Galerkin Approximations to Vertical Temperature Profiles Across the Earth’s Surface

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ABSTRACT

The Galerkin method is used to solve numerically the nonlinear initial-boundary value problem describing the vertical temperature profile for a thermally coupled soil-atmosphere boundary layer in a simple physical setting. The basic physical processes modeled, including conduction and free convection, are simple, yet the solution to the model problem illustrates the Galerkin procedure. The numerical simulation requires special mathematical techniques for handling interior boundary or coupling conditions. The physical interpretation of the internal conditions includes consideration of time-dependent insolation heating and the thermal interaction between the atmosphere and the soil. Comparisons between the exact solutions and the Galerkin solutions for several linear problems show agreement for all cases studied.

1. Introduction

A meteorological problem of current interest is described by two parabolic equations in adjoining regions coupled by an internal boundary condition. Although the most realistic problem is nonlinear, we also consider solutions for linear problems which are special cases obtained for certain choices of the physical parameters. To test the numerical method, the linear problems are compared with known analytic solutions. While the internal boundary condition presents a novel feature, the Galerkin methods described by Douglas and Dupont (1970) are directly applicable to the present problem. Consisting (in the parabolic case) of a variational formulation of the initial-boundary value problem to be solved by a predictor-corrector technique called Crank-Nicolson extrapolation (Douglas and Dupont, 1970, p. 580), their numerical methods have already been applied to a hyperbolic meteorological problem by these authors and others (Wang et al., 1972).

We consider the problem of determining the daytime vertical temperature profiles across the earth’s surface. The physical data employed here will describe quiescent desert conditions, i.e., a relatively clear, calm, and dry atmosphere with no horizontal temperature fluctuation and a moderately smooth interface. We assume no solar energy is absorbed by the atmosphere and that the effect of the longwave radiation from the atmosphere is included in the constant of proportionality $\epsilon$.

Finite difference methods have been employed to solve problems which describe the temperature variation of the soil-atmosphere boundary layer by several authors, e.g., Estoque (1963), Zdunkowski and Trask (1971), and Deardorff (1972). A recent analytical solution using Fourier series has been given by Kuo (1968).

Wang et al. (1972, pp. 743–744) compare the Galerkin with the finite-difference method. They conclude that Galerkin approximations are much more accurate for the same amount of computing effort. They also note that the reduction of time step has a greater effect on the Galerkin method. It is well known that if the time step becomes too small the benefits to be expected from reduced truncation error are negated by increased round-off error. Our results exhibit this phenomenon, since several time steps have been considered which show a reduction in accuracy of the Galerkin approximations (Table 1) with decreasing time step.

2. The equations and the physical model

The potential air temperature, $\theta' = \theta'(z', t)$, and $T = T(z', t)$, the temperature distribution found at the earth’s surface and below, are functions of the vertical distance $z'$ and the time $t$. Data of the problem are $K_n = K_n(z', \theta')$ and $K_s = K_s(z', T)$, the exchange coeffi-

<table>
<thead>
<tr>
<th>Number of time steps</th>
<th>Time step size (sec)</th>
<th>Error—49 basis function</th>
<th>Relative error (\times 10)</th>
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cients for the air and soil, respectively; initial vertical temperature profiles, \( \theta'(z') \), above the surface, and \( T_s(z') \), below the surface; and constant temperatures, \( d \) and \( c_s \), on the upper and lower boundaries, respectively.

The mathematical model, as simplified by the assumptions mentioned earlier, consists of the two second order parabolic partial differential equations

\[
L_1(\theta) = \frac{\partial \theta}{\partial t} - \frac{\partial}{\partial z'} \left( K \frac{\partial \theta}{\partial z'} \right) = 0, \quad 0 < z' < H, \quad t > 0 \tag{1a}
\]

\[
L_2(T) = \frac{\partial T}{\partial t} - \frac{\partial}{\partial z'} \left( K \frac{\partial T}{\partial z'} \right) = 0, \quad -H_1 < z' < 0, \quad t > 0, \tag{1b}
\]

subject to the boundary and initial conditions

\[
\begin{align}
\theta'(H,t) &= d, \quad T(-H_1,t) = c_s; \\
\theta'(z',0) &= \theta'_0(z'), \quad 0 \leq z' \leq H; \\
T(z',0) &= T_0(z'), \quad -H_1 \leq z' \leq 0;
\end{align}
\tag{1c}
\]

and the interface, or interior boundary, condition

\[
0 = -\epsilon_0 \frac{\partial T}{\partial z'} - \epsilon_0 K \frac{\partial \theta'}{\partial z'} + \epsilon_0 T^4 - \mu I \cos \phi, \quad z' = 0. \tag{1d}
\]

The interface condition describes the thermal interaction between the soil and the atmosphere and the forcing mechanisms of insolation heating. It is the balance of the vertical heat flux components across the interface. From left to right, the terms considered are the soil heat flux, the eddy transfer of sensible heat, the net longwave radiation, and the incoming shortwave radiation. The latter describes the special case of insolation observed at the equator during the equinox. Data introduced by this condition are: specific heat constants, \( c_p \) and \( c_s \), and densities, \( \rho_s = \rho_s(z',\theta') \) and \( \rho_a = \rho_a(z') \), of the air and soil respectively; the infrared emissivity, \( \epsilon \), the Stefan-Boltzman constant, \( \sigma \); the solar constant, \( I_a \), the combined albedo and turbidity factor \( \mu \), and the solar hour angle, \( \phi \).

At the earth's surface, continuity must exist between \( T \) and the ambient air temperature \( T_{air}(z',t) \), i.e.,

\[
T_{air}(0,t) = T(0,t), \tag{1e}
\]

where \( \theta' \) and \( T_{air} \) are related by

\[
\theta'(z',t) = T_{air}(z',t) \left( \frac{1000}{P} \right) \tag{2};
\]

where \( P \) is the atmospheric pressure, \( R \) is the gas constant, and \( r = R / \epsilon_0 \).

The exchange, or eddy diffusion, coefficient, \( K_{H} \), has been given different forms by Lumley and Panofsky (1964, pp. 109–110), Estoque (1963), and O'Brien (1970). For the region of free convection Priestley (1959) proposed that

\[
K_H(z',\theta') = \lambda \left( \frac{g}{\theta'} \right)^{1/4} \frac{\partial \theta'}{\partial z'} (z'+z_0)^2; \tag{3}
\]

where \( \theta' \) is the average potential temperature, \( g \) is the force of gravity per unit mass, \( z_0 \) is the roughness parameter, and \( \lambda \) (which varies from 0.9 to 1.4) is a stability factor.

For the first 50 m, we define \( K_H \) as in (3), adding the small constant 0.2 cm² sec⁻¹, which represents the value of the molecular diffusion coefficient. Between 50 and 2050 m, \( K_H \) will be approximated by a cubic spline for which an algorithm is readily available (Carnahan et al., 1969, p. 63; Wendroff, 1966, pp. 27–31). On \( 50 = z_0 < z_1 < \ldots < z_k = b = 2050 \), where \( k = 50 \), we define \( K_H \) to be a set of piecewise cubic polynomials which satisfy

\[
K_{H}(x_i) = K_{H}(x_i, \theta(x_i, 0)) = K_i, \quad i = 0, 1, \ldots, k
\]

\[
K_i = K_{H}(x_0) \sin \left( \frac{2050-x_i}{4000} \pi \right), \quad i = 0, \ldots, 47
\]

\[
K_{47} = 0.833 K_i, \quad i = 47, 48, 49
\]

where the slope and curvature are continuous at the nodes.

The exchange coefficient \( K_s \), which governs the subsurface behavior, depends upon the material and the moisture content (Sellers, 1965, pp. 134–140). We consider a homogeneous material with a constant exchange coefficient, \( K_s = 0.0036 \) cm² sec⁻¹.

To simplify the equations (1), we introduce changes in the original variables. We choose a constant \( b = \theta'_0(0) - T_{air}(0,0) \), which is easily found, and introduce the variable \( \theta(z',t) = \theta'(z',t) - b \). Hence, we see that \( \theta(0,0) = T_{air}(0,0) = T(0,0) \), thus guaranteeing continuity of the dependent variables at the interface. Assuming that the atmospheric pressure remains unchanged at the earth's surface during the study, we also have that \( \theta(0,t) = T_{air}(0,t) \) for \( t \geq 0 \). Define \( z = z' / H_1, 0 \leq z' \leq H \) and \( z = z' / H_1, -H_1 \leq z' \leq 0 \). To gain compactness of notation, we introduce the domain \( J \), the union of \( -1 \leq z < 0 \) and \( 0 \leq z \leq 1 \), and write

\[
L[\Psi] = \frac{\partial \Psi}{\partial t} - A \left( z,t,\Psi, \frac{\partial \Psi}{\partial z} \right) = 0, \quad z \in J, \quad t > 0, \tag{4}
\]

where, setting \( c_1 = d - b \), we have written

\[
\Psi = \begin{cases} \theta + (c_2 - c_1) z/2 - (c_1 + c_2)/2, & \text{if } z \geq 0, \\ T + (c_2 - c_1) z/2 - (c_1 + c_2)/2, & \text{if } z \leq 0, \end{cases} \tag{5}
\]

\[
A \left( z,t,\Psi, \frac{\partial \Psi}{\partial z} \right) = K(z,\Psi) - (c_1 - c_2) K(z,\Psi)/2. \tag{6a}
\]
and

\[ K = \begin{cases} 
K_{h}/H^2, & z > 0, \\
K_{s}/H_1^2, & z < 0.
\end{cases} \]  

(6b)

The other conditions simplify to

\[ \Psi(\pm 1, t) = 0, \; t > 0, \]  

(7a)

\[ \Psi(z, 0) = \Psi_0(z), \]  

(7b)

\[ 0 = c_p H_1 K \left( \frac{\partial \Psi}{\partial z} \right)_0 - \frac{(c_2 - c_1)}{2}, \]  

(7c)

\[ -c_p H K \left( \frac{\partial \Psi}{\partial z} \right)_0 + \left( \frac{c_2 - c_1}{2} \right) \]  

\[ + \epsilon \left[ \Psi + \frac{c_1 + c_2}{2} \right]^4 - \mu I_\infty \cos \phi, \; z = 0, \]  

where the notation 0⁺ and 0⁻ emphasizes the existence of one-sided derivatives at z = 0. The initial function \( \Psi_0(z) \) can be reconstructed from the graphs provided in this paper.

3. Galerkin approximation

To obtain the variational form of (4), we consider the space, \( S \), of all real-valued functions, \( v(z) \), which are piecewise continuously differentiable functions, on \([-1, 1]\). Multiplying (4) by \( v \), and integrating with respect to \( z \) from -1 to 1, yields

\[ \int_{-1}^{1} \left[ \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial z} - \frac{c_1 + c_2}{2} \right] v dz = 0. \]  

Together with the initial and interface conditions, this must be satisfied for all \( v \in S \).

The Galerkin approximation to \( \Psi \) belongs to a finite dimensional subspace, \( M \), of \( S \). Possible choices for such a subspace are considered in Claret et al. (1970), Cavendish et al. (1969). (Cf., also, the bibliography in Wang et al., 1972). For our problem, we choose the chapeau functions, which lead to a tri-diagonal matrix, as a basis. We subdivide the interval \([-1, 1]\) into units of constant length \( h = z_{i+1} - z_i \), by the partition

\[ -1 = z_{-m+1} < \ldots < z_0 = 0 < z_1 < \ldots < z_{m+1} = 1. \]

Thus \( M \) is a 2m+1 dimensional space with basis \( \{v_j\}_{j=m-n}^{n} \), where

\[ \begin{align*}
 v_j &= \begin{cases} 
 \frac{z - z_{j-1}}{h}, & z_{j-1} \leq z \leq z_j, \\
 \frac{z_{j+1} - z}{h}, & z_j \leq z \leq z_{j+1}, \; j = -m, \ldots, n, \\
 0, & \text{otherwise},
\end{cases} \\
 v_j &= \begin{cases} 
 \frac{z - z_{j-1}}{h}, & z_{j-1} \leq z \leq z_j, \\
 \frac{z_{j+1} - z}{h}, & z_j \leq z \leq z_{j+1}, \; j = -m, \ldots, n, \\
 0, & \text{otherwise},
\end{cases}
\end{align*} \]  

(9)

The Galerkin approximation is then defined by

\[ U(z, t) = \sum_{j=-n}^{n} a_j(t)v_j \]  

(10)

into (8) in place of \( \Psi \); and \( v_j \) in place of \( v \).

Following the notation and procedure developed by Douglas and Dupont (1970), we integrate (8) by parts and recall that \( K \) is constant for \( z < 0 \) to obtain an approximation to the variational problem:

\[ \left( \frac{\partial U}{\partial t} \right)_0 + a(w; U, v) = 0, \]  

(11a)

\[ (U, v) = (\Psi_0, v), \]  

(11b)

\[ G_1(U) = \frac{\partial U}{\partial Z}_0^+ - G_2(U) \frac{\partial U}{\partial Z}_0^-, \]  

(11c)

where

\[ a(w, U, v) = \int_{-1}^{1} K(z, w) \frac{\partial U}{\partial t} dv + \int_{-1}^{0} K(z, w) \frac{\partial U}{\partial z} dv \]  

\[ + \frac{(c_2 - c_1)}{2} \int_{0}^{1} K(z, w) \frac{\partial U}{\partial z} dz, \]  

\[ f(z, t; U) = \left[ G_1(U) - G_2(U) \right] \frac{(c_2 - c_1)}{2} \]  

\[ + \epsilon o \left[ U(z, t) + \frac{c_1 + c_2}{2} \right]^4 - \mu I_\infty \cos \phi, \]  

\[ G_1(U) = c_p H \cdot K, \; G_2(U) = c_p H \cdot K. \]

Two linear algebraic systems must be solved at each time \( t_n = n\Delta t, \; n = 0, 1, 2, \ldots \):

\[ \begin{align*}
 \left( W_{m+1} - U_m \right) \frac{\partial}{\partial t} v + a(U_m; U_m, U_{m+1} + \frac{U_m}{2}, v) &= 0, \\
 G_1(U_m) \frac{\partial W_{m+1}}{\partial z} \bigg|_{z_0} - G_2(U_m) \frac{\partial W_{m+1}}{\partial z} \bigg|_{z_0} &= f(0, t, U_m); \tag{12a}
\end{align*} \]

\[ \begin{align*}
 \left( U_{m+1} - U_m \right) \frac{\partial}{\partial t} v + a(U_{m+1}; U_m, U_m + \frac{U_m}{2}, v) &= 0, \\
 G_1(U_{m+1}) \frac{\partial U_{m+1}}{\partial z} \bigg|_{z_0} - G_2(U_{m+1}) \frac{\partial U_{m+1}}{\partial z} \bigg|_{z_0} &= f(0, t, U_{m+1}); \tag{12b}
\end{align*} \]

where

\[ (U_0, v) = (\Psi_0, v); \tag{12c} \]

\[ W_n = \sum_{j=-n}^{n} \beta_j(l_n) v_j, \]

\[ U_m = \sum_{j=-n}^{n} \alpha_j(l_m) v_j. \]
This leads to two systems of matrix equations. The first is

\[ (C + D)\beta(t_{m+1}) = (C - D)\alpha(t_m) + \gamma, \]  

(13a)

where \(\alpha(t_m)\) and \(\beta(t_{m+1})\) are column vectors with entries \(\alpha_n(t_m), \alpha_{n+1}(t_m), \ldots, \alpha_n(t_m)\) and \(\beta_n(t_{m+1}), \beta_{n+1}(t_{m+1}), \ldots, \beta_n(t_{m+1})\), respectively, and

\[ C = [c_{kj}], \quad c_{kj} = \int_{-1}^{1} v_k(v_j dz); \]

\[ D = [d_{kj}], \quad d_{kj} = \frac{\Delta t}{2} \int_{0}^{1} K(z, U_m) v_k v_j dz, \]

\[ + \frac{\Delta t}{2} \int_{-1}^{0} K(z, U_m) v_k v'_j dz; \]

and \(\gamma\) is the column vector with elements

\[ \gamma_k = \frac{\Delta t}{2} (c_{2k} - c_{1k}) \int_{0}^{1} K(z, U_m) v_k dz; \]

for \(k = -n, \ldots, -1, 1, \ldots, n\) and \(j = -n, \ldots, n\).

For \(k = 0\) the equation arising from the interface condition is

\[ -G_2(U_m) v_{-1} \beta_{-1} + (G_1(U_m) v_0) - G_2(U_m) v_0 = 0, \]

\[ + G_1(U_m) v_1 \beta_1 = f(0, t, U_m). \]  

(13b)

The second is

\[ (C + D^*)\alpha(t_{m+1}) = (C - D^*)\alpha(t_m) + \gamma^*, \]  

(14)

where the symbol \(^*\) indicates that \(K[z, (U_m + W_{m+1})/2]\) replaces \(K(z, U_m)\) in the \(D\) and \(\gamma\) above. In the equation corresponding to (13b), the term \(f(0, t, (U_m + W_{m+1})/2)\) replaces \(f(0, t, U_m)\).

The initial condition becomes

\[ (C)^\alpha(t_0) = \Gamma, \]  

(15)

where \(\Gamma\) is the column vector with elements

\[ \gamma_k = \int_{-1}^{1} \Psi_k(z)v_k dz, \]

for \(k = -n, \ldots, n\) and \(j = n, \ldots, n\).

If \(\Psi_k(z)\) is observational data, then the initial profile \(\alpha(t_0)\) will automatically satisfy the interface condition.

4. Numerical results

a. Linear cases

Selected linear cases were chosen to test the accuracy of the method and of the computer program. Of course, in this case the two predictor-corrector equations are identical. When the diffusivities, \(k_1\) and \(k_2\), are constant, the analytical solution to a heat conduction problem which is a special case of the problem considered here is given in Carslaw and Jaeger (1947, p. 265). In this case the interface condition is given by

\[ \frac{\partial T}{\partial z} = K_1 \frac{\partial T}{\partial z}. \]  

(16)

To simplify computations we take \(K_2(k_1)/K_1(k_2) = 1\).

In order to begin the computation at \(t = t^*\) with zero error, we calculate the values of the coefficients in the Galerkin approximation to the solution by using (5) and (10), i.e., we obtain \(\alpha(t^*)\) at the node points of the basis functions from the temperature distribution given by the analytic solution. At \(t = 0\) the temperature is zero everywhere except at \(z = -H_1\) which is a constant \(V\) for \(t > 0\). At \(z = H_2\), the temperature is held at zero for \(t > 0\). All numerical integrations are performed using 10-point Gaussian quadrature.

In the first case considered, we choose slight variation between the thermal conductivities, \(K_1\) and \(K_2\), and between the diffusivity coefficients \(k_1\) and \(k_2\). The values are given in Set 1.

Set 1

\(k_1 = 0.0036\) cm\(^2\) sec\(^{-1}\),

\(k_2 = 0.01\) cm\(^2\) sec\(^{-1}\),

\(K_1 = 0.00018\) cal deg\(^{-1}\) cm\(^{-1}\) sec\(^{-1}\),

\(K_2 = 0.0003\) cal deg\(^{-1}\) cm\(^{-1}\) sec\(^{-1}\),

\(H_1 = 25\) cm,

\(H = 80\) cm,

\(t^* = 50,000\) sec.

The temperature spread between the two boundaries is 20K. A comparison between the analytic solution and the Galerkin approximation is shown in Tables 1 and 2.

Illustrating the error dependence on the size of the time step, Table 1 shows how the interface condition can influence the numerical solution. Since the greatest temperature change occurs at \(z = 0\), the numerical error is largest here. The first two rows in Table 2 illustrate how the error at \(z = 0\) behaves as the number of basis function increases.

A second case was run for a temperature spread of 100K and a large differential in parameters. These are shown in Set 2.

Set 2

\(k_1 = 0.0036\) cm\(^2\) sec\(^{-1}\),

\(k_2 = 18225\) cm\(^2\) sec\(^{-1}\),

\(K_1 = 0.001944\) cal deg\(^{-1}\) cm\(^{-1}\) sec\(^{-1}\),

\(K_2 = 4.374\) cal deg\(^{-1}\) cm\(^{-1}\) sec\(^{-1}\),

\(H_1 = 50\) cm,

\(H = 5000\) cm,

\(t^* = 70,000\) sec.
Larger error magnitudes occur in this case, but the pattern of the error curves is similar to the first case.

The analytic solution to the heat equation with zero initial temperature, and with the surfaces \(z = \pm H\) kept at constant temperature \(V = 300\text{K}\) for \(t > 0\) is also given by Carslaw and Jaeger (1947, p. 83). A comparison with the Galerkin approximation was made for \(H = 1000\) cm and all physical constants set to 1. Table 3 shows the error for values of the dimensionless variable \(z\). Again the error peaks at \(z = 0\) and the error decreases as the number of basis functions increases.

### Table 3. Error \(^\circ\text{K (Galerkin—Exact)}\) for various values of \(z\) after 10 time steps of 8000 seconds each. Relative error \(\times 100\) is given below each error entry.

<table>
<thead>
<tr>
<th>(z) (dimensionless)</th>
<th>Number of basis functions</th>
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<tr>
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<td>0.035</td>
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<td>0.92</td>
<td>0.011</td>
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</table>

**FIG. 1.** Ambient air and soil temperature profiles for the local times 7:20 (initial morning distribution), 9:00, 11:00, and 1:20 (left to right respectively). Parameters \(\epsilon = 0.8\) and \(\mu = 0.6\).
node points of the basis functions $v(z)$ from the initial temperature profile. To satisfy the interior boundary condition the coefficient $\alpha_0(t_0)$ is determined by (11c) once $\alpha_0(t_0)$ and $\alpha_1(t_0)$ have been evaluated. From these coefficients the initial ambient air and soil temperatures are reconstructed and plotted on the left in Figs. 1 and 3.

Eqs. (13) and (14) describe two linear systems involving nonsingular tridiagonal matrices. These are easily solved by Gaussian elimination, given, e.g., in Henrici (1962, p. 354). All numerical integrations are evaluated using a 10 point Gaussian quadrature.

$H$ and $H_1$, representing the height of the boundary layer in the atmosphere and the depth in the soil are assigned the values $H = 2050$ m and $H_1 = 1$. For simplicity the atmospheric pressure used in all calculations corresponds to the NACA Standard Atmosphere. For convenience the parameter $\lambda$ is assigned the value of 1. Other parameters used are $\rho = 0.00129$ gm cm$^{-3}$, $\rho_v = 1.8$ gm cm$^{-3}$, $c_p = 0.24$ cal gm$^{-1}$ deg$^{-1}$, $\sigma = 0.135463 \times 10^{-11}$ cal cm$^{-3}$ K$^{-4}$ sec$^{-1}$, $I_n = 0.0333$ cal cm$^{-2}$ sec$^{-1}$, and $c_s = 0.3$ cal gm$^{-1}$ deg$^{-1}$. Figs. 1 through 3 present the solutions obtained using 149 basis functions with a time step of 20 min.

Before looking at individual cases we first examine the overall pattern illustrated in Figs. 1 through 3. As expected the warming and the cooling trends are pronounced at the interface. Also we note that the maximum temperature is achieved between 1:20 and 1:40 local time. As cooling at the interface progresses it should be noted that warming continues high in the atmosphere and deep in the soil. It should be pointed out that in the present investigation no attempt is made to relate $K_H$ to the Richardson number. Neither is there any adjustment of $K_H$ based on the Rayleigh criterion. In defense of the highly superadiabatic temperature profiles Scorer (1958, p. 179) indicates that under desert conditions the superadiabatic layer can extend 1500 m or more above the ground.

Figs. 1, 2, and 3 have in common the parameters $K_s = 0.0036$ cm$^2$ sec$^{-1}$ (sandy clay) and $s_0 = 0$ cm and they show the model solution for the indicated local times. For Figs. 1 and 2 we used $\varepsilon = 0.8$ and $\mu = 0.6$, whereas in Fig. 3, $\varepsilon = 0.7$ and $\mu = 0.7$. Thus the solutions illustrated in Fig. 3 are for a problem having a larger heat source and smaller heat sink at the interface. This results in a maximum spread of more than 15K between the two solutions. Apparently the model is very sensitive to variations in these important parameters. We conclude, from these figures and from others not presented, that the exact surface albedo, the turbiditiy, and the effect of longwave radiational effects must be emphasized if accurate models are to be developed. To emphasize the sensitivity of the model and to further test the Galerkin solutions comparisons were made with the findings of Zdunkowski and Trask (1971). Even though their model is meteorologically more complicated similar basic patterns for different representative soil types were reproduced.

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**Fig. 2.** Ambient air and soil temperature profiles, continued from Fig. 1, for the local times 2:00, 4:00, 5:00, and 6:00 (right to left respectively).

**Fig. 3.** Ambient air and soil temperature profiles for the local times 7:00 (initial morning distribution, same as Fig. 1) 9:00, 11:00, and 1:20 (left to right respectively). Parameters $\varepsilon = 0.7$ and $\mu = 0.7$. 
We conclude by noting that as various parameters are varied the solutions obtained using the Galerkin method are entirely consistent with natural observations and with previous investigations.

REFERENCES


