A Unified Moisture Mode Theory for the Madden–Julian Oscillation and the Boreal Summer Intraseasonal Oscillation

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(Manuscript received 6 May 2021, in final form 8 October 2021)

ABSTRACT: The Madden–Julian oscillation (MJO) and the boreal summer intraseasonal oscillation (BSISO) are fundamental modes of variability in the tropical atmosphere on the intraseasonal time scale. A linear model, using a moist shallow water equation set on an equatorial beta plane, is developed to provide a unified treatment of the two modes and to understand their growth and propagation over the Indian Ocean. Moisture is assumed to increase linearly with longitude and to decrease quadratically with latitude. Solutions are obtained through linear stability analysis, considering the gravest ($n = 1$) meridional mode with nonzero meridional velocity. Anomalies in zonal moisture advection and surface fluxes are both proportional to those in zonal wind, but of opposite sign. With observation-based estimates for both effects, the zonal advection dominates, and drives the planetary-scale instability. With a sufficiently small meridional moisture gradient, the horizontal structure exhibits oscillations with latitude and a northwest–southeast horizontal tilt in the Northern Hemisphere, qualitatively resembling the observed BSISO. As the meridional moisture gradient increases, the horizontal tilt decreases and the spatial pattern transforms toward the “swallowtail” structure associated with the MJO, with cyclonic gyres in both hemispheres straddling the equatorial precipitation maximum. These results suggest that the magnitude of the meridional moisture gradient shapes the horizontal structures, leading to the transformation from the BSISO-like tilted horizontal structure to the MJO-like neutral wave structure as the meridional moisture gradient changes with the seasons. The existence and behavior of these intraseasonal modes can be understood as a consequence of phase speed matching between the equatorial mode with zero meridional velocity (analogous to the dry Kelvin wave) and a local off-equatorial component that is characterized by considering an otherwise similar system on an f plane.

KEYWORDS: Convection; Intraseasonal variability; Tropical variability

1. Introduction

The Madden–Julian oscillation (MJO) and boreal summer intraseasonal oscillation (BSISO) are the two most prominent tropical intraseasonal modes. The MJO (Zhang 2005) is dominant in northern winter while the BSISO (Kikuchi 2021) is more important in northern summer. These tropical intraseasonal oscillations (ISOs) play large roles in global weather and climate. Since the discovery independently by several authors of the MJO (Xie et al. 1963; Li et al. 2018; Madden and Julian 1971) and the BSISO (Yasunari 1979; Sikka and Gadgil 1980; Krishnamurti and Subrahmanyam 1982), a great deal of effort has been devoted to understanding their dynamics, as well as to numerical modeling, subseasonal prediction, and projection of their behavior under global warming.

Both the MJO and BSISO develop most often in the Indian Ocean (although MJO events in particular can originate at any longitude; e.g., Matthews 2008) and share similar time scales, while their spatial patterns differ markedly. The MJO

is largely symmetric about the equator in this region, with twin cyclonic gyres straddling the equator and trailing the main near-equatorial convection clusters (Rui and Wang 1990), a structure sometimes characterized as a “swallowtail” (Zhang and Ling 2012; Kim and Zhang 2021). On the other hand, the BSISO is mostly present in the Northern Hemisphere, with a southeast–northwest-tilted structure in precipitation spanning from South Asia to the western Pacific Ocean. The MJO primarily propagates eastward, while the BSISO propagates both eastward and northward (Lawrence and Webster 2002; Wang et al. 2018). These intraseasonal oscillations share similar time and space scales, but their morphologies differ substantially. The conceptual boundary between the two phenomena in the literature is arbitrary and confusing: depending on how different authors prioritize time scale versus spatial pattern, and which one is of greater interest, the BSISO is treated either as the northern summer incarnation of the MJO or as a distinct phenomenon.

The current understanding of the MJO has converged to some degree on the crucial roles of moisture in shaping its eastward propagation and planetary-scale instability, with

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DOI: 10.1175/JCLI-D-21-0361.1

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competing theories emphasizing different aspects of convection–circulation interaction (Zhang et al. 2020). Studies of the BSISO, however, have followed a different line historically. Several authors have considered its northward propagation largely independent of eastward propagation, and even used zonally symmetric models to understand its dynamics (Webster 1983; Nanjundiah et al. 1992; Jiang et al. 2004; Drbohlav and Wang 2005; Bellon and Sobel 2008a,b). No existing theory seems to be able to simultaneously explain the BSISO's spatial patterns, temporal and spatial scales, and relationship to the MJO, if any.

The MJO and BSISO both draw energy from the tropical ocean and from the background atmospheric state. Over the Indian Ocean, the large-scale circulation features large-scale ascent in the Maritime Continent and descent over the western Indian Ocean, with low-level mean westerlies and upper-level easterlies completing the local branch of the zonal overturning Walker cell. Consistent with this structure, atmospheric moisture increases eastward over the Indian Ocean, with drier air near the African coast and moister air near the Maritime Continent. The mean meridional circulation changes with the seasons, with ascent and the greatest moisture in the summer hemisphere and a drier descending winter hemisphere. In northern summer in particular, sea surface temperatures in the Bay of Bengal and Arabian Sea become quite warm. Moisture and enthalpy build up near the southern slope of the Tibetan Plateau, and the meridional moisture gradient over the Indian Ocean basin in the Northern Hemisphere becomes very weak or even reverses (i.e., moisture increases with latitude). These mean gradients are relevant to the intraseasonal modes. Recent observational analyses (Jiang et al. 2018; Adames et al. 2016) and numerical modeling (Wang et al. 2021) emphasize the importance of moisture for the BSISO, and suggest that the BSISO, like the MJO, may be thought of as a moisture mode, despite the marked differences in morphology between the two.

Here, we perform a unified analysis of an idealized moist linear model which (we argue) explains some key features of both the MJO and BSISO parsimoniously. In this model, the background meridional moisture gradient controls the morphology of the ISOs. An increase in this gradient, representative of that which occurs from northern summer to northern winter over the Indian Ocean, leads to a transformation in the spatial pattern of the modeled intraseasonal oscillation from a more BSISO-like one to a more MJO-like one.

The rest of this article is structured as follows. Section 2 presents the model and its solutions. Model parameter values are estimated in section 3. Results are discussed in section 4, followed by physical interpretation of this simple model in section 5. A discussion and conclusions are presented in section 6.

2. A linear moist shallow water model

a. Formulation

We start from a moist shallow water equation set over the equatorial β plane. The moisture variable can be thought of as total column moisture, while the other variables represent vertical structures broadly consistent with the first baroclinic mode of

the dry system (details in the appendix). The system is written in nondimensionalized form as

$$\frac{\partial u}{\partial t} - yv = -\frac{\partial \phi}{\partial x} - \alpha_u u, \quad (1)$$

$$\frac{\partial v}{\partial t} + yu = -\frac{\partial \phi}{\partial y}, \quad (2)$$

$$\frac{\partial \phi}{\partial t} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = -\alpha(1+r)q - \alpha_\phi \phi, \quad (3)$$

$$\frac{\partial q}{\partial t} - (\Gamma - 1) \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \mathcal{E}u - \alpha q - u\bar{Q}_x - v\bar{Q}_y. \quad (4)$$

These are equations for horizontal momentum u and v , perturbation pressure ϕ , and moisture q , respectively; x and y denote nondimensional distances in longitude and latitude, respectively, with a scaling constant $L = \sqrt{c/\beta}$, where $\beta = 2.2 \times 10^{-11} \text{ m}^{-1} \text{ s}^{-1}$, and $c = 50 \text{ m s}^{-1}$; t is nondimensional time scaled by $T = 1/\sqrt{\beta c}$. This moist shallow water equation set is written following the conventions in Fuchs and Raymond (2017, hereafter FR17) but with a different assumption regarding the moisture field (see appendix for details). The dimensional phase speed of the dry gravity wave associated with the assumed first baroclinic mode vertical structure is $\sim 50 \text{ m s}^{-1}$ [see Eq. (3) and its dimensional form Eq. (A26)], corresponding to an equivalent depth of $\sim 250 \text{ m}$. The behavior of this linear system is dictated by eight physical processes represented by the following parameters (see Table 1):

- α , moisture relaxation time scale for precipitation (αq);
- r , cloud radiative feedback;
- Γ , nondimensional gross moist stability;
- $\mathcal{E}$, the wind-induced surface heat exchange (WISHE; Emanuel 1987) parameter;
- α_u , Rayleigh friction (used only on zonal wind u ; its effect on v is small and thus neglected);
- α_ϕ , Newtonian damping for perturbation pressure;
- $\bar{Q}_x$, zonal gradient of reference moisture;
- $\bar{Q}_y$, meridional gradient of reference moisture.

This linear shallow water model may be thought of as an extension of the classic dry linear equatorial wave model of Matsuno (1966). All the column physical processes associated with convection, clouds, and radiation are packed into six parameters in the q and ϕ equations. If all of these parameters are set to zero, the model reduces to Matsuno's. Similar moist shallow water equation sets—but generally without horizontal moisture advection—have been analytically solved (e.g., Neelin and Yu 1994; Fuchs and Raymond 2002, 2005, 2017; Sugiyama 2009; Ahmed et al. 2021). For some parameter values, the solutions of these systems display planetary instability on intraseasonal time scales. However, the spatial patterns often lack the off-equatorial cyclonic gyres typical of the observed MJO (Rui and Wang 1990), suggesting that they are really convectively coupled Kelvin waves. Many other studies have obtained solutions by projecting the meridional structure to the first several leading basis

TABLE 1. Model parameters.

Symbol	Parameter	Dimensional value	Nondimensional value
α	Moisture relaxation time scale	1 day	0.35
α_u	Rayleigh friction time scale	10 days	0.03
α_ϕ	Newtonian damping time scale	30 days	0.01
r	Cloud-radiative feedback	$0.2 \times 10^{-5} \text{ s}^{-1}$	0.17
$\overline{Q}_x$	Zonal moisture gradient	$-6 \text{ W m}^{-2} (\text{m s}^{-1})^{-1}$	-0.06
$\mathcal{E}$	WISHE parameter	$3 \text{ W m}^{-2} (\text{m s}^{-1})^{-1}$	0.03
Q^0	Meridional moisture gradient at $y = 1$	$-5 \text{ to } 30 \text{ W m}^{-2} (\text{m s}^{-1})^{-1}$	-0.05 to 0.3
Γ	Normalized gross moist stability	—	0.16

functions associated with the dry equation set (e.g., Majda and Stechmann 2009; Sobel and Maloney 2012, 2013; Adames and Kim 2016; Liu and Wang 2016; Stechmann and Hottovy 2020), thus including (sometimes by construction) MJO-like cyclonic gyres, but no BSISO-like solutions were found.

Here, we include both zonal and meridional moisture advection of the basic state moisture by perturbation winds, namely the last two terms in Eq. (4). We make a specific choice for the meridional structure of the basic state moisture that facilitates analysis, while still being consistent with observations at a level of idealization consistent with the model's other aspects. We do not use the meridional basis functions of the dry system as a basis, instead using the distinct meridional structures that emerge naturally from the moist model. These structures were also identified by Ahmed (2021), who used a very similar model, and whose work we became aware of in the late stages of writing this paper. We find here that these structures are key to explaining the MJO–BSISO differences as a function of the meridional moisture gradient.

Our simplifying assumptions about the horizontal moisture gradients are as follows.

First, we consider zonal advection. Assuming a constant zonal moisture gradient, the zonal advection term is mathematically equivalent to the WISHE term, and they may be combined into a single coefficient, $\mathcal{E} - \overline{Q}_x$ (Sobel and Maloney 2013), that relates the net tendency of moisture or MSE anomalies to the perturbation zonal wind. The time scale and instability of the intraseasonal oscillations predicted by the model rest critically upon the sign and magnitude of this coefficient, which will be estimated later in section 3. We note that a nonzero constant zonal moisture gradient is strictly inconsistent with periodic zonal boundary conditions, but given the distinct behavior of the MJO and BSISO over the Indian Ocean, the quantitative importance of zonal advection there, and the much greater complexity that would be needed to consider model parameters varying with longitude, we view it as acceptable and useful to consider a uniform zonal gradient in order to understand the dynamics over the Indian Ocean.

Next, we consider meridional moisture advection. We assume a reference moisture distribution $\overline{Q}(x, y, z)$ that decreases quadratically with latitude $|y|$ as $Q \sim -Q_0(y^2/2) + Q_e$, where Q_0 and Q_e are constants. Positive Q^0 indicates column moisture decreases from the equator to high latitudes. Note that Q_e denotes the maximum column moisture achieved at $y = 0$, and its value does not enter the problem as only the gradient is used. The same structure was assumed by Ahmed (2021). Then the meridional moisture advection is simplified as

$$-v\overline{Q}_y = v y Q^0. \quad (5)$$

It might seem questionable that the absolute value of the meridional gradient Q_y increases with $|y|$ even at large y . However, assuming boundedness as $y \rightarrow \infty$ will lead us to retain only equatorially trapped solutions in which v decays exponentially with y^2 . This exponential decay dominates over the linear increase in moisture gradient at large y , and justifies the use of $v y Q^0$ as a reasonable approximation for the moisture meridional advection $v\overline{Q}_y$. It also gives the meridional moisture advection term the same form as the Coriolis term under the equatorial β -plane approximation, and allows Eq. (4) to preserve its symmetry about the equator in all terms.

b. Reduction to a second-order ODE for v

We assume plane wave solutions proportional to $\exp[i(kx - \omega t)]$ for u , v , ϕ , and q , where k is the nondimensional planetary zonal wavenumber and ω is the complex frequency, whose real part $\text{Re}(\omega)$ is the frequency of oscillation and whose imaginary part $\text{Im}(\omega)$ is the growth rate. Using the constant zonal moisture gradient ($\overline{Q}_x$) and linear meridional gradient (yQ^0), one may form a single equation for $v = V(y)\exp[i(kx - \omega t)]$, where $V(y)$ is the meridional structure, by first eliminating q from Eqs. (3) and (4), and then combining the equation for ϕ with Eqs. (1) and (2). Algebraic manipulation then leads to a second-order ordinary differential equation for $V(y)$:

$$a_0 \frac{d^2 V}{dy^2} + a_1 y \frac{dV}{dy} + (d_0 + d_2 y^2)V = 0. \quad (6)$$

The coefficients a_0 , a_1 , d_0 , and d_2 are

$$a_0 = \omega_u [(\Gamma_e - 1)\alpha - i\omega_\alpha], \quad (7)$$

$$a_1 = [i(\mathcal{E} - \overline{Q}_x) + \omega_u Q^0]\alpha(1 + r), \quad (8)$$

$$d_0 = a_1 - \frac{k}{\omega_u} a_0 - k^2 \frac{\omega}{\omega_u} a_0 - i\omega\omega_\alpha\omega_u\omega_\phi + ik\omega(\mathcal{E} - \overline{Q}_x)\alpha(1 + r), \quad (9)$$

$$d_2 = i\omega_\phi\omega_\alpha + kQ^0\alpha(1 + r), \quad (10)$$

where $\omega_\alpha = \omega + i\alpha$, $\omega_u = \omega + i\alpha_u$, $\omega_\phi = \omega + i\alpha_\phi$. Here, $\Gamma_e = \Gamma(1 + r) - r$ is the effective gross moist stability. The

algebraic structure of Eq. (6) differs from that of the classical neutral wave equation in the second term, $y(\partial v/\partial y)$, whose coefficient a_1 contains nearly all the physical parameters that affect the moisture field.

The quantities ω_α , ω_u , and ω_ϕ can be used to trace the damping and time tendency terms. The time tendency in equation v may be neglected if the longwave approximation is used (Gill 1980), which results in omission of the last three terms containing ω in Eq. (9); or replacing d_0 with $\tilde{d}_0$ [in Eq. (6)] defined as

$$\tilde{d}_0 = a_1 - \frac{k}{\omega_u} a_0. \quad (11)$$

Rayleigh friction may be added back to v by replacing ω in d_0 with ω_u . A moisture convergence closure is achieved by replacing ω_α with $i\alpha$ (i.e., $\partial q/\partial t = 0$). The weak temperature gradient approximation is obtained by omission of ω_ϕ in equations for d_0 and d_2 ; and geostrophic balance by omission of ω_u and ω . FR17 is recovered by setting Q^0 , $\bar{Q}_x$, α_u , and α_ϕ to zero. The single moisture prognostic equation of the MJO theory in Sobel and Maloney (2012, 2013) and Adames and Kim (2016) may be recovered by setting the ω in d_0 and its value in ω_u , ω_ϕ to zero, and projecting to the $n = 1$ parabolic cylinder function in y . More details of these approximations are shown in section b of the appendix.

The boundary conditions for v are

$$V \rightarrow 0 \quad \text{at } y \rightarrow \pm \infty. \quad (12)$$

c. Wave solutions

We consider solutions to Eq. (6) with y dependence either of the form

$$V = 0, \quad \text{or} \quad (13)$$

$$V \propto H_n\left(\frac{y}{\eta_n}\right) e^{-\xi_n y^2}. \quad (14)$$

The first solution is referred to as the $v = 0$ mode. It is analogous to the equatorial Kelvin wave, but modified by zonal moisture advection, and displaying different dispersion c (see Fig. 2 below). The second one contains n th-order Hermite polynomials H_n in standard form (Abramowitz and Stegun 1964), except that the shape factor ξ_n and the scaling factor η_n are complex numbers to be determined as part of the dispersion relationships for the wave parameters (ω , η_n , ξ_n). The key difference between the above solution and the traditional ones is in these shape and scaling factors. This functional form was also recently obtained by Ahmed (2021), who showed MJO-like solutions (but no BSISO-like solutions). The main differences between our study and Ahmed (2021) result from the choices of the parameter values. The most important of these is the range of meridional moisture gradients considered, particularly in that we consider very small, zero, or even reversed gradients. Another substantial difference is that our convective closure depends only on moisture,

while that of Ahmed (2021) includes temperature dependence as well. The differences in other parameter choices may be quantitatively consequential, but are probably not qualitatively so.

We focus on the $n = 1$ solution in the present study. This simplifies the problem as we need solve for only two wave parameters, (ω, ξ_1) , while retaining the essential dynamics. Higher-order Hermite polynomial solutions will be reported elsewhere. The solutions of u , q , and ϕ are symmetric about the equator. While v is antisymmetric, the use of linear moisture gradient makes the meridional advection term $-vyQ^0$ also symmetric, consistent with the other terms in Eq. (4) for moisture q .

The most novel aspect of the wave solutions in Eq. (14), and one which is critical to understanding our results, is the fact that the ξ_n terms are complex. This means that even the gravest meridional mode oscillates with latitude, and does so increasingly rapidly with increasing latitude. It is easiest to develop intuition by considering just this mode, for which $n = 1$, so that we can ignore the complex part of $H_1(y/\eta_1) = y/\eta_1$ (i.e., ignore the complex number η_1). Combining the solution's zonal and time dependence $\exp[i(kx - \omega t)]$ with its meridional structure (14), the wave phase function of the $n = 1$ mode is

$$\psi = -\text{Im}(\xi_1)y^2 + kx - \omega t. \quad (15)$$

We can define a local wavenumber l_y and meridional phase speed c_y of these oscillations for $y \neq 0$:

$$l_y = \frac{\partial \psi}{\partial y} = -2 \text{Im}(\xi_1)y, \quad \text{and} \quad c_y = \frac{\text{Re}(\omega)}{l_y} = -\frac{\text{Re}(\omega)}{2 \text{Im}(\xi_1)y}. \quad (16)$$

Here c_y is undefined at $y = 0$. In general, the local wavenumber $|l_y|$ increases with $|y|$, while $|c_y|$ decreases with $|y|$. A useful indicator of the wave geometry is the wavenumber ratio l_y/k . For an eastward propagating wave with positive zonal wavenumber k , $l_y > 0$ indicates poleward propagation, and a westward phase tilt; $l_y = 0$, a neutral wave without meridional tilt, as in the Gill problem; and $l_y < 0$, equatorward propagation and an eastward phase tilt. In some of what follows, we will show the local meridional wavenumber at $|y| = 1$, denoted as $l_y|_{y=1}$, to understand how meridional structure varies with various parameters, understanding that the variation of l_y with y follows (16).

The functional form of Eq. (14) indicates that the wave amplitude decreases exponentially in latitude, with the meridional decay scale:

$$L_y = \text{Re}\left[(\xi_1)^{-1/2}\right]. \quad (17)$$

d. The $v = 0$ solution

Before proceeding to discuss the $n = 1$ solution, we briefly examine the simpler $v = 0$ (i.e., $n = -1$) solution, whose counterparts in similar models (without horizontal moisture advection) have been explored extensively by Emanuel (1987), Neelin and Yu (1994), FR17, and many other authors. As we shall see later, the $v = 0$ solution is closely related to

the $n = 1$ solution. The dispersion relation of the $v = 0$ solution, written $\mathcal{A}_{v0}(\omega, k) = 0$, is a third-order polynomial:

$$\mathcal{A}_{v0}(\omega, k) = \omega_u \omega_\phi \omega_\alpha - k^2 \omega - ik^2 \alpha \Gamma_e - k(\mathcal{E} - \overline{Q}_x) \alpha (1 + r) = 0. \quad (18)$$

There is no Q^0 dependence in $\mathcal{A}_{v0}$ as meridional moisture advection vanishes when $v = 0$. The above cubic equation for ω has three roots in general, but only one or two represent solutions that satisfy the boundary condition [Eq. (12)] at different zonal wavenumbers k and only these are considered physical. A derivation and detailed solutions with the present parameter set, including a visualization of the spatial structure, are shown in section d in the appendix.

Two useful relations between the above $\mathcal{A}_{v0}$ expression and the coefficients are shown below:

$$\begin{aligned} d_0 &= a_1 - \frac{k}{\omega_u} a_0 - i\omega \mathcal{A}_{v0} = \tilde{d}_0 - i\omega \mathcal{A}_{v0}, \\ \mathcal{A}_{v0} &= -i\omega_u \left(d_2 - \frac{k}{\omega_u} \tilde{d}_0 \right). \end{aligned} \quad (19)$$

e. The $n = 1$ dispersion relation

We now proceed to the $n = 1$ solution. Its dispersion relationship consists of two parts: one for ω and the other for ξ_1 . We begin with ω . Substitution of the solution ansatz $V(y) = ye^{-\xi_1 y^2}$ into Eq. (6) yields the dispersion relation $\mathcal{A}_1(\omega, k)$ linking ω and k :

$$\mathcal{A}_1(\omega, k) = 2a_1^2 + a_1 d_0 - d_0^2 - 9a_0 d_2 = 0. \quad (20)$$

The algebraic structure of Eq. (20) is similar to Eq. (36) in FR17. Equation (20) is formally an eighth-order polynomial for ω , similar to those found in Fuchs and Raymond (2005, 2017) and Emanuel (2020). No analytic solution exists in general for such high-order algebraic equations.

The difficulty with this dispersion Eq. (20) $\mathcal{A}_1 = 0$ is that multiple wave modes at various frequencies are mixed together with (sometimes nonlinear) interactions among the parameters, making it very difficult to isolate the modes of interest and understand their parameter dependences. One way to simplify the system is to apply the longwave approximation, which eliminates the gravity wave mode and reduces Eq. (20) to fourth order. An accessible form may be obtained by using $\tilde{d}_0$ [Eq. (11)] and Eq. (19) in Eq. (20) and rearranging to obtain

$$\tilde{\mathcal{A}}_1(\omega, k) = \frac{2}{9} \left(a_1 - 2 \frac{k}{\omega_u} a_0 \right)^2 - i \frac{a_0}{\omega_u} \mathcal{A}_{v0}(\omega, k) = 0, \quad (21)$$

where $\tilde{\mathcal{A}}_1(\omega, k) = 0$ is the longwave approximation to $\mathcal{A}_1(\omega, k) = 0$ [Eq. (20)]. The second term on the right-hand side of (21) is closely related to the dispersion relation for the $v = 0$ mode (18), and is similarly independent of Q^0 . The first term has a quadratic dependence on ω , as both a_0 and $(k/\omega_u)a_1$ are linear functions of ω such that

$$\begin{aligned} a_1 - 2 \frac{k}{\omega_u} a_0 &= [2ik + Q^0 \alpha (1 + r)] \omega - 2k \alpha \Gamma_e \\ &\quad + [i(\mathcal{E} - \overline{Q}_x) + i\alpha_u Q^0] \alpha (1 + r). \end{aligned}$$

The above equation isolates the term containing Q^0 so that we can see it as a perturbation to the dispersion relation for the $v = 0$ mode. Another way to look at $\mathcal{A}_1$ is to associate the two individual terms with different meridional structures. As will become clear in the next subsection [Eq. (26)], algebraically, the first part is the coefficient for the structure $y^2 e^{-\xi_1 y^2}$ for ϕ , and the second term is the coefficient of the structure $e^{-\xi_1 y^2}$. If the former is set to zero, one nearly recovers the $v = 0$ dispersion relation. These results indicate that the $v = 0$ and $n = 1$ modes belong to the same wave family and share the same moist instability mechanisms.

Equation (21) also suggests a geometric interpretation of the $\mathcal{A}_1(\omega, k) = 0$ equation: the solution represents the intersection between the hyperplane defined by the first term on the right-hand side and the hyperplane of ω defined by the $v = 0$ mode, the latter of which has no explicit dependence on Q^0 . The $v = 0$ hyperplane may be simplified to obtain lower-order approximations to Eq. (21), which leads to lower-order algebraic equations for ω . To the leading order, we may let $\mathcal{A}_{v0} = 0$, such that $a_1 - 2(k/\omega_u)a_0 = 0$, whose left-hand side is a linear function of ω [see discussion in the preceding paragraph and Eqs. (7) and (8)]. Let r , α_u , and α_ϕ be zero for the sake of argument (these parameters strongly influence the growth rate but only weakly influence the frequency), and solve for ω :

$$\omega = \frac{i(\mathcal{E} - \overline{Q}_x) \alpha - k \Gamma}{\alpha Q^0 - ik}. \quad (22)$$

The values for $\text{Re}(\omega)$ obtained from this equation fall well in the intraseasonal range for values of k representing planetary spatial scales. Further neglecting Q^0 and Γ leads to the first-order approximation $\mathcal{A}_{v0}(\omega, k) = 0$ as derived in FR17 [their Ω_1 , one line below Eq. (32)].

Higher-order or global asymptotic approximations are possible and may provide further insight, but we do not pursue these in the present study. In practice, we solve ω in Eq. (20) for each wavenumber k numerically as in previous studies. We will return to physical interpretation by considering the relationship of the $n = 1$ mode to two simpler modes in section 5. The numerical solutions are constrained by the boundary condition in Eq. (12).

The second part of the dispersion relation is for the shape factor ξ_1 , which may be written

$$\xi_1 = \frac{a_1 + d_0}{6a_0}. \quad (23)$$

The real part, $\text{Re}(\xi_1)$, is the squared decay scale away from the equator, while the imaginary part, $\text{Im}(\xi_1)$, determines the solution's meridional scale and (for a given k) its tilt in the zonal direction. The term ξ_1 may also be simplified using the longwave approximation [$\tilde{d}_0 = a_1 - (k/\omega_u)a_0$]:

$$\xi_1 = \frac{[i(\mathcal{E} - \overline{Q}_x) + \omega_u Q^0] \alpha(1+r)}{3\omega_u(\Gamma_e - i\omega)} - \frac{k}{6\omega_u}. \quad (24)$$

The expression (24) offers some clues regarding the dependence of ξ_1 on Q^0 [which can be traced back to the term $\omega_u Q^0 y(\partial V/\partial y)$ in Eq. (6)]. To the extent that variations of ω with Q^0 are of secondary importance compared to the direct dependence in the numerator, Eq. (24) indicates that increasing Q^0 leads to increases in both $\text{Im}(\xi_i)$ and $\text{Re}(\xi_i)$, thus directly influencing the horizontal tilt and the meridional scale. This result has important consequences, as will be discussed and verified later in section 4 (Figs. 4c,d).

f. Detailed $n = 1$ solution

In general, ω and ξ_1 are obtained by solving the dispersion relation, $\mathcal{A}_1(\omega, k) = 0$, at each k and n . The complete $n = 1$ solution is written as follows:

$$v(x, y, t) = C y e^{-\xi_1 y^2} e^{i(kx - \omega t)}, \quad (25a)$$

$$\begin{aligned} \phi(x, y, t) = \mathcal{A}_{10}^{-1} \left\{ \omega_u (G - i\omega) \frac{\partial v}{\partial y} \right. \\ \left. + [-k(G - i\omega) + iL + \omega_u Q^0 \alpha(1+r)] y v \right\}, \end{aligned} \quad (25b)$$

$$u(x, y, t) = \frac{i}{\omega_u} y v + \frac{k}{\omega_u} \phi, \quad (25c)$$

$$w(x, y, t) = \frac{L}{G - i\omega} u - \frac{\omega_u \omega_\phi}{G - i\omega} \phi, \quad (25d)$$

$$q(x, y, t) = \frac{i\omega_\alpha \phi + w}{\alpha(1+r)}, \quad (25e)$$

where C is an arbitrary constant, $L = (\mathcal{E} - \overline{Q}_x) \alpha(1+r)$, $G = \Gamma_e \alpha$, and $w = -(u_x + v_y)$ is convergence. The expression $\mathcal{A}_{10}$ (lhs of the $v = 0$ dispersion relation) appears in the denominator of Eq. (25b) and is generally nonzero. The meridional structures under the longwave approximation are formally the same as the above Eqs. (25a)–(25e), except that ω is obtained from the dispersion relation $\mathcal{A}_1 = 0$ [Eq. (21)]. Only the real parts of the physical variables in the solutions are considered meaningful. The dispersion equation for ω [(20) and (21)] is unpacked using the symbolic operation “coeffs” and solved via the “roots” function in Matlab. The numerical solutions are cross-validated by balancing the budgets of all the variables (at $t = 0$) in Eqs. (1)–(4) and (6) to numerical precision; that is, the budget of each variable is numerically closed.

For $\phi(x, y, t)$, it is instructive to expand v and $\partial v/\partial y$ and rewrite their meridional structures [shown inside the curly bracket of Eq. (25b)] as follows:

$$\begin{aligned} \phi &\propto \omega_u (G - i\omega) e^{-\xi_1 y^2} \\ &+ [-k(G - i\omega) + iL + \omega_u Q^0 \alpha(1+r) - 2\xi_1 \omega_u (G - i\omega)] y^2 e^{-\xi_1 y^2} \\ &= a_0 e^{-\xi_1 y^2} + \left(a_1 - \frac{k}{\omega_u} a_0 - 2\xi_1 a_0 \right) y^2 e^{-\xi_1 y^2} \\ &\cong a_0 e^{-\xi_1 y^2} + \left(a_1 - 2\frac{k}{\omega_u} a_0 \right) \frac{1}{3} y^2 e^{-\xi_1 y^2}. \end{aligned} \quad (26)$$

The longwave approximation [using Eqs. (19) and (23)] is used to arrive at the last expression. The above equation breaks down $\phi(y)$ into two parts. The first term on the RHS, proportional to $e^{-\xi_1 y^2}$, is trapped near the equator and “Kelvin-like.” The second part, proportional to $y^2 e^{-\xi_1 y^2}$, maximizes in the subtropics and is “Rossby-like.” The “Kelvin–Rossby” ratio may be defined using the corresponding coefficients:

$$RK = \frac{a_1}{a_0} - \frac{k}{\omega_u} - 2\xi_1 \cong \frac{a_1}{3a_0} - \frac{2k}{3\omega_u}. \quad (27)$$

Equation (11) for $\tilde{d}_0$ is used to obtain the second expression, whose RHS is similar to that of $\xi_1 = (a_1/3a_0) - (k/6\omega_u)$. By plugging in a_1 and a_0 , we can see that the influence of Rayleigh (i.e., momentum) damping on the ratio RK is readily apparent, while the Newtonian cooling α_ϕ does not enter a_0 or a_1 , and has no direct influence on it.

3. Parameter estimates for moist physics

The values of the eight free parameters in this linear system— α , r , Γ , $\mathcal{E}$, α_u , α_ϕ , $\overline{Q}_x$, and Q^0 —will be discussed in this section. The moisture relaxation time scale is chosen to be one day (see section d in the appendix), corresponding to a nondimensional value $\alpha \sim 0.35$, while the cloud-radiative feedback parameter $r = 0.17$. These two parameter values are the same as in FR17. The gross moist stability $\Gamma = 0.16$; the Rayleigh friction parameter, $\alpha_u = 0.03$, or a damping time scale of 10 days; and the Newtonian cooling parameter $\alpha_\phi = 0.01$, or a time scale of 30 days. The latter two parameters are important to the spatial pattern in dry dynamics (Wu et al. 2001). These parameter values (or similar values) have been used in the past (Fuchs and Raymond 2005; Sobel and Maloney 2013; Adames and Kim 2016; Fuchs and Raymond 2017).

Next, we estimate the WISHE parameter ε and zonal moisture gradient $\overline{Q}_x$ over the Indian Ocean, where both the MJO and BSISO most frequently initiate. The large-scale overturning circulation is the Walker cell in this region. Accordingly, the low-level zonal winds are westerly (eastward), and moisture increases eastward (i.e., there is a positive zonal moisture gradient $\overline{Q}_x$). As a consequence, WISHE and zonal advection oppose each other in this region: a westerly wind anomaly increases surface turbulent fluxes but also increases the advection of dry air, whereas an easterly wind anomaly decreases surface fluxes but decreases the advection of dry air (equivalently, it advects in anomalously moist air). Using the ERA5 dataset (Hersbach et al. 2020), we compute zonal moisture advection and use surface fluxes from ERA5, and fit linear regressions between these quantities and zonal wind at 850

FIG. 1. (a) Column-integrated zonal MSE advection (Hadv) vs U850 (zonal winds at 850 hPa). (b) Surface latent heat fluxes (LH) vs U850 using the ERA5 dataset. All three variables are averaged over 70°–90°E, 10°S–10°N. A 20–90-day bandpass filtering is applied to all the time series. The gray bars indicate the mean values.

hPa, respectively, in the equatorial Indian Ocean. All three quantities are first averaged within 10°S–10°N, 70°–90°E, and bandpass-filtered to retain 20–90-day periods to reduce noise.

Figure 1 shows that column-integrated zonal advection of MSE [corresponding to $-u\overline{Q}_x$ in Eq. (4)] is negatively correlated with zonal wind with a regression slope $\sim -4.4 \text{ W m}^{-2} (\text{m s}^{-1})^{-1}$, while surface fluxes are positively correlated with zonal wind with a regression slope $\sim 2.1 \text{ W m}^{-2} (\text{m s}^{-1})^{-1}$. The R^2 values are not high (0.27, 0.16), reflecting the limitation of linearization of a highly nonlinear problem. Estimates of the linear slopes vary if different averaging areas are used, but the opposing effect of the two processes is robust. This opposing effect is also present in Wang and Li (2020), who computed the MSE budget of a reanalysis dataset and showed in their Fig. 4 that zonal MSE advection contributes positively to the total tendency, while surface fluxes contribute negatively, and the former is larger in magnitude. One caveat of this analysis in Fig. 1 is that the bandpass-filtered zonal advection (uq_x) includes contributions from both linear and nonlinear terms. No attempt is made to separate high-frequency nonlinear contributions from $u\overline{Q}_x$, the zonal advection of mean moisture by perturbation zonal wind. Alternatively, Q_x may also be estimated from the climatological mean total water vapor field over the Indian Ocean; column moisture increases by an amount $\Delta Q \sim 4 \text{ mm}$ (or kg m^{-2}) over approximately $\Delta L = 20^\circ$ in longitude. This corresponds to -5 W m^{-2} per unit m s^{-1} [as $L_v(\Delta Q/\Delta L)$] for $-\overline{Q}_x$, similar to the linear regression estimate. The nondimensional value is

$$-\overline{Q}_x \approx -0.05 \left(\frac{\Delta Q}{\Delta L} \frac{1}{\sqrt{\beta c}} \frac{1}{\langle \Lambda_q \rangle} \frac{1}{c^2/H_0} \frac{L_v g}{c_p \Theta} \right)$$

(see section a of the appendix).

According to the estimates above, we use a nondimensional value $\mathcal{E} \sim 0.03$ for the WISHE parameter, and $-\overline{Q}_x \sim -0.06$ for the zonal gradient. Together, we have $\mathcal{E} - \overline{Q}_x$ varying from -0.03 to -0.02 . A moist shallow water model with zero values for $\overline{Q}_x$, negative $\mathcal{E}$ has been explored in FR17. Many

previous studies (e.g., Emanuel 1987; Yano and Emanuel 1991; Adames and Kim 2016; Fuchs and Raymond 2005, 2017) use a negative WISHE parameter, arguing that the time-mean zonal wind at low levels is easterly, either in the Pacific Ocean, or in the zonal mean (Sentić et al. 2020). Since the MJO and BSISO both frequently grow in amplitude over the Indian Ocean and decay over the Pacific, we consider the mean state over the Indian ocean to be most relevant to understanding the growth of both phenomena. We also consider them to be moisture modes locally there, with the MJO in particular transitioning to a more Kelvin wave-like structure as it moves over the Pacific (Sobel and Kim 2012). We choose our parameters accordingly, assuming low-level westerlies, such that an easterly perturbation wind moistens by zonal advection more than it dries by reducing the surface fluxes, so that the net effect is moistening by easterlies and drying by westerlies, as discussed above. In this we differ from Adames and Kim (2016), who appear to have assumed mean easterlies (the WISHE parameter C_u is negative in AK16; see their Fig. 2 caption). The assumed negative sign of the difference ($\mathcal{E} - \overline{Q}_x$) here is the key to the large-scale instability in our model. If this quantity is positive, there is no planetary-scale eastward-propagating unstable mode, as discussed further below.

The meridional moisture gradient parameter Q^0 is more difficult to estimate. The term $\overline{Q}_y$ varies with latitude, and has considerable seasonal variability. It is negative in boreal winter and positive in boreal summer, when the moisture maximum is located around 20°N. A rough estimate is that column moisture decreases by about 10 mm over the 10° of latitude between 10° and 20°N in boreal winter (a range centered around nondimensional latitude $y = 1$, corresponding to 15 degrees), which gives a value of $Q^0 \sim 0.2$ – 0.3 , which is 4–6 times larger than $\overline{Q}_x$. During boreal summer, Q^0 varies between ~ -0.05 and zero, typically being 5–10 times smaller than its value in northern winter, but making the meridional moisture gradient comparable to the zonal one. Sensitivity of the solutions to Q^0 will be explored in the next section.

FIG. 2. (a),(d) Frequency, (b),(e) growth rate, and (c),(f) zonal phase speed for the $v = 0$ and $n = 1$ mode with no meridional moisture gradient, for (left) $Q^0 = 0$ and (right) $Q^0 = 0.22$. Other parameters are $\alpha = 0.35$, $\alpha_\phi = 0.01$, $\alpha_u = 0.03$, $\mathcal{E} - \overline{Q}_x = -0.03$, $r = 0.17$, and $\Gamma = 0.16$.

4. Results

a. The $n = 1$ solutions and meridional moisture gradient

The $n = 1$ mode has between one and three physical solutions at different zonal wavenumbers k (see full spectrum in Fig. A2). The left column of Fig. 2 shows the intraseasonal branch of the dispersion relation for the $n = 1$ [Eq. (20)] and $v = 0$ [Eq. (18)] modes with zero meridional moisture gradient $Q^0 = 0$. The other parameter values are $\alpha = 0.35$, $\alpha_\phi = 0.01$, $\alpha_u = 0.03$, $\mathcal{E} - \overline{Q}_x = -0.03$, $r = 0.2$, and $\Gamma = 0.16$. The full spectrum of the wave solution is given in the appendix

(Fig. A2). The maximum growth rate is found at planetary wavenumbers due to negative $\mathcal{E} - \overline{Q}_x$ as mentioned above. The zonal phase speed is 8.2 m s^{-1} for $k = 1$, broadly consistent with the observed zonal propagation phase speed of the ISOs. The $n = 1$ mode has higher growth rates than does the $v = 0$ mode, except at very small wavenumbers $k \leq 0.5$. Both become damped for short waves for k greater than about 1 for the $v = 0$ mode or 2 for the $n = 1$ mode. The two modes share very similar phase speeds for the range of k displayed in Fig. 2c, and their frequencies are similar at low wavenumbers $k \leq 1$. The $v = 0$ mode has a phase speed $\sim 8 \text{ m s}^{-1}$, which is

FIG. 3. Spatial pattern of the $n = 1$ solution with different values of the magnitude of the meridional moisture gradient Q^0 at zonal wavenumber $k = 1$. Perturbation pressure ϕ (blue and red contours); moisture q : black; solid, $0.7 \times \max_q$ (maximum value of q); dashed, $-0.7 \times \max_q$. Horizontal wind vectors are in gray. Units are arbitrary; 1.5 zonal waves are shown for clarity. Parameters are as in Fig. 2 except for the different values of Q^0 .

much slower than FR17 ($\sim 16 \text{ m s}^{-1}$) due to the opposing effects of the zonal moisture gradient and WISHE. Aquaplanet numerical modeling by Shi et al. (2018) and Khairoutdinov and Emanuel (2018) simulated similar modes, although no mean zonal moisture gradient was present in these numerical simulations.

Analogous results for $Q^0 = 0.22$, a value representative of the meridional moisture gradient at $y = 1$ during northern winter (and 4–5 times the zonal gradient), are displayed in the right column of Fig. 2; the other parameters are the same as for the $Q^0 = 0$ solutions shown on the left. For $k \geq 2$, the $n = 1$ solutions are considered unphysical and not shown, as $\text{Re}(\xi_1)$ turns negative with equatorward propagation ($c_y < 0$), and the solution grows with y , which is consistent with the Eq. (24) that increasing k leads to smaller $\text{Re}(\xi_1)$. For smaller k , the $n = 1$ mode's growth rate is greater than that of the $v = 0$ mode. Comparing the two cases indicates that the meridional moisture gradient further contributes to planetary-scale

instability, but stabilizes higher zonal wavenumbers. The zonal phase speed at $k = 1$ is 10.8 m s^{-1} , about 20% higher than for $Q^0 = 0$. The frequency–wavenumber relationship is different for the two values of Q^0 ; for $Q^0 = 0.22$, the frequency of the $n = 1$ mode increases approximately linearly with wavenumber, whereas for $Q^0 = 0$ it varies only weakly below $k = 2$ and decreases with k thereafter, staying closer to the $v = 0$ mode.

Figure 3 shows the spatial structures of the pressure, moisture, and wind fields associated with the $n = 1$ mode at $k = 1$ for six values of Q^0 . When $Q^0 = -0.05$ this represents the positive meridional moisture gradient at $y = 1$ during northern summer, similar in magnitude to the zonal gradient but small compared to the meridional gradient in northern winter. The structures in ϕ and u are symmetric about the equator, and display northwest–southeast tilt for $y > 1$, and southwest–northeast tilt for $y < -1$, under small positive or negative Q^0 values (Figs. 3a,b). We suggest that this pattern

FIG. 4. (a) Growth rate, (b) zonal phase speed, (c) meridional scale $[\text{Re}(\xi_1)]^{-1/2}$, and (d) local meridional wavenumber l_y at $y = 1$ for the six cases shown in Fig. 3.

represents an idealized version of the BSISO without hemispheric asymmetry. Analysis of the moist static energy budget will be further discussed to support this claim in the next subsection. As Q^0 further increases, the swallowtail-like wave pattern (Zhang and Ling 2012; Kim and Zhang 2021) emerges (Figs. 3c,d). Large Q^0 (0.22) leads to nearly a neutral wave pattern at $|y| = 2$ [where by “neutral” we mean $\text{Im}(\xi) = 0$ as in Matsuno’s solutions], with cyclonic gyres straddling the equator, reminiscent of the classical Gill response to steady equatorial heating. The cyclonic gyres tilt farther eastward for even larger Q^0 (Fig. 3f). No zonal wave $k = 1$ solution exists for $Q^0 > 0.54$. Emanuel (2020; see Fig. 3 therein) also showed an MJO-like neutral wave pattern. This shape can be traced back to the WISHE term $[\alpha \text{ in } a_2 \text{ of Emanuel’s Eq. (9)}]$ in the temperature equation [Emanuel’s Eq. (4)], which differs from our Q^0 parameter. Regardless of this difference, both results indicate that it is a term of the form $\omega y(\partial V/\partial y)$ that changes the shape factor in the ξ_1 [Eq. (24)], leading to the MJO-like neutral wave pattern.

Figure 4 shows the growth rate, zonal phase speed, meridional decay scale, and meridional wavenumber at $y = 1$ for zonal wave $k = 1$ as a function of Q^0 over the range discussed above. The growth rate (Fig. 4a) varies nonmonotonically with Q^0 , with a maximum at $Q^0 = 0.15$. The zonal phase speed (Fig. 4b) increases nearly linearly with Q^0 from 8 to

12 m s^{-1} over the range shown. Figure 4c shows the meridional scale $L_y = [\text{Re}(\xi_1)]^{-1/2}$, which varies from 2.2 at $Q^0 = -0.05$ to 1.62 at $Q^0 = 0.3$, indicating that the solution becomes increasingly trapped near the equator as Q^0 increases. The changing wave tilt with Q^0 corresponds to the variations l_y/k . Figure 4d shows that the local meridional wavenumber l_y at $|y| = 1$ decreases from 2.8 to 0.3 as Q^0 increases from -0.05 to 0.3. The neutral wave solution is that for which $l_y = -2\text{Im}(\xi_1) = 0$, whose structure resembles Matsuno’s neutral Rossby wave solution but with opposite propagation direction in longitude, due to the meridional moisture advection. For the parameters in Fig. 3, this neutral wave is achieved at $Q^0 \approx 0.23$. These changes of ξ_1 with respect to Q^0 are consistent with the prior discussion of Eq. (24) (section 3e).

The spatial patterns associated with these $n = 1$ solutions are associated with the two terms in Eq. (26) for ϕ , proportional to $e^{-\xi_1 y^2}$ and $y^2 e^{-\xi_1 y^2}$, respectively. The former is trapped near the equator, while the latter vanishes there and maximizes at $y = \text{Re}[(2\xi_1)^{-1/2}]$ (1.22 for $Q^0 = 0.22$ as in Fig. 3e). These two structures correspond to the spatial patterns in two portions of the domain shown in Fig. 3: the deep tropics (e.g., $|y| < 0.5$), where $v \approx 0$, such that the $v = 0$ mode provides a reasonable approximation; and the subtropics (say, $|y| > 1$), where the structure is plane wave-like, except that

FIG. 5. Time–latitude diagram of low-level convergence for $Q^0 = -0.05, 0.1$, and 0.22 .

meridional wavenumber decreases with y . Variations in Q^0 mostly affect the solution at $|y| > 1$, but not at $|y| < 0.5$ due to the relatively small values of v and meridional moisture gradient $|yQ^0|$ near the equator as expected.

The BSISO-like tilted structures found at large Q^0 have relatively fine meridional scale compared to those of the more MJO-like structures found at small Q^0 . This might be the reason that some other theoretical models have captured the MJO-like pattern but not the BSISO-like pattern: the meridional resolution used in those models, typically $n = 1$ – 3 in the expansion in parabolic cylinder functions (e.g., Gill 1980), is inadequate to resolve the finer meridional structures of the BSISO, whereas these arise naturally here even for $n = 1$ due to the complexity of ξ .

Figure 5 shows the time–latitude diagram of convergence for $k = 1$. The constant C in Eq. (25a) is chosen to be unity for simplicity. The poleward propagation is symmetric about the equator, and its speed varies as a function of latitude [cf.

FIG. 6. Various terms in the zonal momentum budget for (a) $Q^0 = -0.05$ and (b) $Q^0 = 0.22$ at $t = 0$. All quantities are averaged between $y = 1.45$ and 1.55 .

Eq. (16)]. A bulk estimate of this meridional phase speed is ~ 50 – 100 km day^{-1} between Y (distance from the equator) = 1000 and 2000 km when $Q^0 = -0.05$ (Fig. 5a) and 0.1 (Fig. 5b). This value is broadly consistent with observations of the BSISO (e.g., Jiang et al. 2004; Wang et al. 2018), as is the poleward propagation in both hemispheres, as shown in Waliser et al. (2009, their Fig. 6). The poleward speed is faster for higher Q^0 . Wave phase is established at nearly the same time for all latitudes for $Q^0 = 0.22$ (Fig. 5c).

b. Budgets of zonal momentum and moist static energy

In addition to its effect on the wave morphology, the magnitude of the meridional moisture gradient has a strong influence on the dominant zonal momentum balance. Figure 6 shows the zonal moment budget around $y = 1.5$. For small negative $Q^0 = -0.05$, all four terms in Eq. (1)—tendency, Coriolis force, pressure gradient, and friction—have similar magnitudes. For large Q^0 (0.22), the familiar geostrophic balance is struck between the Coriolis and pressure gradient forces, while the tendency and friction terms are small. This indicates that the use of the quasi-steady state assumption for u (Gill 1980) in several MJO theories (Sobel and Maloney 2012, 2013; Adames and Kim 2016) is appropriate.

The moist static energy (MSE) budget is further used to diagnose the roles of diabatic processes in the solutions (e.g., Andersen and Kuang 2012). Combining Eqs. (3) and (4) to

FIG. 7. Projection of different terms in the moist static energy (MSE) budget terms to the MSE tendency for (a) $Q^0 = -0.05$ and (b) $Q^0 = 0.22$. The six terms are Wadv, vertical advection; Uadv, zonal advection; E, surface fluxes; CRF, cloud-radiative feedback; Vadv, meridional advection; and damping in Eq. (28). The sum of these six terms equals 1 due to normalization: $\|\partial \text{MSE} / \partial t\| = 1$. Results are shown separately for $|y| < 0.5$ and $1 < |y| < 2$.

eliminate precipitation (αq) yields the equation for MSE, defined as $h = q - \phi$:

$$\frac{\partial h}{\partial t} = \Gamma \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\mathcal{E} - \overline{Q}_x \right) u + \alpha r q + v y Q^0 + \alpha_\phi \phi. \quad (28)$$

The terms on the right-hand side of Eq. (28) are vertical advection, surface fluxes, zonal advection, cloud-radiative feedback (CRF), meridional advection, and damping, respectively. Projection of these terms onto the tendency (left-hand side) may be used to quantify the contributions of individual processes in Eq. (28):

$$\frac{\left\langle \text{MSE tendencies}, \frac{\partial \text{MSE}}{\partial t} \right\rangle}{\left\langle \frac{\partial \text{MSE}}{\partial t}, \frac{\partial \text{MSE}}{\partial t} \right\rangle}, \quad (29)$$

where $\langle \cdot, \cdot \rangle = \iint dx dy$ denotes the inner product (here only; elsewhere angle brackets denote the mass-weighted vertical integral). We take the ratio $-u \overline{Q}_x / \mathcal{E} u = 2$ (Fig. 1). Figure 7 shows the MSE terms projected onto the tendency for

$Q^0 = -0.05$ and 0.22 in the deep tropics, $|y| < 0.5$; in the subtropics, $1 < |y| < 2$. Within the deep tropics, for both $Q^0 = -0.05$ and 0.22 , zonal moisture advection dominates the other terms, contributing $\sim 200\%$ of the tendency, with the second largest term being surface fluxes, which contribute negatively (-100%). The major difference between the two Q^0 cases is in the subtropics, where zonal moisture advection contributes more than 150% to the eastward propagation for $Q^0 = -0.05$ (Fig. 7a), opposed by other processes, among which surface fluxes dominate. This leading role for zonal advection is consistent with the analysis by Jiang et al. (2018; cf. their Fig. 4), who showed that zonal advection of mean moisture dominates the MSE budget of the BSISO in the Indian Ocean sector in a reanalysis dataset. For $Q^0 = 0.22$, meridional advection plays an important role in the subtropics ($1 < |y| < 2$), secondary only to that of zonal moisture advection (Fig. 7b). In both cases, cloud-radiative feedback contributes slightly positively to growth rate and eastward propagation in the subtropics. Surface fluxes contribute negatively (i.e., slowing down the eastward propagation or reducing the growth rate), but the above MSE budget does not distinguish the two. Sensitivity of the MSE budget to the analysis domain is noted in Wang and Li (2020), which analyzed the MSE budget of the MJO in different domains and noted contrasting results.

To further distinguish the contributions by various processes to the growth rate and frequency/propagation of the MSE tendency, we use the following equation to diagnose these two separately:

$$\begin{aligned} \frac{1}{h} \frac{\partial h}{\partial t} &= \text{Im}(\omega) - i \text{Re}(\omega) \\ &= \frac{1}{h} \left[\Gamma \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left(\mathcal{E} - \overline{Q}_x \right) u + \alpha r q + v y Q^0 + \alpha_\phi \phi \right]. \end{aligned} \quad (30)$$

The first expression follows from the definition of the growing wave solution in the form of $h \sim \exp(-i\omega t)$. The real part of $(1/h) \partial h / \partial t$ is the growth rate $\text{Im}(\omega)$, while the imaginary part $-\text{Re}(\omega)$ denotes frequency/propagation. We thus consider the real and imaginary parts of individual terms in the second expression as relevant to the growth and frequency of oscillation, respectively. Importantly, the spatial structure drops out in $(1/h) \partial h / \partial t$, and the diagnosis of growth rate or frequency is independent of x or y , unlike the diagnostic Eq. (29).

Figure 8 shows the budgets for growth rate and frequency for $Q^0 = -0.05$ and 0.22 for zonal wave $k = 1$, normalized by $\text{Im}(\omega)$ and $-\text{Re}(\omega)$, respectively. The dominant contribution of CRF to the growth rate is borne out for both Q^0 cases (Figs. 8a,c), while vertical advection acts as the leading sink in the growth rate budget. The large contribution by the two processes is due to their phasing relative to the MSE, moisture, and precipitation anomalies; CRF is in phase with precipitation as it is expressed as proportionality to precipitation, while vertical advection is out of phase with precipitation. For frequency and propagation (Figs. 8a,c), zonal advection and surface fluxes oppose each other, while the former dominates the contribution to the frequency and propagation budgets.

FIG. 8. Budget of (left) growth rate and (right) frequency derived from the MSE equation for (a),(b) $Q^0 = -0.05$ and (c),(d) $Q^0 = 0.22$. The six terms are Wadv, vertical advection; Uadv, zonal advection; E, surface fluxes; CRF, cloud-radiative feedback; Vadv, meridional advection; and damping in Eq. (28). The sum of these six terms equals 1 due to normalization.

Meridional advection becomes important for $Q^0 = 0.22$, only secondary to CRF, the leading contributor.

c. Sensitivity to parameters $\mathcal{E} - \overline{Q}_x$, Γ , and α

The parameter space has eight dimensions, a number sufficiently large as to make it impossible to fully explore the space. Here, we only examine parameter sensitivity within limited ranges for $\mathcal{E} - \overline{Q}_x$, Γ , and α by changing one at a time. The sensitivity tests below give permissible ranges of the parameters, and these are entirely empirical at this stage.

Figure 9 shows the sensitivity of the $n = 1$ and $v = 0$ modes using the same control wave parameters as in Fig. 4 to the parameter that combines WISHE and the zonal moisture gradient, $\mathcal{E} - \overline{Q}_x$, for zonal wave $k = 1$. For different values of Q^0 case, and considering the $v = 0$ solution, growth rate (Fig. 9a) and zonal phase speed (Fig. 9b) both decrease as $\mathcal{E} - \overline{Q}_x$ increases from -0.09 to 0 , with the growth rate reaching zero at the upper end of that range, illustrating the key role of this parameter. A value of -0.09 was used in FR17 (using different assumptions than here, in particular considering only WISHE, under a background state of mean easterly winds),

and the phase speed we obtain for that value (17 m s^{-1}) is close to their result, although the interpretation is quite different. The shape factor ξ_1 also varies with $\mathcal{E} - \overline{Q}_x$. Figure 9c shows that the meridional decay scale decreases with that parameter, while the local meridional wavenumber $l_y|_{y=1}$ (Fig. 9d) is positive and increases for $Q^0 = 0$ over the range $(-0.09, 0)$ for $\mathcal{E} - \overline{Q}_x$. As $\mathcal{E} - \overline{Q}_x$ approaches 0 , rapid decrease in L_y and increase in $l_y|_{y=1}$ occur. For $Q^0 = 0.22$, it decreases from 1.2 to near zero at $\mathcal{E} - \overline{Q}_x = -0.035$. Solutions no longer exist for $\mathcal{E} - \overline{Q}_x$ less than 0.02 .

Figure 10 shows the sensitivity of the solutions to effective gross moist stability Γ_e by changing Γ only. For $\Gamma_e < -0.05$, the most unstable mode is short waves instead of planetary wavenumbers (e.g., 0.5 or 1). For $Q^0 = -0.05$, growth rate of the $n = 1$ and $v = 0$ solutions generally decreases with positive Γ_e (Fig. 10a), while it is relatively flat for $Q^0 = 0.22$. Phase speed increases with Γ_e (Fig. 10b). The meridional scale (Fig. 10c) increases significantly with positive Γ_e , and decreases with negative Γ_e . The term $l_y|_{y=1}$ increases with Γ_e .

Figure 11 shows the sensitivity to the convective rate constant α from 0.2 to 0.75 , corresponding to a time scale varying

FIG. 9. Sensitivity of wave parameters to the parameter representing the net effect of zonal moisture advection and WISHE, $\mathcal{E} - \bar{Q}_x$, for zonal wave $k = 1$. Parameters shown are (a) growth rate, (b) zonal phase speed, (c) meridional scale L_y , and (d) local meridional wavenumber l_y at $y = 1$; L_y and l_y are undefined for the $v = 0$ solution.

from 1.7 days to 0.5 days. The growth rate (Fig. 11a) decreases with α while the phase speed increases with α for both the $n = 1$ and $v = 0$ solutions when $Q^0 = 0$, but both vary little with α when $Q^0 = 0.22$. The meridional decay scale increases with α for both Q^0 cases, while $l_y|_{y=1}$ increases gradually within the range where solutions exist (Fig. 11d).

5. Matching between the $v = 0$ mode and f -plane moisture wave

As discussed above, the algebraic structures of the $n = 1$ mode and their associated spatial patterns have two components, one dominant near the equator and the other in the subtropics. This suggests that some understanding might be gained by considering local approximations that represent these two components separately, and then considering under what conditions these might combine to form a single mode. We consider the $v = 0$ mode near the equator; for higher latitudes, say $|y| > 1$, we consider the modes that would exist on an f plane. The spatial patterns of the $v = 0$ and the f -plane wave modes are shown in the appendix (Figs. A3 and A4). We will argue that the structure of the $n = 1$ mode can be thought of as a consequence of phase speed matching between these two more local modes.

The dispersion relation of the $v = 0$ mode is given by Eq. (18). We consider the moisture mode under the f -plane

approximation below. We write the plane wave form solution as $\exp[i(kx + ly - \omega t)]$, where k and l are the nondimensional zonal and meridional wavenumbers. The f -plane moisture wave mode was considered by Sobel et al. (2001) with different assumptions regarding surface fluxes and moisture advection. Using the f -plane approximation and assuming a constant meridional moisture gradient leads to several changes to Eqs. (1)–(5). Replacing the Coriolis terms in Eqs. (1) and (2) with constant f , $y(u, v) \mapsto f(u, v)$, and replacing meridional advection in Eq. (5), $vyQ^0 \mapsto vfQ^0$, so that the meridional moisture gradient (Q^0) is assumed to vary proportionally with f . Setting the time tendency in (2) to zero under the longwave approximation leads to $d_0 = 0$. Using $\partial/\partial y = il$, the dispersion relation is written in a form similar to Eq. (6):

$$-a_0 l^2 + ia_1 f l + f^2 d_2 = 0, \quad (31)$$

where a_0 , a_1 , and d_2 share the same forms as in Eqs. (7), (8), and (10). Equation (31) is a quadratic equation for ω . Defining $\tilde{l} = l/f$, it can be expanded as

$$\begin{aligned} & [i(\tilde{l}^2 + 1)]\omega^2 + \left[-\tilde{l}^2(\alpha\Gamma_e + \alpha_u) + i\tilde{l}Q^0\alpha(1+r) - \alpha - \alpha_\phi\right]\omega \\ & - i\tilde{l}^2\alpha_u\alpha\Gamma_e - [\tilde{l}(\mathcal{E} - \bar{Q}_x + \alpha_u Q^0) - kQ^0]\alpha(1+r) - i\alpha\alpha_\phi = 0. \end{aligned} \quad (32)$$

FIG. 10. Sensitivity to effective gross moist stability Γ_e for zonal wave $k = 1$: (a) growth rate, (b) zonal phase speed, (c) meridional scale, and (d) local meridional wavenumber l_y at $y = 1$.

Setting three parameters that mostly affect the growth rate (but have little effect on phase speed) to zero (i.e., $\alpha_u = \alpha_\phi = r = 0$) leads to

$$[i(\tilde{l}^2 + 1)]\omega^2 + [-\tilde{l}^2\alpha\Gamma + i\tilde{l}Q^0\alpha - \alpha]\omega - [\tilde{l}(\mathcal{E} - \overline{Q_x}) - kQ^0]\alpha = 0. \quad (33)$$

We consider approximate solutions in two limits: $Q^0 = 0$ and large Q^0 . For small k , we can obtain a good estimate of the frequency by ignoring the ω^2 term in the above to obtain

$$\omega \sim -\frac{\tilde{l}(\mathcal{E} - \overline{Q_x})}{(\tilde{l}^2\Gamma + 1)} \quad \text{for } Q^0 = 0, \quad (34)$$

$$\omega \sim kQ^0 \quad \text{for large } Q^0. \quad (35)$$

Using $\tilde{l} = 20$ and $k = 1$, the dimensional zonal phase speed from Eq. (34) is $\sim 6.3 \text{ m s}^{-1}$, which is a reasonable first-order approximation to the phase speed of the $n = 1$ solution. Equation (35) then gives the approximate phase speed for $k = 1$ as $\omega/k = Q^0$, corresponding to dimensional phase speed $\sim 0.22c$, or 11 m s^{-1} , which agrees well with dispersion relationship (Fig. 2). The dispersion equation (35) is nondispersive, which seems to agree with the relationship in Fig. 2d.

We again consider the two cases $Q^0 = -0.05$ and $Q^0 = 0.22$. We do not expect the meridional wavenumber of the f -plane

waves to match those of the beta plane solution (due to complete lack of constraint imposed by spherical geometry). Instead, we ask under what conditions the f -plane and $v = 0$ modes have the same phase speeds, and in particular where the phase speed of the f -plane mode is relatively insensitive to the reference latitude used to choose the value of f so that the matching is robust to this somewhat arbitrary choice, as follows. If coupling between the $v = 0$ and the f -plane moisture wave is a useful characterization of the dynamics of the $n = 1$ mode, the latter should have larger l for $Q^0 = -0.05$ and smaller l for $Q^0 = 0.22$. The relative magnitudes of these two would be consistent with those derived from the local wavenumber using Eq. (16). Figure 12 shows the zonal phase speed for $k = 1$ as a function of meridional wavenumber l for the two Q^0 values with three different f values: 1, 1.5, and 2. Other parameters are kept the same as Fig. 2. In both Q^0 cases, the $v = 0$ solution has the same phase speed: 8.3 m s^{-1} . For $Q^0 = -0.05$, the three f waves approximately agree on the zonal phase speed for $l \sim 13$ with a zonal phase speed $\sim 7.5 \text{ m s}^{-1}$. For $Q^0 = 0.22$, $l \sim 0.5$ – 3 with a zonal phase speed ~ 12 – 13 m s^{-1} . The ratio between the values of l corresponding to these two values of Q^0 for this f -plane wave, $r_f = l|_{Q^0=-0.05}/l|_{Q^0=0.22}$, is about 1/13, or 0.0667. On the other hand, the wave morphology derived from the $n = 1$ solution [Eq. (20)] is controlled by ξ_1 [Eq. (16)] as $l_y = -2 \text{Im}(\xi_1)y$. For $Q^0 = -0.05$, $\text{Im}(\xi_1) = -0.089$; for $Q^0 = 0.22$, $\text{Im}(\xi_1) = -1.33$. The ratio of the local wavenumber l_y

Fig. 11. Sensitivity to convective time scale α for zonal wave $k = 1$: (a) growth rate, (b) zonal phase speed, (c) meridional scale, and (d) local meridional wavenumber l_y at $y = 1$. No solutions exist for larger values of α than those shown.

($r_\beta = l_y|_{Q^0=-0.05}/l_y|_{Q^0=0.22}$) is 0.067. The two ratios, r_f and r_β , agree with each other approximately, suggesting that the f -plane moisture wave approximation helps explain the wave geometry to some extent. This matching also seems to be consistent with the nonexistence of the $n = 1$ solutions at higher zonal wavenumbers for $Q^0 = 0.22$ (Fig. 2d): frequency matching is less likely as the gap between the two modes grows with zonal wavenumber k .

The $v = 0$ mode is a moisture mode (as a direct analog to the Kelvin mode) whose instability results from the net effect of WISHE and zonal moisture advection, while the f -plane moisture mode propagates eastward due to those effects as well as to meridional moisture advection (as in Sobel et al. 2001), although the latter becomes small as the meridional moisture gradient does (as is relevant in northern summer). The two share similar eastward phase speeds, hence phase speed matching between the equatorial $v = 0$ mode and the subtropical wave solution can occur. This matching is best achieved at planetary wavelengths ($k = 0.5$ or 1), and the combined $n = 1$ mode on the β plane is also most unstable at that scale.

It is useful to think about how the phase speed matching between the $v = 0$ moisture mode and the f -plane moisture wave differs from the conventional Kelvin–Rossby paradigm for the MJO. The free dry Rossby wave propagates westward while the free dry Kelvin wave propagates eastward, so the MJO cannot be thought of as resulting from an interaction between these free waves, but only their quasi-stationary

forced counterparts (e.g., Gill 1980). Here, the matching of interest is between true free (moist) modes, and this allows us to explain the relationship of the MJO to the BSISO, something to our knowledge previously not well explained in terms of Rossby and Kelvin waves.

6. Conclusions

We have presented a moist shallow water model, representing a first baroclinic vertical mode structure as is common in theoretical studies of tropical atmosphere dynamics, linearized about a resting atmosphere on an equatorial beta plane. Idealized representations of moist physics are similar to those in many past studies. We assume a basic state with a positive zonal moisture gradient and westerly low-level wind, consistent with the observed mean state over the Indian Ocean (Sobel and Maloney 2013; Ahmed 2021). The moisture distribution is assumed to vary quadratically with latitude, a form that simplifies the analysis while remaining qualitatively consistent with observations.

The behavior of this simple model is interpreted as relevant to the dynamics of the MJO and BSISO over the Indian Ocean, where both modes most frequently develop. Normal mode solutions are obtained through linear stability analysis. The natural meridional structures of the modes are fundamentally different from those of the dry system; for some parameters, the gravest meridional mode with nonzero meridional velocity produces meridional oscillations whose

FIG. 12. (a) Phase speed of the zonal wave-1 f -plane moisture wave for $Q^0 = -0.05$. (b) As in (a), but for $Q^0 = 0.22$. Three f values are used: 1 (blue), 1.5 (green), and 2 (red). Open circles indicate stable solutions; closed symbols are unstable solutions. The horizontal bar in (a) indicates the phase speed 8.2 m s^{-1} of the $n = 1$ mode at $k = 1$ for $Q^0 = -0.05$, and 10.8 m s^{-1} in (b).

wavelength decreases with latitude. Our analysis focuses on this gravest ($n = 1$) mode, as well as the mode with zero meridional velocity (broadly analogous to the Kelvin mode).

It is shown that, under the longwave approximation, the $v = 0$ and $n = 1$ eastward-propagating modes share the same fundamental instability mechanisms. It is essential to both the instability and eastward propagation of the MJO- and BSISO-like modes that the moistening effect of low-level easterly anomalies due to horizontal advection exceeds their drying effect due to reduced surface fluxes (since the wind anomalies are superimposed on a westerly background state). The magnitude of the growth rate is controlled largely by the combination of these two effects, as well as by cloud-radiative instability.

The effect of the meridional moisture gradient is explored systematically, and we argue that this gradient determines

whether the unstable intraseasonal modes are MJO-like or BSISO-like. It is found (Fig. 3) that, with nearly zero or small positive meridional moisture gradient (typical of the Northern Hemisphere in northern summer), the horizontal structure exhibits a northwest–southeast tilt in the Northern Hemisphere (and the opposite tilt in the Southern Hemisphere), consistent with the observed BSISO. As the meridional moisture gradient increases, the horizontal tilt decreases and the pattern transitions to a more swallowtail-like wave pattern. When the meridional moisture gradient reaches values typical of northern winter, the MJO-like structure emerges, with cyclonic gyres straddling the equatorial precipitation maximum in the subtropics of both hemispheres. We conclude that the meridional moisture gradient shapes the horizontal structures, leading to the transformation from the BSISO-like tilted horizontal structure to the MJO-like neutral wave structure, as the gradient increases from small positive values in northern summer to large negative values in northern winter. On the other hand, the meridional moisture gradient does not significantly change the time scale or growth rate of the most unstable mode.

We further argue that the $n = 1$ mode can be thought of as resulting from phase speed matching between two simpler and more local modes: the $v = 0$ mode, which is trapped near the equator, and the waves found on an f plane representing the subtropics. The meridional moisture gradient Q_y does not affect the $v = 0$ mode, since that gradient enters the problem only through meridional advection. The meridional wavenumbers of the f -plane mode whose phase speeds best match those of the $v = 0$ mode vary with Q_y in a manner broadly consistent with the increasing meridional scale of the $n = 1$ mode with Q_y .

While the idealized MJO here is recognizable in its similarity to the observed one, the idealized BSISO is perhaps less obviously so. We summarize its characteristics that compare favorably with observations as follows: 1) It has planetary scale. 2) There is poleward propagation in both hemisphere; see Figs. 3 and 6 of Waliser et al. (2009). 3) There is a northwest–southeast tilt of horizontal structure in Northern Hemisphere. 4) The MSE budget (Figs. 7 and 8) is dominated by zonal advection and surface fluxes, consistent with observational analysis in Fig. 4 of Jiang et al. (2018). 5) It exhibits seasonality: it is active in the boreal summer season with negative or small positive meridional moisture gradient. 6) The idealized BSISO initiates and grows in the Indian Ocean. One discrepancy between the theory and observation is that the present theory has neglected hemispheric asymmetry: the idealized BSISO is symmetric about the equator, while the observed BSISO is highly antisymmetric and mainly active in the Northern Hemisphere. To address this issue, we have performed linear stability analysis using a moisture profile that maximizes at off-equatorial latitude (e.g., 20°N). Our preliminary analysis suggests that while this indeed adds some realism, the fundamental dynamics of the moisture mode instability and propagation remain unchanged compared to the idealized symmetric BSISO mode. The results will be reported elsewhere.

The present study uses estimated parameter values relevant to the Indian Ocean basin to understand the initialization dynamics of the ISOs there. No attempt is made to explore the dynamics over the Pacific Ocean, where the MJO's convection usually weakens while its circulation continues to propagate eastward in a manner often more consistent with a free Kelvin wave (e.g., Sobel and Kim 2012; Powell 2017). Our analysis fits into a conceptual picture in which the MJO and BSISO, as moisture modes per se, exist primarily over the Indian Ocean, and change form over the Maritime Continent and western North Pacific. In the latter case, the mean surface wind changes to easterly in the eastern Pacific Ocean, and the zonal moisture decreases to the east; both are opposite to their respective values in the Indian Ocean. The degree of cancellation between the two warrant careful analysis. The present study assumes constant zonal moisture gradient. While this assumption does not satisfy periodic boundary conditions, we consider it as a useful starting point, given the distinct behavior of the MJO and BSISO over the Indian Ocean and the much greater difficulty of conducting an analysis with a basic state gradient that varies in longitude. A more sophisticated analysis that explicitly accounts for zonal variation of the basic state and the transition from the Indian Ocean to the western Pacific is left for future work.

The limited range of physical processes considered here is not intended to imply that excluded processes are irrelevant. Boundary layer frictional convergence (Wang and Rui 1990) and shallow circulation (Wu 2003), for example, may also play important roles, perhaps by introducing more complex vertical structures and thus influencing the gross moist stability. Some aspects of the large-scale basic flow, neglected here except for its implicit presence in the WISHE term, might be especially important. In particular, the interaction of convection with the vertical wind shear associated with the Asian summer monsoon may be important to the behavior of the real BSISO, as argued in previous studies (e.g., Jiang et al. 2004; Bellon and Sobel 2008a). The trailing stratiform heating and dipolar vertical velocity in climate model simulations (Wang et al. 2017) and cloud-permitting simulations (Wang et al. 2015), together with the leading shallow circulation to the east, may alter eastward propagation (Wang and Li 2020) and the instability. Both effects may be considered by introducing the second baroclinic mode (Mapes 2000; Khouider and Majda 2006; Kuang 2008; Andersen and Kuang 2008) in the theory. Nonetheless, we find it compelling that the model used here appears to explain the different structures of the MJO and BSISO in terms of the observed meridional moisture gradient without including those processes. We view this as a reason to consider the MJO and BSISO as different manifestations of the same underlying dynamics, in the most basic sense, and the meridional moisture gradient as the critical parameter that controls the change in structure from one to the other with the seasons.

Acknowledgments. We thank three anonymous reviewers for their insightful comments. We are thankful to K. A. Emanuel for sharing his moist shallow water model solver, which inspired the present study. This research was supported

by Office of Naval Research Grant N00014-16-1-3073 and National Science Foundation Grant AGS-1543932, and by the Monsoon Mission Project under India's Ministry of Earth Sciences. SW acknowledges support from NSFC 41875066.

Data availability statement. The ERA5 reanalysis dataset and the DYNAMO sounding dataset are publicly available. Matlab code is available at Github (<https://github.com/wangsg2526/mjobsiso>), or upon request to the corresponding author.

APPENDIX

Formulation and Simplifications

a. Formulation of the model

A first-baroclinic moist shallow equation set has been derived by many different authors. Here, we start from the linearized anelastic equation set in a resting atmosphere.

$$\frac{\partial u}{\partial t} - \beta y v = -\frac{\partial \phi}{\partial x} - \alpha_u u, \quad (\text{A1})$$

$$\frac{\partial v}{\partial t} + \beta y u = -\frac{\partial \phi}{\partial y} - \alpha_u v, \quad (\text{A2})$$

$$\frac{\partial \phi}{\partial z} = b, \quad (\text{A3})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + 1\bar{p} \frac{\partial(\bar{p}w)}{\partial z} = 0, \quad (\text{A4})$$

$$\frac{\partial b}{\partial t} + wN^2 = \frac{g}{c_p \bar{\theta}} \frac{\bar{\theta}}{\bar{T}} J - \frac{\partial \bar{w}'b'}{\partial z} - \alpha_b b, \quad (\text{A5})$$

$$\frac{\partial q}{\partial t} + w \frac{\partial \bar{Q}}{\partial z} = e_{\text{vap}} - c_{\text{ond}} - \frac{\partial \bar{w}'q'}{\partial z} - u\bar{Q}_x - v\bar{Q}_y, \quad (\text{A6})$$

where (x, y, z) are zonal, meridional, and vertical distances; t is time; (u, v, w) (m s^{-1}) are horizontal and vertical velocities; ϕ ($\text{m}^2 \text{s}^{-2}$) is perturbation pressure; q (kg kg^{-1}) is the moisture mixing ratio; and b is buoyancy: $b = g(\theta/\bar{\theta})$ (m s^{-2}). All these variables are perturbations linearized about a reference or basic state. The terms $\bar{\theta}(z)$, $\bar{T}(z)$, $\bar{p}(z)$, and $\bar{Q}(x, y, z)$ are potential temperature, temperature, density, and humidity in the reference state, respectively; N^2 is the Brunt-Väisälä frequency $N^2 = (g/\bar{\theta})\partial\bar{\theta}/\partial z$ in the reference state. The terms α_u and α_b are Rayleigh damping on momentum and buoyancy, respectively. The last two terms in Eq. (A6) denote zonal and meridional advection of reference moisture by the perturbation wind. The terms w' , b' , and q' are associated with convective-scale eddies; e and p are rain evaporation and condensation; $\bar{\theta} = 300 \text{ K}$; $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$. The term J denotes diabatic heating, and it consists of two parts: heat release due to phase

transformation during condensation c_{ond} and evaporation e_{vap} , and radiative heating R :

$$J = L_v(c_{\text{ond}} - e_{\text{vap}}) + R = L_v(p - e)(1 + r),$$

where L_v is latent heat of condensation ($L_v = 2.5 \times 10^6 \text{ J kg}^{-1}$) and r is a nondimensional radiative feedback parameter expressing R as a fraction of net condensational heating. Here we take $r = 0.17$.

The vertical structure functions of the first baroclinic mode are as shown:

$$\begin{bmatrix} u \\ v \\ w \\ \phi \\ b \end{bmatrix}(x, y, z, t) = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \phi_1 \\ b_1 \end{bmatrix}(x, y, t) \begin{bmatrix} \Lambda_u \\ \Lambda_u \\ \Lambda_w \\ \Lambda_u \\ \Lambda_b \end{bmatrix}(z). \quad (\text{A7})$$

These vertical structures satisfy the relationships

$$\Lambda_u = H_0 \frac{1}{\bar{\rho}} \frac{\partial(\bar{\rho}\Lambda_w)}{\partial z}, \text{ and} \quad (\text{A8})$$

$$\Lambda_b = -H_0 \frac{\partial\Lambda_u}{\partial z}. \quad (\text{A9})$$

We choose $H_0 \sim 5 \text{ km}$. The value of Λ_w may be analytically specified [e.g., $\bar{\rho}\Lambda_w \propto \sin(\pi z/H)$], or derived from observations. Here we use the Λ_w profile from Wang et al. (2016) using the time mean sounding profile from the Dynamics of the Madden-Julian Oscillation (DYNAMO) northern sounding array (Johnson et al. 2015), assuming a rigid lid at the tropopause, and Λ_w is normalized. Vertical derivatives of Λ_w give Λ_u , and Λ_b , Λ_u , Λ_w , and Λ_b are all nondimensional. As a result, all the dependent variables in Eq. (A7) retain their dimensions.

Using these vertical structure functions, the momentum, hydrostatic, and continuity equations may be written as follows:

$$\frac{\partial u_1}{\partial t} - \beta y v_1 = -\frac{\partial \phi_1}{\partial x} - \alpha_u u_1, \quad (\text{A10})$$

$$\frac{\partial v_1}{\partial t} + \beta y u_1 = -\frac{\partial \phi_1}{\partial y} - \alpha_u v_1, \quad (\text{A11})$$

$$\frac{\phi_1}{H_0} = -b_1, \quad (\text{A12})$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{w_1}{H_0} = 0. \quad (\text{A13})$$

The buoyancy equation is

$$\frac{\partial b_1}{\partial t} \Lambda_b + w_1 \Lambda_w N^2 = \frac{g}{c_p \Theta} \bar{T} J - \alpha_b b_1 \Lambda_b. \quad (\text{A14})$$

We denote mass-weighted vertical integration by $\langle \cdot \rangle = \int_0^H \bar{\rho} dz$, where H is the tropopause height. The vertical integrals of the

vertical structure functions are then $\langle \Lambda_w \rangle \approx \langle \Lambda_b \rangle \sim 10^4 \text{ kg m}^{-2}$, and $\langle \Lambda_u \rangle \sim 0$. Integrating the buoyancy Eq. (A14) and using the continuity Eq. (A13) for the first baroclinic mode yields

$$\langle \Lambda_b \rangle \frac{\partial b_1}{\partial t} + \Gamma_s H_0 \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = -\frac{g}{c_p \Theta} \frac{\bar{\theta}}{\bar{T}} L_v P (1 + r) - \alpha_b \langle \Lambda_b \rangle b_1. \quad (\text{A15})$$

The term Γ_s is the gross dry stratification:

$$\Gamma_s = \langle \Lambda_w N^2 \rangle. \quad (\text{A16})$$

Note that $\Gamma_s / \langle \Lambda_b \rangle = (c^2 / H_0^2) \sim 10^{-4} \text{ s}^{-2}$ based on the DYNAMO time mean soundings. An estimate may be given as follows. Assuming constant N^2 , we have $\Gamma_s / \langle \Lambda_b \rangle = (\langle \Lambda_w \rangle / \langle \Lambda_b \rangle) N^2 \approx N^2$. For $H_0 \sim 5 \text{ km}$, $N \sim 10^{-2} \text{ s}^{-1}$, and the speed of the gravest gravity mode $c \sim 50 \text{ m s}^{-1}$, this gives $\sqrt{\Gamma_s / \langle \Lambda_b \rangle} \approx c / H_0$. Dividing the above equation for b_1 by $\langle \Lambda_b \rangle$ gives

$$\frac{\partial b_1}{\partial t} - \frac{c^2}{H_0} \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = \frac{1}{\langle \Lambda_b \rangle} \frac{L_v g}{c_p \Theta} \frac{\bar{\theta}}{\bar{T}} P - \alpha_b b_1. \quad (\text{A17})$$

Surface sensible heat flux, $\text{SH} = -\langle \partial w' b' / \partial z \rangle$, is neglected due to its smallness, compared to the latent heat flux in observations.

Now we proceed to simplify the moisture equation. We change variables for moisture as

$$(\tilde{q}, \tilde{Q}) \mapsto \frac{L_v g}{c_p \Theta} \frac{\bar{\theta}}{\bar{T}} (q, \bar{Q}). \quad (\text{A18})$$

The factor $[L_v g / (c_p \Theta)] \bar{\theta} / \bar{T}$ is $\sim 80 \text{ m s}^{-2}$ at the surface, 100 m s^{-2} at the tropopause. $\tilde{q}$ has units of m s^{-2} , as does buoyancy b . The tilde ($\tilde{\cdot}$) will be dropped afterward.

Assuming the first baroclinic structure for q is

$$q = q_1(x, y, t) \Lambda_q(z), \quad (\text{A19})$$

we use $\Lambda_q(z) \propto \exp(z/H_q) \sin(\pi z/H)$, where $H_q = 3 \text{ km}$. The exact structure of $\Lambda_q(z)$ is not important. Taking the vertical integral of q [Eq. (A6)] yields

$$\langle \Lambda_q \rangle \frac{\partial q_1}{\partial t} - \Gamma_q \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \right) = E - P - u \bar{Q}_x - v \bar{Q}_y \quad (\text{A20})$$

The terms $E = -\langle \partial w' q' / \partial z \rangle$ and $P = \langle p - e \rangle$ are surface evaporation and precipitation, respectively. The gross moisture stratification (Neelin and Yu 1994) is

$$\Gamma_q = \left\langle \Lambda_w \frac{\partial \bar{Q}}{\partial z} \right\rangle. \quad (\text{A21})$$

Note that Γ_q is negative. Defining normalized gross moist stability as $\Gamma = (\Gamma_s + \Gamma_q) / \Gamma_s$, an estimate based on the time mean of the DYNAMO sounding array data (Johnson et al. 2015) gives $\Gamma = 0.18$. Other estimates in the past suggest that Γ ranges from small negative values to ~ 0.2 .

Rewrite Γ_q as

$$\begin{aligned}\Gamma_q &= -(1 - \Gamma)\Gamma_s \\ &= -(1 - \Gamma)\frac{c^2}{H_0^2}\langle\Lambda_b\rangle.\end{aligned}$$

Let $\langle\Lambda_q\rangle = \langle\Lambda_b\rangle$ without loss of generality. The left-hand side (lhs) of the equation for q_1 is

$$\text{lhs} = \frac{\partial q_1}{\partial t} + (1 - \Gamma)\frac{c^2}{H_0}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right). \quad (\text{A22})$$

We use a simple quasi-equilibrium convective parameterization in which temperature dependence is neglected (FR17), so that precipitation depends only on moisture:

$$P = \alpha\langle q\rangle = \alpha\langle\Lambda_q\rangle q_1, \quad (\text{A23})$$

where the constant α is 1 day⁻¹ in this study.

The vertically integrated equations for b_1 and q_1 simplify to

$$\frac{\partial b_1}{\partial t} - \frac{c^2}{H_0}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = \alpha q_1(1 + r) - \alpha_b b_1, \quad (\text{A24})$$

$$\begin{aligned}\frac{\partial q_1}{\partial t} - (1 - \Gamma)\frac{c^2}{H_0}\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) &= \mathcal{E}u_1 - \alpha q_1 \\ &\quad - \frac{\langle\Lambda_u\overline{Q}_x\rangle}{\langle\Lambda_q\rangle}u_1 - \frac{\langle\Lambda_u\overline{Q}_y\rangle}{\langle\Lambda_q\rangle}v_1.\end{aligned} \quad (\text{A25})$$

Changing the variables as $\langle\Lambda_u\overline{Q}_x\rangle/\langle\Lambda_q\rangle \mapsto \overline{Q}_x$, and $\langle\Lambda_u\overline{Q}_y\rangle/\langle\Lambda_q\rangle \mapsto \overline{Q}_y$ simplifies the form of this equation. Linearization of E leads to $E = \mathcal{E}u_1$, where $\mathcal{E}$ is the proportionality constant.

Replacing b_1 in (A24) with ϕ_1 using Eq. (A12) leads to

$$\frac{\partial \phi_1}{\partial t} + c^2\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = -\alpha H_0 q_1(1 + r) - \alpha_b \phi_1. \quad (\text{A26})$$

Equations (A10), (A11), (A25), and (A26) comprise a closed system with four prognostic variables. This is the moist analog of the classical dry shallow water equation set (Matsuno 1966) with moisture as an additional variable. It may be nondimensionalized as follows:

$$\begin{aligned}t &\mapsto (\beta c)^{-1/2} t^*, & x &\mapsto (c/\beta)^{1/2} x^*, & y &\mapsto (c/\beta)^{1/2} y^*, \\ u_1 &\mapsto c u_1^*, & v_1 &\mapsto c v_1^*, & w_1 &\mapsto H_0(\beta c)^{1/2} w_1^*, \\ \phi_1 &\mapsto c^2 \phi_1^*, & b_1 &\mapsto \frac{c^2}{H_0} b_1^*, & q_1 &\mapsto \frac{c^2}{H_0} q_1^*.\end{aligned} \quad (\text{A27})$$

This leads to the nondimensional moist shallow water equation set at the beginning of section 2 (after dropping the subscript 1 and subscript *). A similar form can also be derived from the primitive equation set in Neelin and Zeng (2000), or a simplified Boussinesq equation set in

(a) Qv EOF1

(b) Column-integrated q_1 vs rain

FIG. A1. (a) Λ_q as derived from the leading EOF of the moisture anomalies from the DYNAMO northern sounding array. (b) Column-integrated first EOF anomaly $\langle q_1 \rangle$ and precipitation P anomalies. Both quantities are daily averaged.

Fuchs and Raymond (2005, 2017). It is also broadly similar to the bulk model in Emanuel (1987, 2020), or the 1.5-layer model in Liu and Wang (2016, 2017) and Wang and Chen (2017).

b. Approximations of the equation set

For a plane wave solution $\exp[i(kx - \omega t)]$, we have $\partial/\partial t = -i\omega$. Rewrite Eq. (1) as follows:

FIG. A2. Spectrum of the solution for the values of Q^0 used in Fig. 3. Open circles indicate stable solutions; closed symbols are unstable solutions. The sizes of the symbols indicate growth rate.

$$\begin{aligned}
 -i\omega_u u - yv &= -\frac{\partial \phi}{\partial x}, \\
 -i\omega_v + yu &= -\frac{\partial \phi}{\partial y}, \\
 -i\omega_\phi \phi + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= -\alpha(1+r)q, \\
 -i\omega_\alpha q - (\Gamma - 1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= \varepsilon u - u\bar{Q}_x - v\bar{Q}_y.
 \end{aligned} \tag{A28}$$

This shows that the first term (the ω terms) can be used to trace various approximations by setting it zero or replacing it with the corresponding damping coefficient. The geostrophic approximation, longwave approximation, weak temperature gradient approximation, and moisture convergence closure are obtained respectively as follows:

$$\begin{aligned}
 -yv &= -\frac{\partial \phi}{\partial x}, \\
 +yu &= -\frac{\partial \phi}{\partial y}, \\
 +\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= -\alpha(1+r)q, \\
 +\alpha q - (\Gamma - 1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= \varepsilon u - u\bar{Q}_x - v\bar{Q}_y.
 \end{aligned} \tag{A29}$$

The leading-order balance in the moisture equation is between convergence (second term on the lhs) and rain (first term on the lhs). Further neglecting the right-hand side leads to conventional moisture convergence closure. Evaluation of these approximations would be straightforward with the same solver and parameters.

c. Rain and column-integrated moisture

We derive the linear relationship between precipitation and first moisture mode q_1 using the DYNAMO northern sounding dataset. First, empirical orthogonal analysis is performed on 6-hourly moisture anomalies. The leading empirical orthogonal function (EOF; Fig. A1a) explains more than 66% of total variances. Second, the column-integrated first EOF of moisture anomalies is regressed to surface precipitation. Regression coefficient gives α , which varies from 1 to 1.3 days with different rain rate estimate (Fig. A1b). This estimate agrees with FR17, who used a 1-day time scale.

d. The $v = 0$ mode

The $v = 0$ mode shares the same instability mechanisms with more complex solutions. Because of its fundamental importance, the dispersion relation $\mathcal{A}_{v0}(\omega, k) = 0$ is rederived below. The reduced equation set of the $v = 0$ mode for plane wave solution $\exp[i(kx - \omega t)]$ is written as follows:

$$-i\omega_u u = -ik\phi, \quad (\text{A30a})$$

$$+yu = -\frac{\partial \phi}{\partial y}, \quad (\text{A30b})$$

$$-i\omega_\phi \phi + iku = -\alpha(1+r)q, \quad (\text{A30c})$$

$$-i\omega_\alpha q - (\Gamma - 1)iku = (\mathcal{E} - \overline{Q}_x)u. \quad (\text{A30d})$$

From Eq. (A30d), we have

$$q = \frac{i[(\mathcal{E} - \overline{Q}_x) + ik(\Gamma - 1)]u}{\omega_\alpha}. \quad (\text{A31})$$

Let $S = (\mathcal{E} - \overline{Q}_x) + ik(\Gamma - 1)$. Substitution of $q = iSu/\omega_\alpha$ to (A30c) leads to

$$-\omega_\phi \omega_\alpha \phi = [-\alpha(1+r)S - k\omega_\alpha]u. \quad (\text{A32})$$

Multiplying it with (A30a) yields

$$\omega_\phi \omega_\alpha \omega_u + k[-\alpha(1+r)S - k\omega_\alpha] = 0. \quad (\text{A33})$$

Using S and rearranging it gives $\mathcal{A}_{v0}(\omega, k) = 0$.

Next we show the polarization relationship. Combining Eqs. (A30a) and (A30b) yields

$$i\omega_u \frac{\partial \phi}{\partial y} = -ik\phi y. \quad (\text{A34})$$

The solution is written as

$$\phi = \exp\left(-\frac{k}{2\omega_u} y^2\right) \exp[i(kx - \omega t)]. \quad (\text{A35})$$

Other variables are

FIG. A3. The spatial structure of $v = 0$ mode, showing ϕ (shading), convergence (black contour), and wind vectors. Parameters are as in Fig. 2a.

$$u = -\frac{k}{2\omega_u} \phi, \quad (\text{A36})$$

$$w = -i\frac{k^2}{2\omega_u} \phi, \text{ and} \quad (\text{A37})$$

$$q = \frac{i\omega_\phi \phi + w}{\alpha(1+r)}. \quad (\text{A38})$$

Figure A3 shows the spatial structure of the $v = 0$ mode using the following parameters: $\alpha = 0.35$, $\alpha_\phi = 0.01$, $\alpha_u = 0.03$, $\mathcal{E} - \overline{Q}_x = -0.03$, $r = 0.17$, and $\Gamma = 0.16$ (as in Fig. 2a).

e. The f -plane moisture wave

The f -plane moisture wave solution under the longwave approximation is written as

$$v \propto \exp[i(kx + ly - \omega t)], \quad (\text{A39})$$

$$\phi = \frac{-if^2}{il\omega_u + fk} v, \quad (\text{A40})$$

$$u = -il\phi/f, \quad (\text{A41})$$

$$w = -iku - ilv, \quad (\text{A42})$$

$$q = \frac{i\omega_\phi \phi + w}{\alpha(1+r)}. \quad (\text{A43})$$

Figure A4 shows the spatial structure of the f -plane moisture wave using the following parameters: $\alpha = 0.35$, $\alpha_\phi = 0.01$, $\alpha_u = 0.03$, $\mathcal{E} - \overline{Q}_x = -0.03$, $r = 0.17$, and $\Gamma = 0.16$ (as in Fig. 2a).

FIG. A4. The spatial structure of f plane wave for $k = 1$, $l = 15$: ϕ (shading), convergence (black contour; 1.1, solid, -1.1, dashed), and wind vectors. Parameters are as in Fig. 2a.

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