly, the term $L(w)$ has led to a significant value of v in this case.

It will be seen, from the model construction in section 7, that the effects considered are significant for perturbations of medium wavelength. Finally, one finds, upon trial, that appreciable departures from undisturbed zonal flow occur even for vertical-velocity patterns of the horizontal scale of one to two degrees of latitude. This is the scale of perturbations due to the passage of air over mountains. Thus, with assurance that the effects discussed are of significant order, we next pass to illustration of their application.

6. Inferences from the differential equations

Perhaps the simplest application of the theory to synoptic problems is the direct use of relations taken from the differential equations.

For example, the third equation of (13),

$$V = L(W) = (r/\cos\phi)\,\partial W/\partial Z,$$

together with the assumption of zero or negligible vertical velocity at the lower boundary of the atmospheric region considered, yields a correlation between the V and W components in the lower layers of the atmosphere. For if the friction layer is either absent or else contributes no large-scale, systematic vertical velocities, at a level not too high [since

$$W \approx (\partial W/\partial Z)\,\Delta Z$$

for this sort of level], from the third equation of 13,

$$V \approx (r/\cos\phi)(1/\Delta Z)W, \qquad (32)$$

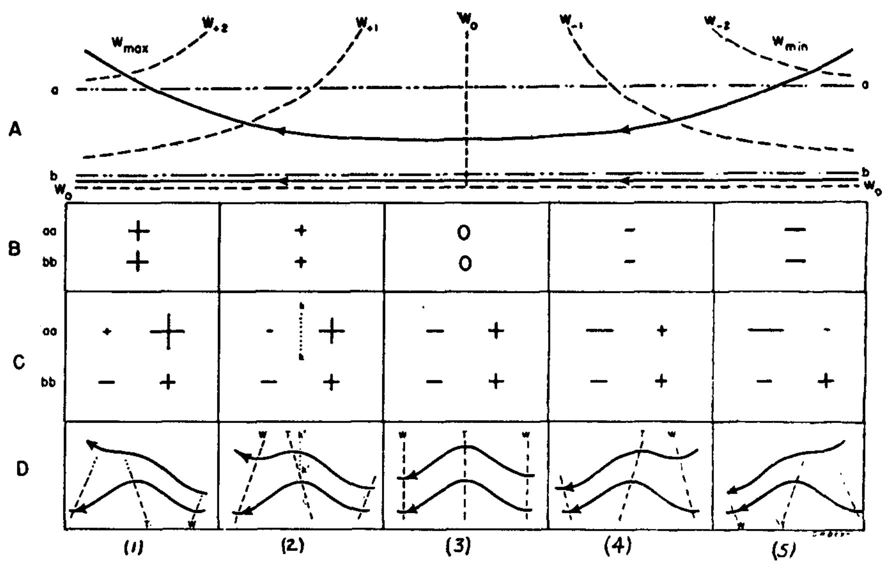

FIG. 3. Effect of vertical and meridional motions, associated with southern portion of northern-hemisphere subtropical anticyclone, on streamline pattern of symmetrical equatorial wave.

where ΔZ is measured parallel to the axis of the earth.[4] The coefficient of W is essentially positive; thus we have a correlation between V and W—a correlation presumably fair, on grounds of continuity, even for motions only nearly of the type here studied.

Thus, one would deduce that poleward motion is accompanied by upward motions in the lower atmosphere. Qualitatively, this places rain areas just ahead of westerly troughs and just behind easterly ones. Further, since the V component is greatest near the inflection points of the streamlines, the center of the rain area is near the inflection points, just east of both westerly and easterly troughs. Of course, one is also led to expect areas of subsidence in the lower atmosphere near the inflection points to the west in

[4] Hence (32) applies only to the region S (see fig. 1).

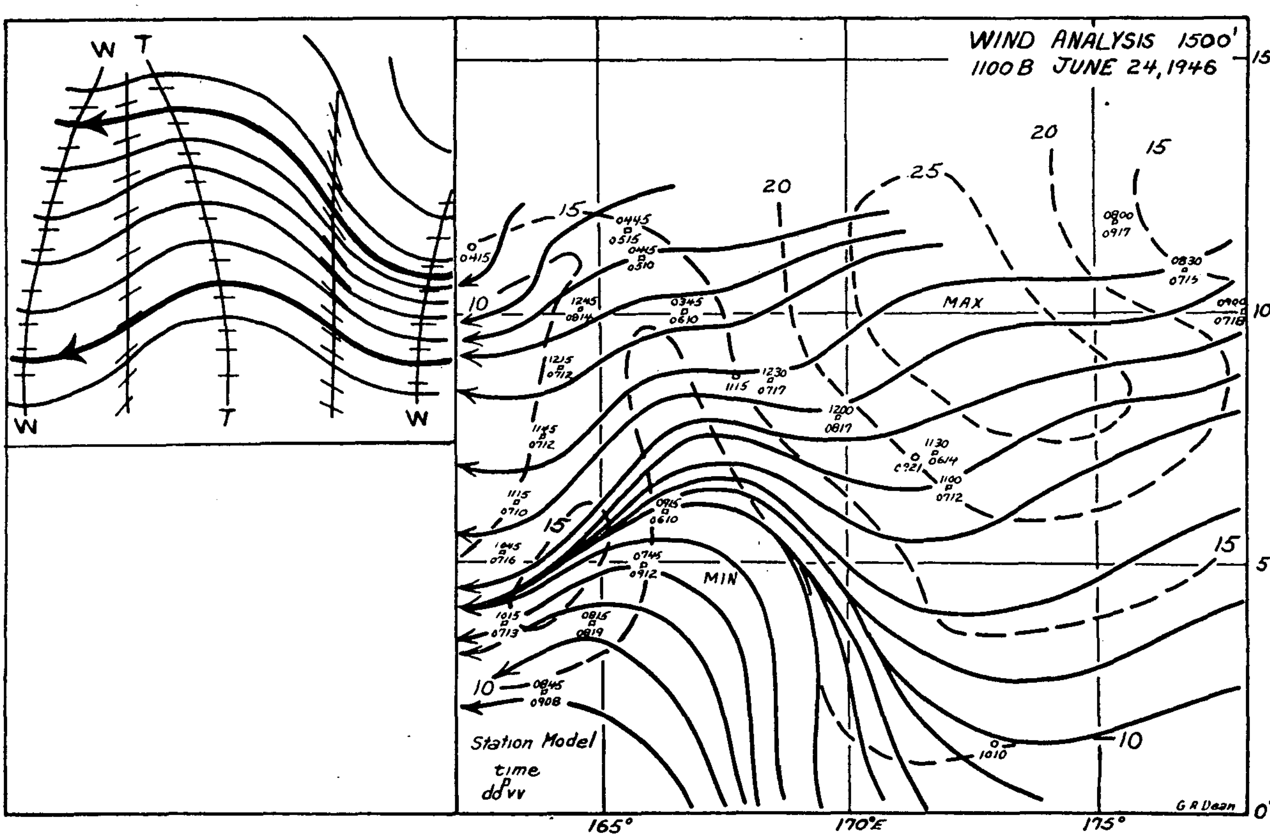

FIG. 4. A-wave on basic current having south component; left (a): schematic presentation; right (b): synoptic case.

both cases. Finally, we note that the relationships concern relative momenta; hence, for fast waves in the westerlies one expects a reversal of the pattern: subsidence ahead, rain or cloud behind.

We next give an illustration of the qualitative treatment of a zonal perturbation which is not relatively steady, and which superficially does not appear to be zonally propagated. Consider a wave-like perturbation, moving from east to west in the equatorial portion of a large oceanic anticyclone, as that shown schematically as section A of fig. 3.[5] It has been seen that the pattern A is a consequence, via (32), of the vertical patterns B in fig. 3. We now take the view that we may superpose the vertical-velocity fields of a perturbation (fig. 3C [3]) and the basic field (section B of fig. 3) for the corresponding latitude, and arrive at the vertical-velocity fields (section C of fig. 3) approached at each longitude as a perturbation, consisting of an area of more subsident air than usual followed by an area of greater upward motion than usual, moves zonally from east to west. By use of (32), the flow patterns shown as section D of fig. 3 are deduced. As might be expected, these patterns, obtained from a model of the vertical-velocity pattern of an easterly wave, are the same as one would expect from the consideration of the horizontal pattern—*i.e.*, from the direct superposition of part D(3) of the figure upon section A.

In fig. 4a, the passage from part C(2) of fig. 3 to part D(2) is illustrated. The trough and wedge lines have been indicated and ruled as 90 deg isogons. The trough line is very nearly north-south near the equator, but tilts toward the west as one proceeds poleward. This follows from the fact that the vertical-velocity field is greater than zero on kk of part C(2) of fig. 3, and increasingly so polewards. This implies that, on $k'k'$ of part D(2) of fig. 3, we have positive components and, so, are east of the trough. The wedge lines are constructed similarly. Ruled lines have also been sketched midway between the wedge and trough lines, with cognizance of the general algebraic increase in W and, hence, in V toward the pole. Finally, the streamlines have been entered. The construction of the other portions of section D of fig. 3 was similarly accomplished.

We return now, briefly, to the discussion of section D of fig. 3. There are illustrated the varying tilts of troughs and wedges to be expected in waves, at various longitudes, in the trade-wind portion of an oceanic anticyclone—that is, if we use as part of our definition of trough and wedge lines that they be loci of points of zonal wind. It has also been indicated that, with this definition of trough and wedge lines, they may well be

[5] It is recognized that the streamlines spiral outward from the center of such an anticyclone, and, hence, that the western portion of the corresponding oceanic high is being considered.

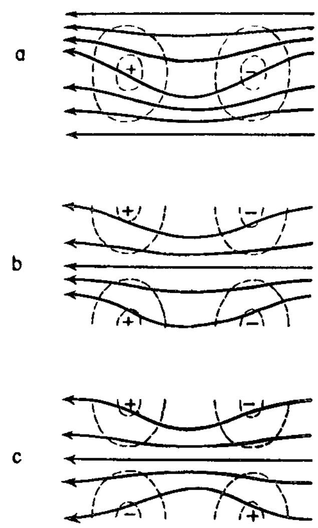

FIG. 5. Tropical-wave classification on basis of v components; + indicates ascending motion; − indicates descending motion. Top (a): A-wave (antisymmetric); middle (b): extraordinary A-wave; bottom (c): S-wave (symmetric).

expected to lose their identity in the major troughs, between the underlying anticyclones. However, in spite of this, and the qualitative differences in appearance and apparent direction of motion of the patterns of section D of fig. 3, it is to be remembered that the model was constructed by superposition of the *same* disturbance at different longitudes of the trades. It is thus plausible that waves which seem to lose themselves as they enter a polar trough may well appear downstream, at a later date, as apparently new disturbances of the tropical easterlies. In actual synoptic experience, extrapolation at a uniform zonal speed of disturbances in the easterlies, regardless of apparent changes of their intensity or shape or of lack of fixes due to sparsity of data, has proven extremely successful [3].

Illustration of the direct use of the differential equations is concluded by showing, in fig. 5, a system for the classification of waves that has been provisionally adopted by the workers of the Tropical Project at the University of California at Los Angeles. If we consider the possible symmetric and skew-symmetric vertical-velocity patterns about a wave axis, we obtain, using (32), the streamline patterns of fig. 5. These have been drawn without regard for the effects of variation of the zonal component of motion.

The first two patterns, symmetric in W, are asymmetric in V (and thus, approximately, in the horizontal streamlines); the opposite is true for the last pattern. Also, relation (32), correlating the V and W fields, is valid only for the lower layers of the atmosphere. Accordingly, the workers on the Tropical Project have adopted the terminology A-wave (or antisymmetric wave) and S-wave (or symmetric wave) to indicate the three-dimensional streamline perturbations having, at the 1500-ft level, the patterns shown in fig. 5(a) and (c), respectively. These wave classifications are only limiting cases. It is clear that intermediate patterns are possible as the parts of the wave on either side of the central latitude are shifted longitudinally with respect to each other. Also, waves found in practice are not as simple as those illustrated in fig. 5. As an example, the wave shown in fig. 4(b) is an A-wave, superimposed upon a basic current having a south component. Finally, we note that the extraordinary A-wave has been so named primarily because it has not been of synoptic importance so far.

In the next section, we shall consider the solutions to the differential equations. Such solutions will give more detail and will be better suited for consideration of the effects superimposed from outside the region in which the assumptions are valid, than are the considerations of this section. However, the sort of reasoning here illustrated has the advantage of greater dependability, in that here the approximations have not been integrated. Simply because certain terms of a system of differential equations are small relative to others, it does not follow that their integrated effect will also be comparatively small.

7. Models

One of the most useful methods of studying meteorological problems is the construction of models. Particularly is this so where, as in this paper, attention is confined to problems in which perturbation methods may be used; for in such cases, the principle of superposition may be used. (Strictly speaking, it is the linearity of the perturbation equations, *coupled with linear boundary-conditions*, that permits this.) Thus, we may study separately the influences introduced through each boundary condition, and then combine them to attain the total solution. In the simple model constructed in this section, a constant basic current will be assumed. The modifications which would be introduced by variations in the basic zonal current are obvious and will not be elaborated upon. However, we do pause to note one important effect of variations in the basic current. Because of the decrease of streamline amplitude which results from an increase in the basic current, one should expect wave formations to be apparently centered not along the locus of greatest cyclonic shear of the basic current, but rather displaced somewhat to the lower-speed side.

For illustration, a model based upon the solution (27) has been chosen. We specify

$$\rho\omega r \cos\phi + U(r, \phi, \psi_0) = -5\times10^{-3} \text{ ton m}^{-2}\text{ sec}^{-1},$$
$$W(r, \phi, \psi) = -f(r)\,[\cos 36\phi + \cos 24(\psi-\psi_0)] \times[1+\{4(\psi-\psi_0)\}^2]^{-1}, \quad (33)$$

where it will be noted that we have reintroduced the term $\rho\omega r\cos\phi$, and so are back in coordinates fixed to the earth. Mention of conditions on ρ has been omitted. It is practicable graphically to determine the pressure field from the fields ρ and $\mathbf{v}$, either from the equation of motion [16] or by treating the pressure as essentially hydrostatic and integrating ρ with respect to height. However, only the construction of the streamline pattern is illustrated here.

Returning to the discussion of (33), we note that it represents a simple wave in the vertical-momentum component, having a latitudinal wavelength of 10 deg and a longitudinal one of 15 deg. This pattern is displayed in fig. 6. The amplitude factor $-f(r)$ has been so chosen as to yield maximum vertical components which correspond to vertical velocities of the order of $2\frac{1}{2}$ cm/sec at the 6-km level. This level will be referred to as $\bar{r}$. It is the true level of non-divergence of the model, but differs slightly from the level of horizontal non-divergence.[6] Such a choice for the

[6] The equation of the level of non-divergence [obtained by setting $\nabla_H\cdot\rho\mathbf{v} = 0$ in the fourth equation of (13)] is $\partial W/\partial r = 0$,

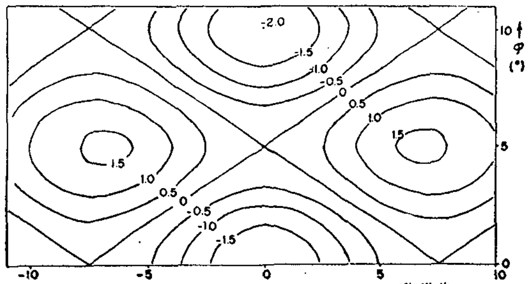

FIG. 6. Vertical momentum pattern of model,
$$\frac{W(r,\phi,\psi)}{-f(r)} = \frac{\cos 36\phi + \cos 24(\psi-\psi_0)}{1+[\pi(\psi-\psi_0)/45]^2}.$$
Amplitude factor $f(r)$ is presented in fig. 7.

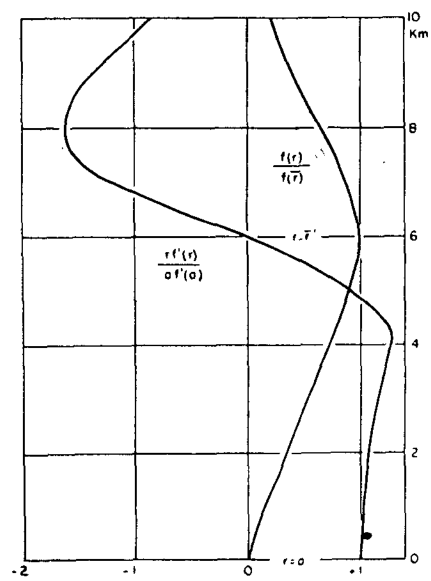

FIG. 7. Coefficients $f(r)/f(\bar{r})$ and $r\,f'(r)/a\,f'(a)$. $f(r)$ curve has been specified such that $f(a) = 0$, $f(\bar{r}) = -1.5\times10^{-5}$, $a\,f'(a) = -5\times10^{-2}$, and level of maximum $f(r)$ is 6 km.

maximum vertical-velocity somewhat stretches the requirement that the fields be perturbations, for ordinarily the maximum values of w found synoptically are of the order 10–15 cm/sec [13], only 4 to 6 times those chosen here. However, this larger amplitude was chosen in order to have sensible deviations from zonal flow. The choice of the 6-km level for the level of non-divergence was prompted by wind time-sections presented by Palmer [11], showing a change in phase of the v component at 18,000 ft. These were time sections of a situation from June for the Bikini area ("Operation Crossroads" data). Thus, the model is appropriate to this season and area. For the same area in March, a level of non-divergence is sometimes found at about 8000 ft [12]; thus, our modeling might well be repeated with other and lower levels of non-divergence. The amplitude factor $-f(r)$ is displayed in fig. 7.

Finally, it is noted that we have chosen the simplest possible basic current and an exceedingly simple field $W(r, \phi, \psi)$. This choice was prompted by the desire for ease of computation—an admissible consideration, since we are merely illustrating the solution (27).

Straightforward application of (27) to (33) yields

$$
\begin{aligned}
U(r, \phi, \psi) &= -\tfrac{1}{2} r f'(r)[17 \sin\phi \sin 36\phi - \cos 37\phi] \tan^{-1} 4(\psi - \psi_0) \\
&\quad - \tfrac{1}{2} f(r)\,[629 \cos\phi \cos 36\phi + 18 \cos 37\phi] \tan^{-1} 4(\psi - \psi_0) \\
&\quad + \left\{ 2[r f'(r) + f(r)] \cos\phi \int_{\psi_0}^{\psi} \cos 24(\xi - \psi_0)\,[1 + \{4(\xi - \psi_0)\}^2]^{-1} \right\} d\xi, \\
V(r, \phi, \psi) &= -r f'(r) \tan\phi[\cos 36\phi + \cos 24(\psi - \psi_0)][1 + \{4(\xi - \psi_0)\}^2]^{-1} \\
&\quad + \{f(r)\, 36 \sin\phi\,[1 + \{4(\psi - \psi_0)\}^2]^{-1}\}, \\
W(r, \phi, \psi) &= -f(r)\,[\cos 36\phi + \cos 24(\psi - \psi_0)][1 + \{4(\psi - \psi_0)\}^2]^{-1}.
\end{aligned}
\tag{34}
$$

Because these are all linear combinations of $r f'(r)$ and $f(r)$, the three-dimensional momentum field, $\boldsymbol{m}(r, \phi, \psi) = \rho\, \boldsymbol{v}(r, \phi, \psi)$, is simply presented as:

$$
\boldsymbol{m}(r, \phi, \psi) \approx \frac{r f'(r)}{a f'(a)} \boldsymbol{m}_H(a, \phi, \psi) + \left[\frac{f(r)}{f(\bar{r})} U(\bar{r}, \phi, \psi) - \left(1 - \frac{r f'(r)}{a f'(a)}\right)(5 \times 10^{-3})\right]\boldsymbol{i} + \frac{f(r)}{f(\bar{r})} W(\bar{r}, \phi, \psi)\boldsymbol{k}. \tag{35}
$$

In this expression, the terms of (34) which are enclosed in braces have been neglected. These terms are small compared to those retained. Thus, to obtain the horizontal component of $\boldsymbol{m}(r, \phi, \psi)$ at any point (r, ϕ, ψ), one enters fig. 7 to obtain the ratios $r f'(r)/a f'(a)$ and $f(r)/f(\bar{r})$, and figs. 8 and 9 for $\boldsymbol{m}_H(a, \phi, \psi)$ and $U(\bar{r}, \phi, \psi)$; these are then entered in (35). It is clear from the zonal nature of the basic current and the virtually zonal nature of the momentum field at $\bar{r}$, that the sign of the v component is determined by the term $[r f'(r)/a f'(a)]\boldsymbol{m}_H$ of (31). Hence, there is a phase shift as expected at the level of non-divergence, where the coefficient $r f'(r)/a f'(a)$ changes sign.

Figure 8 was constructed with the smoothing usual in synoptic analysis. The isogon and isovel fields were first computed from (34) for U and V; the isogons were ruled, and streamlines were then drawn. The $U(a, \phi, \psi)$ and $V(a, \phi, \psi)$ fields are displayed in figs. 10 and 11.

In summary, we note that the model displays certain features over and above a simple sinusoidal wave. As is to be expected, the correlation between the V and W components, which was deduced in the preceding section, is valid. This correlation applies only to the lower levels; reference to figs. 5 and 7 provides a ready check of the relationship. Ultimately, with

while that for the level of horizontal non-divergence (obtained by setting $\nabla_H \cdot \rho \boldsymbol{v}_H = 0$) is $\partial W/\partial r + 2W/r = 0$. For our model, these become $f'(r) = 0$ and $f'(r) + 2f(r)/r = 0$, respectively. The first is satisfied at 6 km, the second some 15 m higher.

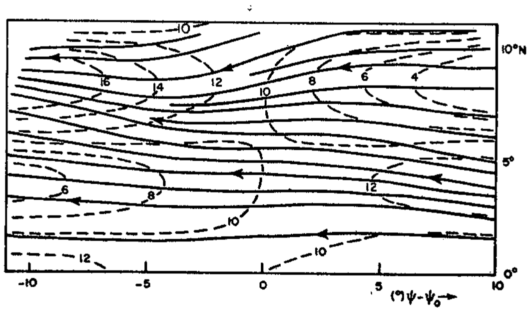

FIG. 8. Horizontal-momentum field of model at surface $r = a$. Isovels labelled in knots $\times\ \bar{\rho}$ ton/m³.

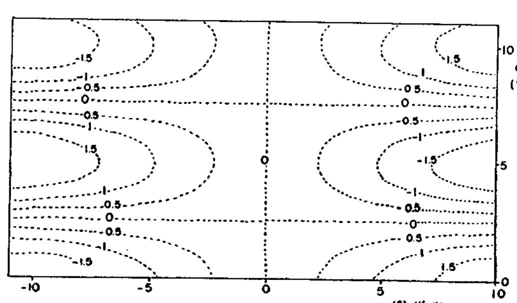

FIG. 9. Zonal component of momentum perturbation at level of non-divergence. $U(\bar{r}, \phi, \psi)$ in knots $\times\ \bar{\rho}$ ton/m³.

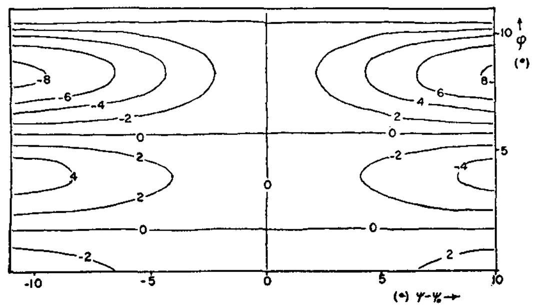

FIG. 10. $U(a, \phi, \psi)$ in knots $\times\ \bar{p}$ ton/m³.

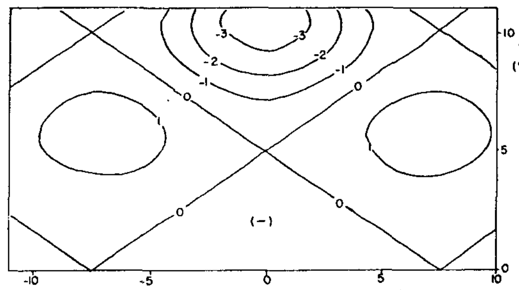

FIG. 11. $V(a, \phi, \psi)$ in knots $\times\ \bar{p}$ ton/m³.

height, this correlation fails, since the streamlines of the model tend to become zonal and finally to reverse in phase above the level of non-divergence. This emphasizes a feature of the model which resembles reality—$f(r)$ has been specified in such a fashion as to yield a level of non-divergence. Any model not having such a level will necessarily have either horizontal-divergence fields of unrealistically low magnitudes or vertical velocities which ultimately, with height, become unrealistically large. Finally, we note that the wedge and trough lines of the model (if they are defined as *loci* of east winds) have the tilts to be expected for a wave superimposed upon a basic current of converging east-northeasterly and east-southeasterly winds. The wedge and trough lines are the edges of the diamond-shaped areas of fig. 11 (lines of $V = 0$). In particular, the northern edges of those diamonds centered near 5°N constitute a saw-toothed curve of trough and wedge lines through our wave at about 7.5°N (see fig. 7). Such an arrangement of trough and wedge lines is characteristic of an equatorial wave [11]. Thus the model, based upon boundary values chosen primarily for simplicity of computation (but with some regard to reality), has several characteristics of an actual synoptic wave. Indeed, fig. 8 would not startle a practiced synoptic analyst, were it presented to him as an actual situation.

Acknowledgments.—The author wishes to thank Professors Forsythe, Holmboe and Palmer for their careful criticism of this paper. In addition to considerations of content, they have made many suggestions which have improved the presentation. At the same time, the author also wishes to thank Mr. Dean, who not only analyzed the synoptic streamline-fields referred to in this paper but also prepared most of the figures.

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