Global and Regional Entropy Production by Radiation Estimated from Satellite Observations

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ABSTRACT
Vertical profiles of shortwave and longwave irradiances computed with satellite-derived cloud properties and temperature and humidity profiles from reanalysis are used to estimate entropy production. Entropy production by shortwave radiation is computed by the absorbed irradiance within layers in the atmosphere and by the surface divided by their temperatures. Similarly, entropy production by longwave radiation is computed by emitted irradiance to space from layers in the atmosphere and surface divided by their temperatures. Global annual mean entropy production by shortwave absorption and longwave emission to space are, respectively, 0.852 and 0.928 W m\(^{-2}\) K\(^{-1}\). With a steady-state assumption, entropy production by irreversible processes within the Earth system is estimated to be 0.076 W m\(^{-2}\) K\(^{-1}\) and by nonradiative irreversible processes to be 0.049 W m\(^{-2}\) K\(^{-1}\). Both global annual mean entropy productions by shortwave absorption and longwave emission to space increase with increasing shortwave absorption (i.e., with decreasing the planetary albedo). The increase of entropy production by shortwave absorption is, however, larger than the increase of entropy production by longwave emission to space. The result implies that global annual mean entropy production by irreversible processes decreases with increasing shortwave absorption. Input and output temperatures derived by dividing the absorbed shortwave irradiance and emitted longwave irradiance to space by respective entropy production are, respectively, 282 and 259 K, which give the Carnot efficiency of the Earth system of 8.5%.

1. Introduction
The hydrological cycle and dynamics in the Earth system are driven by energy received from the sun. Once averaged over a year and over the entire globe, about 71% of the solar irradiance is absorbed by Earth (e.g., Stephens et al. 2012). While dynamics in the atmosphere and ocean distributes energy absorbed by Earth, energy is converted into different forms. A part of shortwave irradiance absorbed by ocean and land is used to evaporate water vapor. Water vapor enters the atmosphere and heats the atmosphere when it condenses. Energy transport is also associated with heating and cooling by radiation, dynamics, and water vapor phase change, which in turn alter entropy of the Earth system.

Entropy is produced by heating and cooling by irreversible processes. In addition, entropy is carried by radiation. Entropy produced by a blackbody is, therefore, the sum of entropy produced by radiative cooling and entropy carried by the blackbody radiation (Planck 1913). For the Earth system, entropy is imported by radiation emitted by the sun. Heating due to absorption of solar radiation and scattering (Wu and Liu 2010) within the Earth system produce entropy. In addition to diabatic heating by irreversible processes occurring within the system, radiative cooling and heating by longwave radiation contribute to entropy production. Longwave radiation emitted to space then exports entropy. Earlier studies refer to entropy produced by irreversible processes within the Earth system as material entropy (e.g., Bannon 2015; Bannon and Lee 2017).

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Global entropy budget was first estimated by Peixoto et al. (1991). Their estimate was primarily based on in situ surface and upper-atmosphere data taken between 1963 and 1973 (Peixoto and Oort 1992). Stephens and O’Brien (1993) used satellites’ broadband radiance measurements to estimate entropy production by longwave radiation emitted to space. Observational estimates of entropy production within the Earth system by absorption of shortwave radiation, however, have not been updated since the estimate by Peixoto et al. (1991) was made. Goody (2000) extended work by Peixoto et al. (1991) and estimated entropy produced by irreversible processes within the Earth system. Lucarini et al. (2011) used climate models and estimated entropy production by nonradiative irreversible processes. The analysis of thermodynamics of the climate system is also used to understand how general circulation changes under a warming climate (Kjellsson 2015; Laliberté et al. 2015).

In this study, we revise entropy budget estimated by earlier studies using satellite observations that have been taken since March 2000. We use cloud and aerosol properties derived from satellites to estimate entropy production by the absorption of shortwave irradiance and by the emission of longwave irradiance to space. Radiation also carries entropy, but we primarily focus on entropy produced by heating and cooling within the atmosphere and at Earth’s surface in this study.

Before we present entropy production estimates in section 4, we use a 1D model to illustrate the relationship between entropy produced by radiation and entropy produced by irreversible processes within the system in section 2. Section 3 explains data products used in this study and how entropy production is computed. Section 5 utilizes the results with the concept illustrated by the 1D model to discuss how entropy production changes when the energy input to Earth changes.

2. Conceptional 1D model

We use a 1D isothermal atmosphere model discussed by Bannon (2015) to describe the relationship between energy budget and entropy budget of the Earth climate system (Fig. 1). The 1D model also helps to understand how entropy produced by shortwave absorption and longwave emission is related to irreversible processes within the Earth system. In addition, we extend the model to help relating entropy produced by nonradiative irreversible processes and entropy produced by heating and cooling by radiation within the Earth system.

We let the incoming solar irradiance at top of the atmosphere (TOA) be \( F_0 \). The shortwave irradiance absorbed by the atmosphere is \( \beta F_0 \), where \( \beta \) is the absorptivity of the atmosphere. We assume that the atmosphere does not reflect shortwave irradiance, but the surface reflects shortwave irradiance \( \alpha F_0 \), where \( \alpha \) is the reflectivity of the surface. The rest of the shortwave irradiance \( (1 - \alpha - \beta)F_0 \) is absorbed by the surface. Excluding scattering by the atmosphere does not affect establishing the relationship between entropy produced by radiation and nonradiative processes. The surface is assumed to be black to longwave radiation and emits the longwave irradiance proportional to the fourth power of the temperature \( F_{sfc} = \sigma T_{sfc}^4 \), where \( T_{sfc} \) is the surface temperature. The emissivity of the atmosphere is \( \epsilon \) so that the atmosphere emits the irradiance \( F_{atm} = \epsilon \sigma T_{atm}^4 \), both downward and upward, where \( T_{atm} \) is the temperature of the isothermal atmosphere. The atmosphere absorbs the longwave irradiance \( \epsilon F_{sfc} \) emitted by the surface and the rest \( (1 - \epsilon)F_{sfc} \) is transmitted to space. Energy is also transported by turbulence from the surface to the atmosphere at the rate of \( F_{tur} \), which represents a nonradiative irreversible process in the model.

The energy balance at TOA, the surface, and for the atmosphere is, respectively,

\[
F_{TOA}^{\text{net}} = (1 - \alpha)F_0 - F_{atm} - (1 - \epsilon)F_{sfc},
\]

\[
F_{sfc}^{\text{net}} = (1 - \alpha - \beta)F_0 + F_{atm} - F_{sfc} - F_{tur}, \quad \text{and}
\]

\[
F_{atm}^{\text{net}} = \beta F_0 + \epsilon F_{sfc} + F_{tur} - 2F_{atm}.
\]

The relationship among these net energy fluxes is

\[
F_{TOA}^{\text{net}} = F_{sfc}^{\text{net}} + F_{atm}^{\text{net}}.
\]

At the top of the atmosphere, the system receives entropy of radiation \( J_{sun} = (4/3)F_0/T_{sun} \) by a blackbody radiation from the sun. A 4/3 factor appears in the expression for the entropy carried by radiation (denoted by \( J \)). Feistel (2011) shows that when entropy exchange between two black bodies is considered, \( (4/3)\alpha T^3 \) is the
only mathematical expression for entropy carried by blackbody radiation to be consistent with the second law of thermodynamics. In addition, Yourgrau et al. (1982, sections 3–9) show, using a photon gas, that the 4/3 factor is necessary for the entropy carried by radiation. A simple example treating both processes, entropy carried by radiation and entropy produced by heating and cooling, is discussed in section 102 of Planck (1913).

Reflecting the shortwave irradiance and converting direct to diffuse radiation produce entropy \( J_{\text{ref}} \) (Wu and Liu 2010). Emission of the longwave irradiance to space carries entropy, \( J_{\text{atm}} = (4/3)(F_{\text{atm}}/T_{\text{atm}}) \) and \( (1 - \varepsilon)J_{\text{sf}} = (1 - \varepsilon)(4/3)(F_{\text{sf}}/T_{\text{sf}}) \). Therefore, entropy export to space by radiation is

\[
J_{\text{TOA}} = J_{\text{ref}} - J_{\text{atm}} - (1 - \varepsilon)J_{\text{sf}} + (1 - \alpha)J_{\text{sun}}. \tag{5}
\]

Entropy produced by radiation exchange and heating at the surface is

\[
\dot{S}_{\text{sf}} = \frac{F_{\text{net}}^{\text{sfc}}}{T_{\text{sfc}}} = \frac{(1 - \alpha - \beta)F_0 + F_{\text{atm}} - F_{\text{sf}} - F_{\text{tur}}}{T_{\text{sfc}}}, \tag{6}
\]

and entropy produced within the atmosphere is

\[
\dot{S}_{\text{atm}} = \frac{F_{\text{net}}^{\text{atm}}}{T_{\text{atm}}} = \frac{\beta F_0 + e F_{\text{sf}} + F_{\text{tur}} - 2F_{\text{atm}}}{T_{\text{atm}}} \tag{7}
\]

The sum of Eqs. (5)–(7) is equal to the total entropy produced by the Earth system \( \dot{S}_{\text{tot}} \),

\[
\dot{S}_{\text{tot}} = J_{\text{TOA}} + \dot{S}_{\text{sf}} + \dot{S}_{\text{atm}}. \tag{8}
\]

We can also express entropy production separated by individual heating and cooling processes, absorbed shortwave irradiance \( \dot{S}_{\text{abs}} \), emitted longwave irradiance \( \dot{S}_{\text{emt}} \), and turbulent flux \( \dot{S}_{\text{tur}} \),

\[
\dot{S}_{\text{abs}} = \frac{(1 - \alpha - \beta)F_0}{T_{\text{sfc}}} + \frac{\beta F_0}{T_{\text{atm}}}, \tag{9}
\]

\[
\dot{S}_{\text{emt}} = \frac{e F_{\text{sf}} - 2F_{\text{atm}}}{T_{\text{atm}}} + \frac{F_{\text{atm}} - F_{\text{sf}}}{T_{\text{sfc}}}, \tag{10}
\]

\[
\dot{S}_{\text{tur}} = \frac{F_{\text{tur}}}{T_{\text{atm}}} - \frac{F_{\text{tur}}}{T_{\text{sfc}}}. \tag{11}
\]

Similar to the sum of Eqs. (5)–(7), the sum of Eqs. (9)–(11) and entropy exported through TOA [i.e., Eq. (5)] is the entropy produced by the Earth system,

\[
J_{\text{TOA}}^{\text{net}} + \dot{S} = J_{\text{TOA}}^{\text{net}} + \dot{S}_{\text{abs}} + \dot{S}_{\text{emt}} + \dot{S}_{\text{tur}} = J_{\text{TOA}}^{\text{net}} + \frac{F_{\text{net}}^{\text{sfc}}}{T_{\text{sfc}}} + \frac{F_{\text{net}}^{\text{atm}}}{T_{\text{atm}}} \tag{12}
\]

where \( F_{\text{net}}^{\text{sfc}} \) and \( F_{\text{net}}^{\text{atm}} \) are given by Eqs. (2) and (3) and \( J_{\text{TOA}}^{\text{net}} \) is given by Eq. (5). Equation (12) states that the entropy produced by the system is the sum of the entropy carried by radiation through the top boundary (TOA) and heating or cooling by net energy flux at the surface and to the atmosphere. The last two terms on the right side of Eq. (12) can be written as

\[
\frac{F_{\text{net}}^{\text{sfc}}}{T_{\text{sfc}}} + \frac{F_{\text{net}}^{\text{atm}}}{T_{\text{atm}}} = \left( \frac{(1 - \alpha - \beta)F_0 + \beta F_0}{T_{\text{sfc}}} \right) - \left( (1 - \varepsilon)F_{\text{sf}} + F_{\text{atm}} \right) \frac{1}{T_{\text{atm}}} + \frac{F_{\text{atm}} - e F_{\text{sf}} - F_{\text{atm}} - F_{\text{tur}} + F_{\text{tur}}}{T_{\text{sfc}}} + \frac{F_{\text{atm}} - e F_{\text{sf}} - F_{\text{atm}} - F_{\text{tur}} + F_{\text{tur}}}{T_{\text{atm}}}. \tag{13}
\]

Terms in the first and second square brackets on the right side of Eq. (13) are, respectively, entropy produced by absorption of shortwave irradiance and by emission of longwave irradiance to space. Terms in the third bracket are entropy produced by radiation exchange between the surface and atmosphere and by the turbulent flux.

Using Eqs. (12) and (13), we can express the rate of entropy change of the system due to heating and cooling by radiation through TOA and by irreversible process within the system as

\[
\frac{dS}{dt} = \dot{S} - \frac{Q_a}{T_a} - \frac{Q_e}{T_e} + \dot{S}_{\text{tur}}, \quad \text{where} \tag{14}
\]

\[
Q_a = (1 - \alpha)F_0, \tag{15}
\]

\[
Q_e = (1 - \varepsilon)F_{\text{sf}} + F_{\text{atm}}, \tag{16}
\]

\[
T_a = \frac{Q_a}{(1 - \alpha - \beta)F_0 + \beta F_0}, \tag{17}
\]

\[
T_e = \frac{Q_e}{(1 - \varepsilon)F_{\text{sf}} + F_{\text{atm}}}, \quad \text{and} \tag{18}
\]

\[
\dot{S}_{\text{tur}} = \frac{F_{\text{atm}} - e F_{\text{sf}} - F_{\text{atm}} - F_{\text{tur}} + F_{\text{tur}}}{T_{\text{sfc}}} + \frac{F_{\text{atm}} - e F_{\text{sf}} - F_{\text{atm}} - F_{\text{tur}} + F_{\text{tur}}}{T_{\text{atm}}}. \tag{19}
\]

At a steady state when \( F_{\text{net}}^{\text{sfc}} = F_{\text{net}}^{\text{atm}} = 0 \), \( dS/dt = 0 \). Because of heating of ocean and land and melting ice and glacier, the global annual mean TOA net irradiance is estimated to be 0.71 ± 0.1 W m\(^{-2}\) (Johnson et al. 2016; Loeb et al. 2018a). When entropy production due to a positive net TOA irradiance can be handled by adding entropy storage term in Eq. (14) and subsumed into entropy production within the system (Bannon and Lee 2017), then,
at a steady state, the difference of the entropy produced by longwave emission to space and shortwave absorption balances with entropy production within the system is

$$\frac{Q_e}{T_c} - \frac{Q_a}{T_a} = \dot{\Sigma}_{irr}. \quad (20)$$

Although $Q_e/T_c - Q_a/T_a$ equals $\dot{\Sigma}_{irr}$, vertical radiation exchange and energy flux by turbulence affect entropy production in the column, as indicated by Eq. (13), because temperature varies with height. This is different from energy balance described by Eq. (4) and makes somewhat difficult to interpret regional $Q_e/T_c - Q_a/T_a$.

We further separate entropy produced by vertical exchange of longwave radiation and by nonradiative processes in $\dot{\Sigma}_{irr}$. Rewriting longwave terms in the third square bracket in the right side of Eq. (13) separating the entropy produced by net longwave surface and atmosphere irradiances and the entropy produced by longwave irradiance emitted to space

$$\frac{F_{\text{atm}} - \varepsilon F_{\text{sfc}} + \varepsilon F_{\text{sfc}} - F_{\text{atm}}}{T_{\text{atm}}} = \left[ \frac{F_{\text{atm}} - F_{\text{sfc}} + \varepsilon F_{\text{sfc}} - 2F_{\text{atm}}}{T_{\text{atm}}} \right] - \left[ \frac{- (1 - \varepsilon) F_{\text{sfc}} + F_{\text{atm}}}{T_{\text{atm}}} \right]. \quad (21)$$

The first term of the right side of Eq. (21) is the entropy produced by net longwave irradiance and the second term of the right side of Eq. (21) is the entropy produced by longwave irradiance emitted to space. When Eq. (21) is used to rewrite Eq. (14),

$$\frac{dS}{dt} = \left[ \frac{(1 - \alpha - \beta) F_{0}}{T_{\text{sfc}}} + \beta F_{0} \right] \frac{T_{\text{atm}}}{T_{\text{sfc}}} \left[ \frac{F_{\text{atm}} - F_{\text{sfc}} + \varepsilon F_{\text{sfc}} - 2F_{\text{atm}}}{T_{\text{atm}}} \right] - \frac{F_{\text{tur}}}{T_{\text{sfc}}} + \frac{F_{\text{tur}}}{T_{\text{atm}}}. \quad (22)$$

At a steady state when $dS/dt = 0$, entropy production by nonradiative irreversible process is

$$\dot{\Sigma}_{\text{tur}} = - \left[ \frac{F_{\text{net,sfc,SW}}}{T_{\text{sfc}}} + \frac{F_{\text{net,atm,SW}}}{T_{\text{atm}}} + \frac{F_{\text{net,sfc,LW}}}{T_{\text{sfc}}} + \frac{F_{\text{net,atm,LW}}}{T_{\text{atm}}} \right]. \quad (23)$$

where $F_{\text{net,sfc,SW}}$ and $F_{\text{net,atm,SW}}$ are, respectively, net shortwave irradiance at the surface and within the atmosphere and $F_{\text{net,sfc,LW}}$ and $F_{\text{net,atm,LW}}$ are, respectively, net longwave irradiance at the surface and within the atmosphere. Therefore, at the steady state, the sum of entropy production by shortwave absorption and longwave emission within the system and entropy production by nonradiative processes is zero. This expression is used by Lucarini et al. (2011) [their Eq. (10)], Lucarini et al. (2014), and Lembo et al. (2019) in estimating entropy production by irreversible processes in climate models. In this study, as discussed in section 3, we consider vertical temperature and radiative heating rate profiles and their correlation in an hourly temporal resolution and a $1^\circ \times 1^\circ$ spatial resolution in computing entropy production.

3. Data products

We use Clouds and the Earth’s Radiant Energy System (CERES) Edition 4.1 SYN1deg-Month data product (Rutan et al. 2015a; Kato et al. 2018). The SYN1deg-Month product includes shortwave and longwave irradiance profiles computed hourly for every equal-area grid. The angle size of the equal-area grids is $1^\circ \times 1^\circ$ between $45^\circ$N and $45^\circ$S and increases toward poles (Doelling et al. 2013). Cloud and aerosol properties are derived from Moderate Resolution Imaging Spectroradiometer (MODIS, collection 5 March 2000 through February 2016 and collection 6.1 from March 2016 onward for clouds and collection 6.1 throughout for aerosols) on Terra and Aqua except for the period from March 2000 through June 2002 when only MODIS on Terra is used. In addition, cloud properties are also derived from geostationary satellites hourly over regions between $60^\circ$N and $60^\circ$S to account for the diurnal cycle of clouds.

Temperature and humidity profiles used in producing SYN1deg-Month are taken from the Goddard Earth Observing System Data Assimilation System reanalysis product (Rienecker et al. 2008), version 5.4.1 (GEOS-5.4.1). Temperature and relative humidity profiles are 6 hourly in the equal-area grid. The temporal resolution of temperature and humidity profiles is increased to hourly below 10 m from the surface. For surface properties, we primarily use MODIS-derived spectral surface albedos except over ocean. Modeled spectral surface albedos are used for ocean (Jin et al. 2004) and other surface types when the MODIS spectral albedo is not available. Surface spectral emissivities are based on Wilber et al. (1999). Further details of inputs and algorithm to produce SYN1deg-Month is given in Rutan et al. (2015a) and Kato et al. (2018).

Computation of entropy production

We compute shortwave and longwave upward and downward irradiances hourly at 35 levels in the atmosphere (plus cloud-top and cloud-base heights, depending on retrieved heights) for every equal-area grid with a radiative transfer model (Fu and Liou 1993; Rose et al. 2006, 2013). For shortwave irradiance computations, a
gamma distribution is used to express the distribution of hourly cloud optical thickness in an equal-area grid separated by 4 different cloud types. We use two parameters to define a gamma distribution, the mean optical thickness and shape factor that is the mean optical thickness divided by the standard deviation of optical thicknesses. We infer the shape factor from linear and logarithmic means of cloud optical thickness (Wilks 1995). When the shape factor is less than 10, we use the algorithm that analytically integrates two-stream solutions of an integro-differential equation of radiative transfer over a gamma distribution (Barker 1996; Kato et al. 2005). When the shape factor is greater than or equal to 10, we use a four-stream model (Liou et al. 1988). The four-stream model is also used for clear-sky irradiance computations. For longwave computations, we use a combination of two- and four-stream models (Fu et al. 1997).

The net irradiance at the $i$th level is defined as the downward irradiance minus upward irradiance, where the top of the atmosphere is defined as level $= 0$ and the level number increases downward. The top layer between level $= 0$ and level $= 1$ is layer 1 and the layer number also increases downward. The absorbed irradiance by the $i$th layer $F_{i,\text{abs}}$ is the net irradiance at $i - 1$th level minus the net irradiance at $i$th level,

$$F_{i,x}^{\text{abs}} = F_{i-1,x}^{\text{net}} - F_{i,x}^{\text{net}} = (F_{i-1,x}^1 - F_{i-1,x}^\uparrow) - (F_{i,x}^1 - F_{i,x}^\uparrow),$$

where upward and downward arrow indicate the direction of the irradiance and subscript $x$ is either shortwave SW or longwave LW. The top-of-atmosphere shortwave $+ \,$ longwave net irradiance is therefore

$$F_0^{\text{net}} = F_{\text{TOA,SW}}^{\text{net}} + F_{\text{TOAL, LW}}^{\text{net}} = F_{\text{TOA}}^{\text{net}}.$$

The entropy production by absorption of shortwave irradiance is then

$$\dot{\Sigma}_{\text{SW}} = \sum_{i=1}^N \frac{F_{i,\text{abs}}^{\text{SW}}}{T_i} + \frac{F_{\text{sfc,SW}}^{\text{net}}}{T_{\text{sfc}}},$$

where $T_i$ is the temperature of the $i$th layer, $F_{\text{sfc,SW}}^{\text{net}}$ is the net shortwave irradiance at the surface, and $T_{\text{sfc}}$ is the surface skin temperature. To compute the entropy production by longwave irradiance emitted to space, Stephens and O’Brien (1993) divide the upward longwave irradiance at TOA by the effective emission temperature. In this study, we utilize the two-stream source function technique discussed in Toon et al. (1989) and the appendix. Toon et al. (1989) replace the radiance in computing the source function by the radiance computed with a two-stream approximation. With this approximation, the source function can be written explicitly. When the source function is specified, the azimuthally averaged radiances at a two-stream angle can also be written explicitly. We compute the downward irradiance starting from TOA moving downward toward the surface with this expression. We subsequently compute the upward irradiance starting at the surface moving upward toward TOA with the specified surface emissivity. We compute the entropy flux in the same way that the irradiance is computed except that the source function is divided by the temperature of the layer such that

$$\dot{\Sigma}_{\text{LW}} = \sum_{i=1}^N \frac{t_i F_{i,\text{LW}}}{T_i} + \frac{t_{\text{sfc}} F_{\text{sfc, LW}}}{T_{\text{sfc}}}. \tag{26}$$

where $t_i$ is the transmission of the irradiance from $i$th level to TOA, $F_{i,\text{LW}}$ is the upward longwave irradiance at the $i$th level and $T_i$ is the temperature of the $i$th layer, and $N$ is the number of layers ($\approx 35$). Once net irradiances are computed, entropy produced by nonradiative irreversible processes is computed by

$$\dot{\Sigma}_{\text{eur}} = -\sum_{i=1}^N \left( \frac{F_{i,\text{SW}}^{\text{abs}}}{T_i} + \frac{F_{i,\text{LW}}^{\text{net}}}{T_i} \right) - \frac{F_{\text{sfc,SW}}^{\text{net}}}{T_{\text{sfc}}} - t_{\text{sfc}} \frac{F_{\text{sfc, LW}}^{\text{net}}}{T_{\text{sfc}}}. \tag{27}$$

4. Results

In this section, we discuss the result of global annual mean entropy production followed by spatial distribution of the production.

4.1 Global annual mean entropy production

Table 1 summarizes computed global mean shortwave and longwave irradiances averaged from July 2005 through June 2015. Because computed TOA irradiances from SYN1deg-Month are not constrained by observed ocean heating rates averaged over a decade by the method discussed in Loeb et al. (2018a), the net TOA irradiance is 1.3 W m$^{-2}$, which is larger than the net TOA irradiance derived from CERES observations of 0.71 W m$^{-2}$ (Johnson et al. 2016; Loeb et al. 2018a). Table 2 summarizes the annual global mean entropy production by shortwave and longwave irradiances computed from a 10-yr period (July 2005 through June

<table>
<thead>
<tr>
<th>Irradiance (W m$^{-2}$)</th>
<th>Top of atmosphere</th>
<th>NET Shortwave</th>
<th>NET Longwave</th>
<th>Surface</th>
<th>NET Shortwave</th>
<th>NET Longwave</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>241 (0.21)</td>
<td>−240 (0.18)</td>
<td></td>
<td>164 (0.28)</td>
<td>−53.0 (0.33)</td>
<td></td>
</tr>
</tbody>
</table>
The entropy produced by nonradiative irreversible processes is 0.049 W m$^{-2}$ K$^{-1}$. This gives the entropy production by nonradiative irreversible processes that increase entropy by 0.043 W m$^{-2}$ K$^{-1}$.

For comparisons, entropy production estimated by Peixoto et al. (1991) is also included in Table 2. Our values are different from those estimated by Peixoto et al. (1991). The difference of entropy production by the longwave irradiance emitted to space is largely caused by partitioning the entropy production by the surface and atmosphere. However, entropy production by shortwave absorption are significantly different, especially at the surface.

Nonradiative irreversible processes that increase entropy considered by Goody (2000) include turbulent enthalpy transport of 0.0024 W m$^{-2}$ K$^{-1}$, frictional dissipation of 0.0113 W m$^{-2}$ K$^{-1}$, and irreversible phase change, transport, and precipitation of water of 0.0188 W m$^{-2}$ K$^{-1}$. Pauluis and Dias (2012) suggest that the atmospheric heating by the frictional dissipation of falling raindrops is nearly equivalent to the frictional dissipation of turbulence in the atmosphere. We therefore replace Goody’s estimate of entropy production by raindrops of 0.0013 m W m$^{-2}$ K$^{-2}$ by 0.010 W m$^{-2}$ K$^{-1}$, which gives the frictional dissipation of 0.020 W m$^{-2}$ K$^{-1}$. Entropy produced by ocean estimated by Bannon and Najjar (2018) is 0.0016 W m$^{-2}$ K$^{-2}$. When these productions are added, entropy production by nonradiative irreversible processes is 0.043 W m$^{-2}$ K$^{-1}$. Our estimate of entropy produced by nonradiative irreversible process of 0.049 W m$^{-2}$ K$^{-1}$ agrees reasonably well to within 15% with the sum of the components. Also, our estimate agrees with the estimate of 0.055 W m$^{-2}$ K$^{-1}$ by Lucarini et al. (2011) using climate models, although their estimate is based on a simple expression [i.e. Eq. (23)] that ignores the vertical irradiance and temperature profiles in the atmosphere. The result of Lembo et al. (2019),
who estimated entropy produced by nonradiative irreversible processes using CMIP5 models with the simple expression, shows a substantial spread among models and dependence on the method used for the estimate (i.e., direct and indirect methods). Nevertheless, our estimate is close to the mean value from seven CMIP5 models estimated by the direct method of 0.045 W m$^{-2}$ (Kato and Rose 2011; Novak and Teilleux 2018), the efficiency that are proposed (e.g., Pauluis and Held 2002; Lucarini et al. 2011) also shows a smaller effective emission temperature when Eq. (29) is used. Therefore, the reason that our estimate shown in Table 2 is slightly larger than the estimate by Stephens and O’Brien (1993), who used Eq. (29), is probably due to calibration differences between ERBE and CERES instruments.

As discussed in Stephens and O’Brien (1993), the sum of entropy production by the surface and atmosphere by longwave irradiances emitted to space is approximately equal to the longwave irradiance emitted to space divided by the effective temperature,

$$\frac{F_{\text{TOA, LW}}^\uparrow}{T_e} = \sigma^{1/4} (F_{\text{TOA, LW}}^\uparrow)^{3/4},$$  \hspace{1cm} (28)

where the effective temperature is defined as

$$T_e = \left( \frac{F_{\text{TOA, LW}}^\uparrow}{\sigma} \right)^{1/4},$$ \hspace{1cm} (29)

and $\sigma$ is the Stefan–Boltzmann constant. The annual global mean TOA upward longwave irradiance is 240 W m$^{-2}$, which gives the entropy production of 0.941 W m$^{-2}$ K$^{-1}$ by Eq. (28), while the global annual mean entropy production from SYN1deg-Month is 0.928 W m$^{-2}$ K$^{-1}$. The difference is presumably caused by the translucent atmosphere combined with decreasing temperature with height in the atmosphere. Longwave radiation emitted by the lower atmosphere transmits through the atmosphere in the wavelength regions where the atmosphere is not entirely opaque. Because of this difference, the effective emission temperature computed from dividing longwave irradiance emitted to space by longwave entropy production using SYN1deg-Month values is 259 K and larger than 255 K computed by Eq. (29) (Table 3). The result of Stephens and O’Brien (1993) also shows a smaller effective emission temperature when Eq. (29) is used. Therefore, the reason that our estimate shown in Table 2 is slightly larger than the estimate by Stephens and O’Brien (1993), who used Eq. (29), is probably due to calibration differences between ERBE and CERES instruments.

The effective absorption temperature of the shortwave irradiance computed with dividing the TOA net shortwave irradiance of 241 W m$^{-2}$ by entropy production by shortwave of 0.853 W m$^{-2}$ K$^{-1}$ is 282 K. While different definitions of the thermodynamic efficiency are proposed (e.g., Pauluis and Held 2002; Lucarini et al. 2011; Novak and Teilleux 2018), the efficiency that can be computed with these effective temperatures is the Carnot efficiency $\eta$ of the Earth system defined by

### Table 3. Global annual mean values that appear in Eq. (2) derived from SYN1deg-Month and the ideal energy balanced model.

<table>
<thead>
<tr>
<th></th>
<th>Ed4 SYN EBAF-TOA</th>
<th>Ideal energy balanced model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Downward shortwave irradiance $F_0$ (W m$^{-2}$)</td>
<td>340</td>
<td>340</td>
</tr>
<tr>
<td>Net shortwave irradiance $F_{\text{sw}}$ (W m$^{-2}$)</td>
<td>241</td>
<td>241</td>
</tr>
<tr>
<td>Emitted longwave irradiance $F_{\text{lw}}$ (W m$^{-2}$)</td>
<td>240</td>
<td>240</td>
</tr>
<tr>
<td>Shortwave absorptivity $\alpha$</td>
<td>0.708</td>
<td>0.709</td>
</tr>
<tr>
<td>Effective absorption temperature $T_a$ (K)</td>
<td>282 ($= F_{\text{sw}}/\alpha$)</td>
<td>283$^a$</td>
</tr>
<tr>
<td>Effective emission temperature $T_e$ (K)</td>
<td>259 ($= F_{\text{lw}}/\alpha$)</td>
<td>255 ($= F_{\text{sw}}/\alpha^{1/4}$)</td>
</tr>
<tr>
<td>$\delta F_{\text{lw}}/\delta a$ (W m$^{-2}$)</td>
<td>125$^b$</td>
<td>340 ($= F_0$)</td>
</tr>
<tr>
<td>$\delta F_{\text{sw}}/\delta a$ (W m$^{-2}$)</td>
<td>85$^b$</td>
<td>0</td>
</tr>
<tr>
<td>$\delta T_a/\delta a$ (K)</td>
<td>80$^b$</td>
<td>90 $[= T_a/(4\alpha)]$</td>
</tr>
<tr>
<td>$\frac{F_{\text{TOA, LW}}^\uparrow}{T_e} \left( \frac{1}{F_{\text{TOA, LW}}^\uparrow} \frac{\partial F_{\text{TOA, LW}}^\uparrow}{\partial a} - \frac{1}{T_e} \frac{\partial T_a}{\partial a} \right)$ (W m$^{-2}$ K$^{-1}$)</td>
<td>0.197$^c$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\frac{F_{\text{net, TOA, LW}}/T_a}{1 \left( \frac{1}{T_a} - \frac{1}{T_e} \frac{\partial a}{\partial a} \right)}$ (W m$^{-2}$ K$^{-1}$)</td>
<td>0.948$^d$</td>
<td>1.201$^e$ (0.828$^d$)</td>
</tr>
<tr>
<td>$\frac{\partial F_{\text{TOA, LW}}^\uparrow}{\partial a} (\text{W m}^{-2} \text{K}^{-1})$</td>
<td>0.197$^b$</td>
<td>0.888$^b$</td>
</tr>
<tr>
<td>$\frac{\partial F_{\text{TOA, LW}}^\uparrow}{\partial a} (\text{W m}^{-2} \text{K}^{-1})$</td>
<td>0.694$^b$</td>
<td>0.201$^e$ (0.172$^f$)</td>
</tr>
</tbody>
</table>

$^a$ Derived from Ed4 SYN.
$^b$ Derived from linear regression.
$^c$ Computed with $\delta F_{\text{lw}}/\delta a = 132$ W m$^{-2}$ and $\delta T_a/\delta a = 71$ K.
$^d$ Computed with setting $T_a/\delta a = 59$ K.
$^e$ Computed with setting $\delta T_a/\delta a = 0$ K, that is, $F_0/T_a$.
$^f$ $1.0 \sim 0.828$ W m$^{-2}$ K$^{-1}$. 

The global annual mean values that appear in Eq. (2) derived from SYN1deg-Month and the ideal energy balanced model.
Bannon and Lee (2017). Based on our results, the Carnot efficiency is

\[ \eta = \frac{T_u - T_e}{T_u} = 0.085. \]  

(30)

b. Regional entropy production

Figure 2 shows regional annual mean entropy production by absorption of solar irradiance and by emission of longwave irradiance to space. Entropy produced by shortwave absorption is larger in the tropics and decreases toward poles. The spatial distribution of entropy produced by shortwave absorption resembles the spatial pattern of shortwave absorption, indicating that the spatial distribution is driven by shortwave absorption. The production is larger in the tropics and decreases toward poles. Absorption of shortwave irradiance produces more entropy at the surface than in the atmosphere. Similarly, entropy production by longwave emission is large over tropics and decreases toward poles. The effect of clouds is evident over the western Pacific, central Africa, and the Amazon. Similar to entropy production by shortwave, the spatial distribution is driven by the spatial distribution of emitted longwave irradiance to...
space. A larger contribution, however, comes from the atmosphere than from the surface. Once entropy production by shortwave absorption is subtracted from entropy production by longwave emission to space, the resulting entropy production increases toward poles (Fig. 3). This is equal to entropy produced by irreversible processes [Eq. (20)]. The regional difference of entropy produced by longwave emission to space and shortwave absorption is the net effect of entropy produced by frictional dissipation, entropy produced by diabatic heating by water phase change associated with water vapor transport (i.e., evaporation and condensation temperature differs), frictional dissipation of falling raindrops, horizontal and vertical enthalpy transport by irreversible processes, and energy transport by radiation exchange. As indicated by Eq. (13) and unlike regional energy balance in the column, vertical energy transports by irreversible processes and radiation exchange affect entropy production in the column.

Figure 4 shows the standard deviation of annual regional mean entropy produced by shortwave absorption and longwave emission to space. The variability of tropical convections causes large variability of both shortwave and longwave entropy production (Fig. 4 top). While both atmosphere and surface contribute the variability of longwave entropy production, the atmospheric contribution to the shortwave entropy production is smaller than the surface contribution. Clouds and water vapor variabilities are responsible for the variability of entropy production by longwave. The variability of clouds also causes the variability of entropy production by shortwave at the surface, but water vapor and aerosol variabilities primarily drive the variability of shortwave entropy production in the atmosphere. Once entropy production by shortwave absorption is subtracted from entropy production by longwave emission to space, the variability due to tropical convection is small (Fig. 5), which is analogous to a small net TOA irradiance over convective cloud regions due to partial cancellation of their shortwave and longwave radiative effects. As a result, large variabilities of the difference of entropy produced by shortwave absorption and longwave emission occur mostly over stratocumulus regions. Similar to Fig. 3, the variability shown in Fig. 5 is the result of variability of frictional dissipation, radiation exchange, and heating and cooling by water phase change in the column.

Figure 6 shows the mean and standard deviation of annual mean entropy produced by nonradiative irreversible process, which is equal to entropy produced by longwave cooling in the atmosphere and surface minus entropy produced by shortwave absorption in the atmosphere and surface [Eq. (23)]. The spatial pattern is similar to Figs. 3 and 5.

5. Discussion

At a steady state, the sum of entropy produced by absorption of solar irradiance and emission of longwave irradiance to space and entropy produced by irreversible processes in the Earth system is zero. In addition, the 1D model suggests that entropy produced by the net shortwave and longwave irradiances in the atmosphere and at the surface and entropy produced by nonradiative irreversible process is zero at a steady state. The nonradiative irreversible processes include frictional dissipation of turbulence, diabatic heating by water phase change associated with water vapor transport (i.e., evaporation and condensation temperatures differ), and frictional dissipation of falling raindrops. Although the 1D model provides insight into how entropy is produced, the 1D model can be misleading when it is used to understand the sensitivity of entropy production to energy input to Earth. Once observations are combined, however, the 1D model can provide an insight of how entropy production changes when energy input to the Earth system is changed. This point is demonstrated in the remaining of this section.

To understand the sensitivity of entropy production to energy input to Earth, we start with showing the relationship between global annual TOA emitted longwave irradiance and net shortwave irradiance anomalies derived from Edition 4.1 Energy Balanced and Filled (EBAF) product (Loeb et al. 2018a) in Fig. 7. The slope of the line derived by regressing net longwave irradiance anomalies versus shortwave irradiance anomalies is 0.39. The TOA emitted longwave irradiance, therefore, increases (decreases) with increasing (decreasing) shortwave absorption. At an annual global scale, however, the magnitude of emitted longwave irradiance anomalies is generally smaller than the magnitude of shortwave
absorption anomalies. Standard deviations of global annual anomalies of the absorbed shortwave irradiance and emitted longwave irradiance are, respectively, 0.40 and 0.25 W m$^{-2}$. Similarly, when we plot anomalies of the annual global mean shortwave and longwave entropy production as a function of anomalies of the annual global mean TOA absorptivity $a$ (TOA net shortwave irradiance divided by $F_0$) (Fig. 8), the slope is 0.948 W m$^{-2}$ K$^{-1}$ for $\partial/\partial a(F_{\text{net,SW}}^{\text{TOA}}/T_a)$ and 0.197 W m$^{-2}$ K$^{-1}$ for $\partial/\partial a(F_{\text{TOA,LW}}^{\text{TOA}}/T_e)$. Positive slopes indicate that both shortwave and longwave entropy productions increase with shortwave absorption. Positive slopes also imply that when the shortwave absorption increases, the relative increase of $T_a$ or $T_e$ is smaller than the relative increase of $F_{\text{net,SW}}^{\text{TOA}}$ or $F_{\text{TOA,LW}}^{\text{TOA}}$. The increase of shortwave entropy production is, however, larger than the increase of longwave entropy production.

At a global annual scale, most energy is used for heating ocean so that interannual variability of TOA net irradiance agrees with the variability of ocean heating rate (Johnson et al. 2016). When absorbed shortwave irradiance is perturbed, shortwave anomalies are damped by ocean so that the magnitude of emitted longwave anomalies is smaller than the magnitude of absorbed shortwave anomalies. These smoothing or
damping processes seem to be responsible for a negative \( \frac{S_{\text{irr}}}{T_{a}} - \frac{S_{\text{net}}}{T_{a}} = \frac{\partial S_{\text{irr}}}{\partial a} \). A negative slope \((-0.694 \text{ W m}^{-2} \text{ K}^{-1})\) derived by a linear regression applied to the \( S_{\text{irr}} \) and \( a \) relationship (Fig. 8) is consistent with the difference of the slopes of \( \frac{\partial S_{\text{irr}}}{\partial a}(F_{\text{TOA,SW}}^1/T_{a}) \) and \( \frac{\partial S_{\text{irr}}}{\partial a}(F_{\text{TOA,LW}}^1/T_{a}) \). Although the slope derived from linear regression is not a partial derivative because other variables are not held constant, we use the slope as a proxy of the partial derivative and describe the difference of entropy production sensitivity derived from observations and the 1D model in the remainder of this section.

We take the derivative of entropy production with respect to shortwave absorptivity,

\[
\frac{\partial S_{\text{irr}}}{\partial a} = \frac{F_{\text{TOA,LW}}^1}{T_{e}} \left( \frac{1}{F_{\text{TOA,LW}}^1} \frac{\partial F_{\text{TOA,LW}}^1}{\partial a} - \frac{1}{T_{e}} \frac{\partial T_{e}}{\partial a} \right)
- \frac{F_{\text{TOA,LW}}^1}{T_{a}} \left( \frac{1}{a} - \frac{1}{T_{a}} \frac{\partial T_{a}}{\partial a} \right),
\]

(31)

and estimate the numerical value of the derivative. Table 3 compares derivative values derived from CERES data products using regression (second column) and theoretical estimates using the 1D model (third column). The 1D model does not predict the change of \( T_{a} \) with shortwave absorption so that observations are needed to estimate \( \partial T_{a}/\partial a \). If we set \( \partial T_{a}/\partial a = 0 \), the derivative \( \partial S_{\text{irr}}/\partial a \) from the 1D model is \(-0.201 \text{ W m}^{-2} \text{ K}^{-1}\). The estimate of \( \partial S_{\text{irr}}/\partial a \) by Bannon and Lee (2017) is \(-0.236 \text{ W m}^{-2} \text{ K}^{-1}\). The reason for the difference between our value of \(-0.201 \text{ W m}^{-2} \text{ K}^{-1}\) and their value is that we use \( T_{a} = 283 \text{ K} \) instead of \( 275 \text{ K} \). When we use the value of \( \partial T_{a}/\partial a \) estimated from the CERES data products, the derivative is \(-0.172 \text{ W m}^{-2} \text{ K}^{-1}\). The difference of the sensitivity derived from the CERES data products and 1D model arises because of the difference of \( \partial F_{\text{TOAL,SW}}^1/\partial a \) or \( \partial F_{\text{TOAL,SW}}^1/\partial F_{\text{TOAL,SW}}^1 \). The 1D model assumes that \( \partial F_{\text{TOAL,SW}}^1/\partial F_{\text{TOAL,SW}}^1 = 1 \) while the CERES observations indicate that the slope of the relationship of annual global mean emitted longwave anomalies as a function of annual global mean net shortwave anomalies is less than 1.

We test whether Eq. (31) can also be used to estimate the critical absorptivity to make \( \partial S_{\text{irr}}/\partial a = 0 \) with an assumption that all values are unaltered when the shortwave absorption is altered. If we use \( \partial S_{\text{irr}}(F_{\text{TOA,SW}}^1/T_{e}) = 0.197 \text{ W m}^{-2} \text{ K}^{-1} \) derived by directly fitting \( F_{\text{TOA,SW}}^1/T_{e} \) versus \( a \), and use \( F_{\text{TOA,SW}}^1/T_{e} \), and \( \partial T_{a}/\partial a \) derived from the CERES data products (the second column of Table 3), \( \partial S_{\text{irr}}/\partial a \) is always negative for the physically possible range of the absorptivity, \( 0 \leq a \leq 1 \). Therefore, entropy production by irreversible processes always decreases with increasing absorption of shortwave irradiance for the possible range of the planetary albedo.

6. Summary and conclusions

We used satellite observations of cloud and aerosol properties, and temperature and humidity profiles from
reanalysis to compute entropy production within the Earth system. We focused entropy production by heating and cooling by radiation that, at a steady state, balances with entropy production by irreversible processes within the system.

Entropy produced by longwave emission to space estimated in this study, $0.929 \text{ W m}^{-2} \text{ K}^{-1}$, is close to that of $0.925 \text{ W m}^{-2} \text{ K}^{-1}$ estimated by Peixoto et al. (1991). In addition, entropy carried by longwave irradiance emitted to space estimated in this study is in a close agreement with that estimated by Stephens and O’Brien (1993) using ERBE data. However, entropy produced by absorption of shortwave irradiance at the surface and by the atmosphere are significantly different from those estimated by Peixoto et al. (1991). With a steady-state assumption, entropy production by irreversible processes within the Earth system is estimated to be $0.076 \text{ W m}^{-2} \text{ K}^{-1}$ and by nonradiative irreversible processes to be $0.049 \text{ W m}^{-2} \text{ K}^{-1}$.

Spatial distribution of entropy production is similar to the spatial distribution of net shortwave irradiance and longwave irradiance emitted to space. The larger standard deviation of annual mean of regional entropy productions by shortwave irradiance absorption and longwave irradiance emission to space is caused by the variability due to convections in the tropics. Once the entropy production by absorption of shortwave irradiance is subtracted from the entropy production by longwave irradiance emission to space, the variability of the difference is larger over stratuscumulus regions, similar to the spatial distribution of the variability of net TOA irradiances.

CERES observations suggest that, at a global annual scale, entropy produced by emission of longwave irradiance to space and by absorption of shortwave irradiance increases with increasing energy input to (absorptivity of) the Earth system. Increasing the entropy produced by the shortwave absorption is larger than increasing the entropy produced by the longwave emission, which makes entropy production by irreversible processes decrease when absorption of shortwave irradiance increases.

**FIG. 7.** Global annual anomalies of TOA emitted longwave irradiance vs global annual anomalies of TOA shortwave net irradiance. Eighteen years of data from March 2000 to February 2018 are used. The solid line is linear regression line with the slope of 0.39. The slope of the dashed line is 1: $R$ and $m$ are, respectively, correlation coefficient and slope.

![Figure 7](image_url)

**FIG. 8.** Annual global anomalies of entropy production by (left) longwave irradiance emission to space, (middle) shortwave irradiance absorption, and (right) longwave irradiance emission minus shortwave absorption as a function of annual global anomalies of absorptivity (i.e., absorbed shortwave irradiance divided by TOA downward shortwave irradiance). Eighteen years of data from March 2000 to February 2018 are used; $R$ and $m$ are, respectively, correlation coefficient and slope.

![Figure 8](image_url)
Acknowledgments. We thank two anonymous reviewers who provided constructive comments that improved the manuscript. This research was supported by the NASA CERES project. The Ed4.1 CERES EBAF–TOA (Loeb et al. 2018b) and SYNdeg-Month (Rutan et al. 2015b) datasets were downloaded from https://ceres.larc.nasa.gov/order_data.php.

APPENDIX

Computation of Material Entropy Production by Longwave Irradiance Emitted to Space

Following Toon et al. (1989), we use two approximations in addition to a two-stream approximation. First, the Planck function is represented by

\[ B_n(\tau) = B_{0n} + B_{1n}\tau, \]

(A1)

where \( B_{0n} \) is the Planck function evaluated at the temperature of the top of the \( n \)th computation layer with the optical thickness of \( \tau_n \) and

\[ B_{1n} = \frac{[B(T_{0n}) - B_{0n}]}{\tau_n}, \]

(A2)

where \( B(T_{bn}) \) is the Planck function evaluated at the temperature of the bottom of the \( n \)th layer \( T_{bn} \). Second, we use the radiance computed with a two-stream approximation instead of the true radiance in computing source function. With these approximations, the source function \( S_{st} \) of longwave irradiances propagating upward and downward through the atmosphere are [Toon et al. 1989, their Eqs. (53) and (54)], respectively,

\[ S_{st}^+ = G e^{-\lambda \tau_n} \exp(\lambda) + H e^{-\lambda \tau} + \alpha_1 + \alpha_2 \tau, \]

(A3)

\[ S_{st}^- = J e^{-\lambda \tau_n} \exp(\lambda) + K e^{-\lambda \tau} + \sigma_1 + \sigma_2 \tau, \]

(A4)

where \( G, H, J, K, \alpha, \) and \( \sigma \) are defined in Table 3 of Toon et al. (1989) for the hemispheric mean two-stream approximation and superscripts + and − indicate upward (propagating from the surface to TOA) and downward (propagating from TOA to the surface). We also assume that the temperature changes linearly within the \( n \)th computational layer with the optical thickness of \( \tau_n \) so that

\[ T_n(\tau) = T_{0n} + T_{1n}\tau, \]

(A5)

and \( T_{bn} \) and \( T_{0n} \) are the temperature at the bottom and top of the \( n \)th layer. The entropy produced by the upward longwave irradiance by the \( n \)th layer is

\[ \int_0^{\tau_n} \frac{S_{st}^+}{T_{0n} + T_{1n}\tau} \exp(-t/\mu) \, dt = A + \frac{1}{T_{0n} + T_{1n}\mu} \exp(\mu) - \frac{2\mu^2}{T_{0n} + T_{1n}\mu - 1} \]

\[ + \frac{1}{T_{0n} + T_{1n}\mu + 2\mu^2} \exp(\mu) \frac{1}{T_{0n} + T_{1n}\mu + 3\mu^2} \exp(\mu) + \cdots \]

(A7)

Substituting Eqs. (A3) and (A4) in Eq. (A7) leads to

\[ \int_0^{\tau_n} \frac{S_{st}^+}{T_{0n} + T_{1n}\tau} \exp(-t/\mu) \, dt = A + \frac{1}{T_{0n} + T_{1n}\mu} \exp(\mu) - \frac{2\mu^2}{T_{0n} + T_{1n}\mu - 1} \]

\[ + \frac{1}{T_{0n} + T_{1n}\mu + 2\mu^2} \exp(\mu) \frac{1}{T_{0n} + T_{1n}\mu + 3\mu^2} \exp(\mu) + \cdots \]

(A8)
\[
\int_0^\tau S_{et} \frac{e^{-\delta t}}{T_{0n}} - \frac{S_{et} T_{ln}}{T_{0n}} e^{-\delta t} + \frac{S_{et} (T_{ln})^2}{T_{0n}^2} T_{c} e^{-\delta t} - \cdots \frac{dt}{\mu} = A^{-}\left[ \frac{1}{T_{0n}} + \frac{T_{ln}}{T_{0n}} \frac{\mu}{\lambda \mu + 1} + \frac{T_{3n}}{T_{0n}} \frac{2\mu^2}{\lambda \mu - 1} \right]
\]

\[
+ \sum \frac{T_{ln}^n}{T_{0n}^{n+1}} \left( \frac{n! \mu^n}{(\lambda \mu + 1)^n} \right) B^{-}\left[ \frac{1}{T_{0n}} - \frac{T_{ln}}{T_{0n}} \frac{\mu}{\lambda \mu - 1} + \frac{T_{3n}}{T_{0n}} \frac{2\mu^2}{\lambda \mu - 1} \right] + \sum (-1)^n \frac{T_{ln}^n}{T_{0n}^{n+1}} \left( \frac{n! \mu^n}{(\lambda \mu + 1)^n} \right) + \cdots
\]

\[
C^{-}\left[ \frac{1}{T_{0n}} + \frac{T_{ln}}{T_{0n}} \frac{\mu}{\lambda \mu - 1} + \frac{T_{3n}}{T_{0n}} \frac{2\mu^2}{\lambda \mu - 1} \right] + \sum (-1)^n n! \mu^n + \sum \frac{J}{\lambda \mu - 1} \tau_n \left[ -\frac{T_{ln}}{T_{0n}} + \frac{T_{3n}}{T_{0n}} \left( \tau_n + \frac{2\mu}{\lambda \mu + 1} \right) + \cdots \right]
\]

\[
+ \frac{K}{\lambda \mu - 1} \tau_n e^{-\lambda \tau_n} \left[ -\frac{T_{ln}}{T_{0n}} + \frac{T_{3n}}{T_{0n}} \left( \tau_n + \frac{2\mu}{\lambda \mu + 1} \right) + \cdots \right]
\]

\[
+ \alpha_1 \tau_n \left[ \frac{T_{ln}}{T_{0n}} + \frac{T_{3n}}{T_{0n}} (\tau_n + 2\mu + \cdots) \right] + \alpha_2 \tau_n^2 \left[ -\frac{T_{ln}}{T_{0n}} + \frac{T_{3n}}{T_{0n}} (\tau_n + 3\mu + \cdots) \right], \quad \text{where (A9)}
\]

\[
A^+ = \frac{G}{\lambda \mu - 1} \left( e^{-\tau_{c}/\mu} - e^{-\tau_{c}/\lambda} \right), \quad \text{(A10)}
\]

\[
B^+ = \frac{H}{\lambda \mu + 1} [1 - e^{-[1+(1/\mu)]\tau_n}], \quad \text{(A11)}
\]

\[
C^+ = \alpha_1 (1 - e^{-\tau_c/\mu}), \quad \text{(A12)}
\]

\[
D^+ = \alpha_2 [\mu - (\tau_n + \mu) e^{-\tau_c/\mu}], \quad \text{(A13)}
\]

\[
A^- = \frac{J}{\lambda \mu + 1} [1 - e^{-[1+(1/\mu)]\tau_n}], \quad \text{(A14)}
\]

Expressions for an exponential assumption of the Planck function within a layer [Fu et al. 1997, Eq. (2.25)] are
\[
\int_0^z \frac{S_{\nu}}{T_{0\nu}} e^{-\nu/\mu} - \frac{S_{\nu}}{T_{0\nu}} \frac{T_{m\nu}}{T_{0\nu}} e^{-\nu/\mu} + \frac{S_{\nu}}{T_{0\nu}} \left( \frac{T_{m\nu}}{T_{0\nu}} \right)^2 t^2 e^{-\nu/\mu} - \ldots \, dt = A^- \left( \frac{1}{T_{0\nu}} + \frac{T_{m\nu}}{T_{0\nu}} \frac{\mu}{\lambda \mu + 1} \frac{2\mu^2}{T_{0\nu}^2 (\lambda \mu + 1)^2} \right)
\]
\[
+ \sum \frac{T_{m\nu}^n}{T_{0\nu}^n} (\lambda \mu + 1)^n + B^- \left( \frac{1}{T_{0\nu}} + \frac{T_{m\nu}}{T_{0\nu}} \frac{\mu}{\lambda \mu - 1} \frac{2\mu^2}{T_{0\nu}^2 (\lambda \mu - 1)^2} \right) + \sum (-1)^n \frac{T_{m\nu}^n}{T_{0\nu}^{n+1}} (\lambda \mu - 1)^n
\]
\[
+ C^- \left( \frac{1}{T_{0\nu}} + \frac{T_{m\nu}}{T_{0\nu}} \frac{\mu}{\beta \mu + 1} - \frac{T_{m\nu}^2}{T_{0\nu}^2} (\beta \mu + 1)^2 - \sum (-1)^n \frac{T_{m\nu}^n}{T_{0\nu}^{n+1}} (\lambda \mu - 1)^n \right) + J \frac{1}{\lambda \mu + 1} \frac{n!}{\tau_n} \left( \frac{T_{m\nu}}{T_{0\nu}} + \frac{T_{m\nu}^2}{T_{0\nu}} \frac{T_{m\nu}}{T_{0\nu}^2} \frac{\mu}{\beta \mu + 1} + \ldots \right)
\]
\[
- K \frac{1}{\lambda \mu - 1} \frac{n!}{\tau_n} e^{-\chi_n} \left( \frac{T_{m\nu}}{T_{0\nu}} + \frac{T_{m\nu}^2}{T_{0\nu}^2} \frac{T_{m\nu}}{T_{0\nu}^2} \frac{\mu}{\lambda \mu - 1} + \ldots \right) + \eta B_\nu \frac{\beta \mu}{\beta \mu + 1} \frac{n!}{\tau_n} \left( \frac{T_{m\nu}}{T_{0\nu}} + \frac{T_{m\nu}^2}{T_{0\nu}^2} \frac{T_{m\nu}}{T_{0\nu}^2} \frac{\mu}{\beta \mu + 1} + \ldots \right),
\]
(A19)

where \(C^\pm\) are replaced by
\[
C^+ = \frac{\zeta}{1 - \mu \beta} [B(0) - B(\tau_n) e^{-\chi_n/\beta}],
\]
(A20)

and
\[
C^- = \frac{\eta}{1 + \mu \beta} [B(0) - B(\tau_n) e^{-\chi_n/\beta}],
\]
(A21)

where \(\zeta, \eta, \text{ and } \beta\) are defined in Fu et al. (1997). We only keep terms with up to second power of \(T_{0\nu}\) in the denominator because high-order terms are smaller in computing entropy production by Eqs. (A18) and (A19). These expressions compute entropy produced by longwave radiation emitted to space. These expressions, however, do not explicitly compute the transmission from each layer to space and hence do not explicitly keep track of the emission temperature of radiation emitted to space.

REFERENCES


