Azimuthally Averaged Transport and Budget Equations for Storms: Quasi-Lagrangian Diagnostics 1

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ABSTRACT

The azimuthally averaged transport and budget equations for a translating storm volume are derived in generalized coordinates. The mean and eddy lateral modes of transport by rotational and irrotational motion are contrasted in symmetric and asymmetric vortices. By contrasting the transport relations in isobaric, cartesian, and isentropic coordinates, the results establish that hydrostatic-rotational regimes of atmospheric motion are typified by eddy modes of transport in isobaric and cartesian coordinates, while both mean and eddy modes may be present in isentropic coordinates. This requirement for a "handover" from an eddy mode of transport in the hydrostatic-rotational environment of a vortex to a mean mode of transport via irrotational motion within a vortex is discussed.

Evidence for the existence of mean meridional circulation in isentropic coordinates for the Midwest extratropical cyclone of 22–24 April 1968 is presented. The inward mass transport in the lower troposphere and outward mass transport in the upper troposphere are coupled to vertical mass transport through isentropic surfaces associated with the release of latent heat in the middle troposphere.

1. Introduction

In his survey, "Fronts and Cyclones," Eliassen (1966) places in perspective the contributions of The Norwegian School, quasi-geostrophic theory, and instability theory to the development of a comprehensive picture of frontal phenomena and extratropical cyclones. His statements that "we certainly cannot yet claim to have a complete theory of fronts and cyclones" even with the considerable progress of recent years and "clearly there is still much to be done before we can claim to have a fully satisfactory theory of fronts and cyclones" stand in contrast to the views by some atmospheric scientists that this basic problem of the middle latitudes has been explained.

For the operational meteorologist who is faced with the problem of daily weather prediction for middle latitudes, the academician who teaches the dynamics of extratropical disturbances and the atmospheric scientist who ponders the complexities of the interaction between waves, vortices, and embedded baroclinity, Eliassen's perspective concerning prior contributions and their interrelationships provides a sense of balance in judgments on the magnitude of the remaining effort.

To a certain degree the question of whether the extratropical cyclone is a wave or a vortex still remains unresolved. Norwegian cyclone theory considered fronts to be primary features of the circulation and cyclones to be secondary developments in the process of wave development on the frontal zones (Bjerknes and Solberg, 1922). However, within quasi-geostrophic and instability theory, the extratropical cyclone is regarded primarily as a wave phenomenon in a baroclinic fluid. Theory does not provide insight into V. Bjerknes' *et al.* (1933) question of what forces the transition of amplifying waves into cyclonic vortices with their maturation through the occlusion process.

Three dimensionality partially accounts for the different emphasis. While wave structures dominate in the upper troposphere, anticyclonic and cyclonic vortices dominate the lower troposphere. The existence of a strong interaction between the regimes is immediately inferred from the condition that in nearly all instances wave amplification of the upper troposphere is coincident with cyclonic and anticyclonic development in the low troposphere. However, since the late 1930's with the contributions by J. Bjerknes (1937) and Rossby (1939) attention has been primarily focused on the dynamics of wave motion. From the emphasis on the wave regime baroclinic and barotropic instability theories have dominated the search for the forcing of extratropical cyclones.

In this and two additional papers we shall formulate a quasi-Lagrangian diagnostic framework based on the budget approach to study development in the low troposphere and present the mass and angular momentum budgets for a case study of cyclogenesis (Johnson

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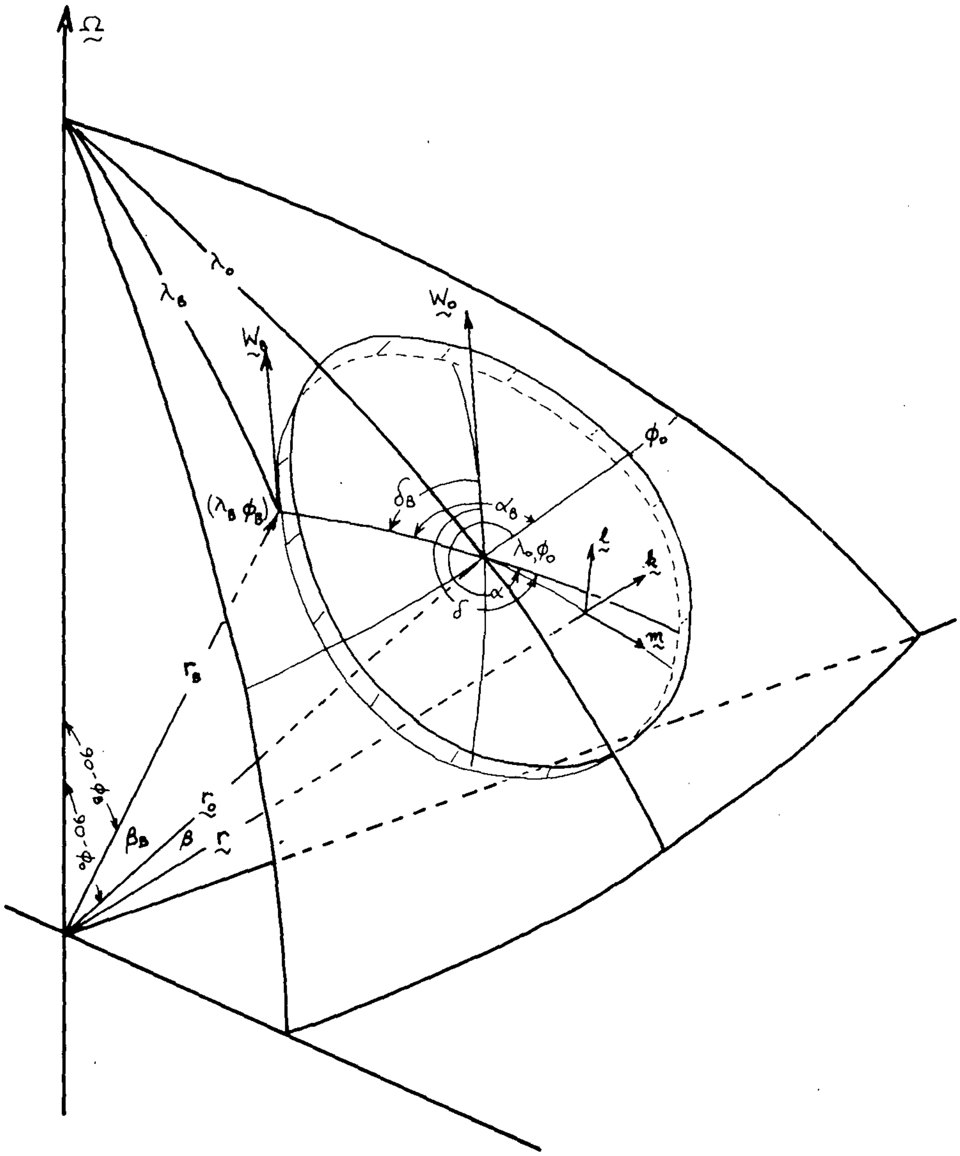

FIG. 1. The local coordinate system.

and Downey, 1975; Johnson and Downey, 1976). Emphasis will be placed on the analysis of properties essential to vortex formation within the wave regime, their transport from the environment, and the modes of vertical redistribution during development.

2. Budget equations for storms

In regard to why the formulation and use of the budget approach for studies of the secondary scales of atmospheric circulation, our motivation partially stems from the vortex nature of some of these scales of motion. All secondary and smaller scales are embedded within the larger planetary scale, and attempts to ascertain basic aspects of the development and maintenance of these circulations involve an interaction with the larger scale environment. Most techniques of scale decomposition for both theoretical and diagnostic studies are based on the spectral representation of atmospheric motion. This decomposition is most appropriate for studies of wave motion and wave interactions. For studies of vortices, however, the alternative of the budget approach merits consideration.

The use of the budget approach leads to a different perspective of the interaction of the secondary scale with its environment external to the budget volume. The underlying physical principles of the interaction are quantified through the transport relations and other governing equations, which in conjunction with the geometry of the budget volume focuses attention on the exchange of mass, momentum, and energy through

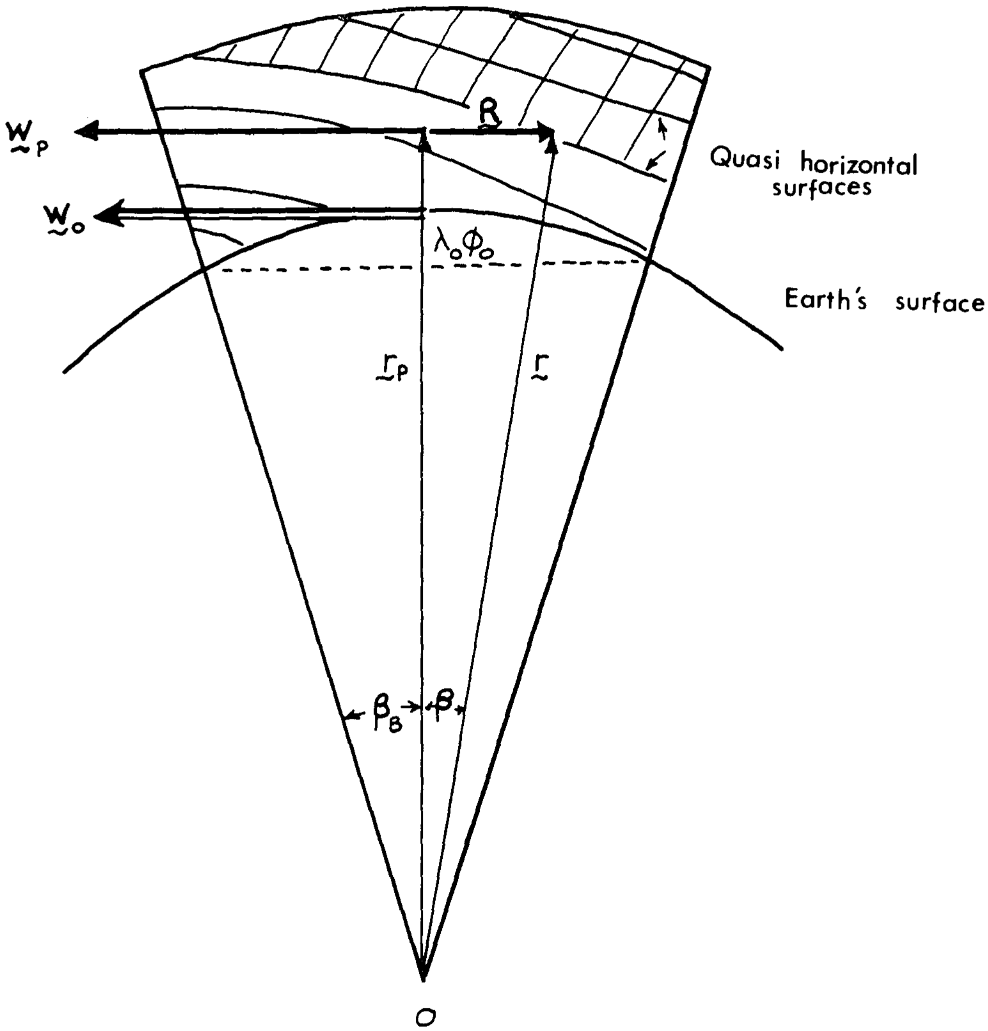

FIG. 2. A schematic cross section of the local coordinate structure through the storm budget volume.

the boundaries. Sources (or sinks) of basic properties are identified with dynamical processes within the volume. In contrast, the role of transport of basic properties in addition to the exchange of internal energy by pressure interaction across the lateral boundaries is de-emphasized in most studies based on a spectral decomposition of atmospheric motion due to the assumption of cyclic continuity. In the wave models the modes for exchange between scales becomes primarily identified with nonlinear interaction between waves.

Some insight on the value of an alternative method of studying the interaction of secondary and primary scales with an emphasis on transport phenomena and open thermodynamic systems is best illustrated by the concept of the steady state hurricane. For a budget volume of approximately 1500 km, the generation of storm available potential energy (equal to the generation of internal energy in a steady state system) serves to offset the kinetic energy dissipation without requiring an inward transport of energy into the system (Anthes and Johnson, 1968). However, from the concept of absolute angular momentum, an inward transport of this property from the environment is required to maintain the steady state vortex. With the need for inward transport the steady state and the hydrostatic relation between mass and pressure require an equal outward transport of mass. Thus, from a mass and energy viewpoint of steady hurricanes, the sphere of influence of the environment need not exceed the 1500 km radius, while from the dynamical aspects of angular momentum, the sphere of influence may be larger.

In the series of three papers, attention is focused on the interaction of the extratropical cyclone with its environment. In contrast to the hurricane example, the uncertainty of the mass, angular momentum, and energy budgets for the extratropical cyclone is greater, primarily because of the complicated structure and intensity of the transport of properties associated with the jet stream and baroclinic waves. The decision to use transport theory and the budget approach to study the role of the jet stream and baroclinic waves in cyclones stems from an analysis of the azimuthally averaged equations. Through this analysis the important processes forcing development in a stratified atmosphere are isolated from the inertial effects associated with the translation of an existing system.

a. The generalized coordinates for the storm budget

The volume moving with the storm and centered about the reference axis $\mathbf{r}_0$, a radius vector from the earth center to the reference latitude-longitude (λ_0,ϕ_0) is illustrated in Fig. 1. The volume of integration is formulated in a variation of spherical coordinates (α,β,η) with the coordinate surfaces being generated by the vertical planes, α = constant; the cones, β = constant; and the quasi-horizontal surfaces, η = constant. The generalized vertical coordinate η is used since in later sections the differing viewpoints of the cartesian $(\eta=z)$, isobaric $(\eta=p)$ and isentropic $(\eta=\theta)$ frameworks are contrasted (see Fig. 2).

An orthogonal right-handed set of unit vectors **m**, **l**, **k** are defined such that

$$\begin{aligned}\mathbf{m}&=\mathbf{l}\times\mathbf{k}\\ \mathbf{l}&=\mathbf{k}\times(\mathbf{r}-\mathbf{r}_0)/|\mathbf{k}\times(\mathbf{r}-\mathbf{r}_0)|\\ \mathbf{k}&=\mathbf{r}/|\mathbf{r}|.\end{aligned}\qquad(1)$$

The convention of projecting from the quasi-spherical coordinates onto a local sphere is utilized in the mapping of horizontal fields.

The relative velocity in component form is

$$\mathbf{U}=\frac{d\mathbf{r}}{dt}=U_\alpha\mathbf{l}+U_\beta\mathbf{m}+U_\eta\mathbf{k},\qquad(2)$$

where

$$U_\alpha=\mathbf{l}\cdot\mathbf{U};\quad U_\beta=\mathbf{m}\cdot\mathbf{U};\quad U_\eta=\mathbf{k}\cdot\mathbf{U}.\qquad(3)$$

In Fig. 2, the horizontal velocity for the reference axis which moves with a characteristic feature of the storm circulation, e.g., the central pressure, is

$$\mathbf{W}_0=\frac{d\mathbf{r}_0}{dt}\qquad(4)$$

The lateral vertical boundary of the volume contains the atmosphere region within the cone, β_B = const, concentric about $\mathbf{r}_0$, and the velocity of this boundary is defined to be

$$\mathbf{W}_B=\frac{d\mathbf{r}_B}{dt}.\qquad(5)$$

The relation between the boundary and the reference axis velocity is presented in Appendix A, while the relations between geographical positions (λ,ϕ) and the storm coordinates (α,β) are presented in Appendix B.

b. The generalized budget equation

The budget integral F for the shaded region in Fig. 2 bounded by the cone β_B and the lower surface η_B is defined in the quasi-spherical coordinates to be

$$F(\beta_B,\eta_B,t)=\int_{\eta_B}^{\eta_T}\int_0^{\beta_B}\int_0^{2\pi}\rho Jfr^2\sin\beta d\alpha d\beta d\eta,\qquad \eta_s\le\eta_B\le\eta_T\qquad(6)$$

$$=\int_{V_\eta}\rho JfdV_\eta,\qquad(7)$$

where ρ is the density, J is the Jacobian of transformation $J(x,y,z/\alpha,\beta,\eta)$ equal to $|\partial r/\partial\eta|r^2\sin\beta$, f is the specific property of interest, dV_η is $d\alpha d\beta d\eta$, η_T represents the top of the atmosphere and $\eta_s=\eta(\alpha,\beta,r_s,t)$ is the earth's surface.

The time derivative of (7) using a three dimensional analog of Leibnitz's rule is

$$\frac{dF}{dt}=\int_{V_\eta}\frac{\partial}{\partial t_\eta}(\rho Jf)dV_\eta+\int_{\eta_B}^{\eta_T}\int_0^{2\pi}\rho Jf\frac{d\beta_B}{dt}d\alpha d\eta\Big|_{\beta_B}-\int_0^{\beta_B}\int_0^{2\pi}\rho Jf\frac{d\eta_B}{dt}d\alpha d\beta\Big|_{\eta_B}\qquad(8)$$

where $d\beta_B/dt$ and $d\eta_B/dt$ are boundary speeds. The subscripted vertical bar denotes the value of the independent variable over which the surface integral is evaluated. The generalized forms of the equation of continuity and the total derivative are given by (9) and (10) respectively:

$$\frac{\partial}{\partial t_\eta}\rho J+\frac{\partial}{\partial\alpha}(\rho JU_\alpha/r\sin\beta)+\frac{\partial}{\partial\beta}[\rho J(\sin\beta)U_\beta/r]+\frac{1}{r^2}\frac{\partial}{\partial\eta}\left(\rho Jr^2\frac{d\eta}{dt}\right)=0\qquad(9)$$

$$\frac{df}{dt}=\frac{\partial f}{\partial t_\eta}+\frac{U_\alpha}{r\sin\beta}\frac{\partial f}{\partial\alpha}+\frac{U_\beta}{r}\frac{\partial f}{\partial\beta}+\frac{d\eta}{dt}\frac{\partial f}{\partial\eta}.\qquad(10)$$

Multiplication of (9) by f and (10) by ρJ and combination give

$$\frac{\partial}{\partial t_\eta}(\rho Jf)=\rho J\frac{df}{dt}-\frac{\partial}{\partial\alpha}(\rho JU_\alpha/r\sin\beta)-\frac{\partial}{\partial\beta}(\rho JU_\beta/r)-\frac{\partial}{\partial\eta}\left(\rho J\frac{d\eta}{dt}\right).\qquad(11)$$

Substitution of (11) in (8) yields

$$\frac{d}{dt}F(\beta_B,\eta_B,t)=\int_{V_\eta}\rho J\frac{df}{dt}dV_\eta - \int_{\eta_B}^{\eta_T}\int_0^{2\pi}\rho Jf[\mathbf{m}\cdot(\mathbf{U}-\mathbf{W})/r]d\alpha d\eta\bigg|_{\beta_B} + \int_0^{\beta_B}\int_0^{2\pi}\rho Jf\left(\frac{d\eta}{dt}-\frac{d\eta_B}{dt}\right)d\alpha d\beta\bigg|_{\eta_B}, \quad (12)$$

where the boundary conditions for the free atmosphere and the earth's surface $\eta_s=\eta(\alpha,\beta,r_s,t)$ are

$$\left.\begin{aligned}&\frac{d\eta_B}{dt}=0;\quad \eta_s(\alpha,\beta,t)<\eta_B\le\eta_T\\&\frac{d\eta_B}{dt}=\frac{d\eta_s}{dt}(\alpha,\beta,t);\quad \eta_s\ge\eta_B.\end{aligned}\right\} \quad (13)$$

With this convention the relative flux $(d\eta/dt-d\eta_B/dt)$ vanishes at the earth's surface.

Equation (12) is the generalized form of a quasi-Lagrangian budget for any specific budget quantity f. The first integral represents the source term $S(f)$, since it sums the internal sources or sinks of the property f. For the mass budget this term vanishes, for the angular momentum budget this term relates to internal torques, while for the energy budget the term relates to generation, conversion, dissipation and/or the mechanism of pressure work. The second term in (12), the lateral transport $\mathrm{LT}(f)$, quantifies the interaction of the storm with its environment through the lateral transport of mass, angular momentum, moisture, and energy into the vortex. The last term measures the vertical transport $\mathrm{VT}(f)$ through the lower boundary of the budget volume as long as it is not contingent with the earth's surface.

Two important features of the storm budget formulated in generalized coordinates are:

1) Equation (12) denotes a budget formulation for a volume which moves with the storm. The spatially fixed budget volume is a special case (W equal zero). However, with such a formulation for translating systems the effects of movement and true development remain inseparable.

2) For a bounded volume by the earth's surface and the top of the atmosphere the integrals, $S(f)$ and $\mathrm{LT}(f)$, computed in any coordinate system must be invariant. However, if the overall budget volume is subdivided by different vertical co-ordinates, i.e., by constant level (z), isobaric (p), or isentropic (θ) surfaces, the form and interpretation of the role of the source and transport terms within layers is different. This aspect is now considered in greater detail.

3. Partitioning of the budget's time rate of change into mean and eddy modes

The statistical partitioning of thermodynamic and dynamic processes into changes associated with average and eddy circulations for the global scale has been emphasized during the last two decades. In this section some differences for the role of mean and eddy modes in the development, maintenance, and decay of the vortex of the extratropical cyclone vortex will emerge. In order to clarify the root of the contradistinction, the storm budget must be studied in generalized coordinates. These results will show that the statistical results are not contradictory, but that the contrasting viewpoints evolve from a non-uniqueness between statistics computed for fully developed asymmetric circulations and certain dynamical, physical concepts emanating from symmetric models.

The azimuthal mass weighted average of a function f and its deviation are defined by

$$\langle f\rangle^\alpha=\frac{\overline{\rho J_\eta f}^\alpha}{\overline{\rho J_\eta}^\alpha} \quad (14)$$

$$f^*=f-\langle f\rangle^\alpha, \quad (15)$$

where the azimuthal average of a function f and its deviation are

$$\bar{f}^\alpha=\frac{1}{2\pi}\int_0^{2\pi}fd\alpha \quad (16)$$

$$f'=f-\bar{f}^\alpha. \quad (17)$$

For simplicity in this definition, J_η is defined to be $|\partial r/\partial\eta|$ and the variation of the length at the radius vector $\mathbf{r}$ is assumed to be independent of any variable on a quasi-spherical surface.

With the definition

$$(U-W)_\beta=\mathbf{m}\cdot(\mathbf{U}-\mathbf{W}_B) \quad (18)$$

and the substitution of the azimuthally mass weighted average and its deviation for f and $(U-W)_\beta$, the transport at the lateral and lower boundaries becomes

$$LT(f)=-\int_{\eta_B}^{\eta_T}\int_0^{2\pi}\overline{\rho J_\eta}^\alpha[\underbrace{\langle f\rangle^\alpha\langle(U-W)_\beta\rangle^\alpha}_{\text{MEAN MODE}} + \underbrace{\langle(U-W)_\beta{}^*f^*\rangle^\alpha}_{\text{EDDY MODE}}]r\sin\beta d\alpha d\eta\bigg|_{\beta_B} \quad (19)$$

$$VT(f)=\int_0^{\beta_B}\int_0^{2\pi}\overline{\rho J_\eta}^\alpha\left[\underbrace{\langle f\rangle^\alpha\left\langle\frac{d\eta}{dt}\right\rangle^\alpha}_{\text{MEAN MODE}} + \underbrace{\langle f^*\rangle^\alpha\frac{d\eta^*}{dt}}_{\text{EDDY MODE}}\right]r^2\sin\beta d\alpha d\beta\bigg|_{\eta_B}. \quad (20)$$

Now the boundary flux of f is partitioned into a mean mode associated with the transport by the azimuthally averaged mass circulation and an eddy mode defined by the covariance of azimuthal deviations.

In the analysis of the modes of lateral transport, it is important to realize that the total transport of an atmospheric property through the entire lateral boundary extending from the earth's surface to the top of the atmosphere must be invariant with respect to the different coordinates. However, where only a limited extent of the vertical boundary or the vertical distribution of the lateral transport is considered, the results computed from different coordinates may vary because the vertical extent $\Delta\eta$ along the line of integration from α equal 0 to 2π encompasses different segments of the conical boundary. This difference already provides a point of departure for the viewpoints of dynamical processes studied in different coordinates.

With the partitioning into mean and eddy modes of transport, however, a new and more subtle degree of freedom for the variance of one's viewpoint is introduced. The components $\langle(U-W)_\beta\rangle$ and $\langle d\eta/dt\rangle$ in the mean mode of transport are uniquely related through the equation of continuity to the azimuthally averaged mass circulation. The specification of different independent vertical coordinates, η_i, specifies different dependent fields, $f(\alpha,\beta,\eta_i,t)$, in the computation of $\langle f\rangle$ and of the mean mode of transport through use of the azimuthal average. Thus, $\langle f\rangle$, $\langle(U-W)_\beta\rangle$, $\langle d\eta/dt\rangle$, and the products specifying the mean mode of transport computed in different coordinate representations of the same vortex may vary. Because the total lateral transport is invariant while the mean mode of transport is coordinate dependent, the eddy mode must also be coordinate dependent. Thus our viewpoint of the dynamical role of mean and eddy modes in the formation of the vortex becomes coordinate dependent. It is important that the types of vortices for which viewpoints are harmonious and the conditions under which viewpoints diverge be established. In the subsequent sections, the modes of lateral transport within axisymmetric and asymmetric vortices is studied and the relation between the mean mode of transport and the azimuthally averaged mass circulation is contrasted for the isobaric, isentropic, and cartesian systems.

4. Modes of lateral transport in symmetric and asymmetric vortices

By Helmholtz's theorem the velocity field $U_H(\alpha,\beta,\eta,t) = (\mathbf{i}u+\mathbf{j}v)$ of the quasi-horizontal surface η may be partitioned into rotational (non-divergent) and irrotational (divergent) components, defined by

$$\mathbf{U}_H=\mathbf{U}_\chi+\mathbf{U}_\psi=\underset{\text{(DIVERGENT)}}{\nabla_\eta\chi}+\underset{\text{(NON-DIVERGENT)}}{\mathbf{k}\times\nabla_\eta\psi} \tag{21}$$

where χ and ψ are scalar potential and stream functions and the subscripts define the velocity component associated with each field of motion.

With this definition the azimuthal mass weighted average normal velocity is

$$\langle(U-W)_\beta\rangle^\alpha=\frac{\overline{\rho J_\eta(U_{\beta\psi}+U_{\beta\chi}-W_\beta)}^\alpha}{\overline{\rho J_\eta}^\alpha}. \tag{22}$$

Consider the case of a translating vortex with *total axial symmetry.* Equation 22 becomes

$$\langle(U-W)_\beta\rangle^\alpha=U_{\beta\chi}(\beta,\eta,t), \tag{23}$$

while the total transport is given by

$$\langle(U-W)_\beta\rangle^\alpha\langle f\rangle^\alpha=U_{\beta\chi}f. \tag{24}$$

In a symmetric vortex the lateral transport is solely by the mean mode through the divergent component of the wind, i.e., there is no transport associated with the non-divergent component nor from the effects of translation. In addition, the unique relation for transport on the boundary given by

$$[\langle U_\chi-W)_\beta\rangle^\alpha\langle f\rangle^\alpha](\eta)=[U_{\beta\chi}f][z(\eta)]=[U_{\beta\chi}f][p(\eta)] =[U_{\beta\chi}f][\theta(\eta)] \tag{25}$$

leads to the viewpoint that the meridional mass circulation in an axisymmetric vortex is unique and independent of the coordinate system.

Now consider the case of a translating vortex with symmetry restricted to the mass distribution. Equation (22) reduces to

$$\langle(U-W)_\beta\rangle^\alpha=\overline{U_{\beta\chi}}^\alpha \tag{26}$$

and the total transport becomes

$$\langle(U-W)_\beta f\rangle^\alpha=\bar{U}_{\beta\chi}\bar{f}+\overline{(U_{\beta\chi}'+U_{\beta\psi}'-W_\beta')f'}. \tag{27}$$

In a vortex where the symmetry is restricted to the mass distribution the lateral transport will be by the eddy mode through the divergent and non-divergent components of the wind and from the effect of translation, while transport by the mean mode is restricted to the divergent component of the wind.

In the case where there is a general asymmetry of both the mass and wind fields, the mean and eddy modes of transport retain their most general form with the mean mode of transport given by

$$\langle(U-W)_\beta\rangle^\alpha\langle f\rangle=(\langle U_{\beta\chi}\rangle^\alpha+\langle U_{\beta\psi}\rangle^\alpha-\langle W_\beta\rangle^\alpha)\langle f\rangle \tag{28}$$

and the total transport by

$$\langle(U-W)_\beta f\rangle^\alpha=(\langle U_{\beta\chi}\rangle^\alpha+\langle U_{\beta\psi}\rangle^\alpha-\langle W_\beta\rangle)^\alpha\langle f\rangle +\langle(U_{\beta\chi}{}^*+U_{\beta\psi}{}^*-W_\beta{}^*)f^*\rangle^\alpha. \tag{29}$$

In an asymmetric vortex, lateral transport involves the divergent and non-divergent components of the wind as well as the effects of translation in both the mean and eddy modes.

5. On the mean mode of lateral transport and the internal mass redistribution

Now that certain constraints between transport by the rotational and irrotational components of motion and mean and eddy mode for symmetric and asymmetric circulations have been ascertained, the coupling of the mean mode of lateral transport to the internal mass redistribution is studied. From the relation between the local time derivative $\partial/\partial t$ and the "local" time derivative $\delta/\delta t$ in a frame of reference translating with a relative velocity $\mathbf{W}=\mathbf{W}(t)$ of the storm axis

$$\frac{\delta}{\delta t_\eta}(\ )=\frac{\partial}{\partial t_\eta}(\ )+\mathbf{W}\cdot\nabla_\eta(\ ) \tag{30}$$

and the equation of continuity for relative motion

$$\frac{\partial}{\partial t_\eta}(\rho J_\eta)+\nabla_\eta\cdot(\rho J_\eta \mathbf{U})+\frac{\partial}{\partial\eta}\left(\rho J_\eta\frac{d\eta}{dt}\right)=0, \tag{31}$$

the equation of continuity in the translating system becomes

$$\frac{\delta}{\delta t_\eta}(\rho J_\eta)+\nabla_\eta\cdot[\rho J_\eta(\mathbf{U}-\mathbf{W})]+\frac{\partial}{\partial\eta}\left(\rho J_\eta\frac{d\eta}{dt}\right)=0. \tag{32}$$

With an areal integration of (32) the azimuthal mass weighted average normal component is given by

$$\langle(U-W)_\beta\rangle^\alpha|_{\beta_B}=\frac{r(1-\cos\beta_B)}{\overline{\rho J_\eta}^\alpha|_{\beta_B}\sin\beta_B}\left[\frac{\delta}{\delta t_\eta}(\overline{\rho J_\eta}^{\alpha\beta_B})+\frac{\partial}{\partial\eta}\left(\overline{\rho J_\eta}^{\alpha\beta_B}\left\langle\frac{d\eta}{dt}\right\rangle^{\alpha\beta_B}\right)\right], \tag{33}$$

where the areal mass weighted average and its deviation are defined by

$$\langle f\rangle^{\alpha\beta_B}=\overline{\rho J_\eta f}^{\alpha\beta_B}/\overline{\rho J_\eta}^{\alpha\beta_B};\quad f^{**}=f-\langle f\rangle^{\alpha\beta_B}, \tag{34}$$

while the areal geometric average is

$$\overline{f}^{\alpha\beta_B}=\frac{1}{2\pi r^2(1-\cos\beta_B)}\int_0^{\beta_B}\int_0^{2\pi}fr^2\sin\beta d\alpha d\beta. \tag{35}$$

The mean mode of transport given by the product of (33) with $\langle f\rangle|_{\beta_B}$ is

$$\langle(U-W)_\beta\rangle^\alpha\langle f\rangle^\alpha|_{\beta_B}=\frac{r(1-\cos\beta_B)}{\overline{\rho J_\eta}^\alpha|_{\beta_B}\sin\beta_B}\left[\frac{\delta}{\delta t_\eta}\overline{\rho J_\eta}^{\alpha\beta_B}+\frac{\partial}{\partial\eta}\left(\overline{\rho J_\eta}^{\alpha\beta_B}\left\langle\frac{d\eta}{dt}\right\rangle^{\alpha\beta_B}\langle f\rangle\right)\right]\Bigg|_{\beta_B}. \tag{36}$$

Equation (36) indicates that the mean mode of transport through the translating boundary is coupled to "local" changes of the area averaged mass distribution or the divergence of the vertical mass transport. In a stationary vortex or for a non-stationary vortex with mass symmetry $\rho J_\eta(\alpha,\beta,\eta)=\rho J_\eta(\beta,\eta)$, the component $\langle W_{\beta_B}\rangle^\alpha$ of the mean mode of transport vanishes and the relation between the lateral mass transport on the boundary and the internal mass redistribution becomes independent of the effects of translation. This effect is consistent with the result that the mean mode of lateral transport in a mass symmetric vortex is solely by the divergent component of the motion. However, in a vortex with an asymmetric mass distribution the mean mode of transport which involves both the rotational and irrotational components of motion is coupled to the vertical redistribution of mass within the vortex and/or local changes.

Two internal characteristic conditions of large scale atmospheric flow are quasi-hydrostatic and quasi-geostrophic balance. Even within intense vortices the assumption of hydrostatic balance is still an excellent approximation.

A third characteristic state of internal balance may be inferred from the steadiness of atmospheric flows, which suggests that the area averaged internal redistribution of mass $(\delta\overline{\rho J_\eta}^{\alpha\beta_B}/\delta t_\eta$ tends to vanish at certain stages in the storm's life history. However, it is important to remember that the first two conditions of internal balance depend on atmospheric response and are independent of the selection of a coordinate system, while the steady state mass condition will be coordinate dependent.

The hydrostatic defect (the imbalance) after Dutton and Johnson (1967) is

$$\chi=\alpha\frac{\partial p}{\partial z}+g, \tag{37}$$

which becomes

$$\rho J_\eta=-\frac{1}{g}\frac{\partial p}{\partial\eta}+\frac{\chi}{g}\rho J_\eta. \tag{38}$$

The substitution of (38) into (33) yields

$$\langle(U-W)_\beta\rangle^\alpha|_{\beta_B}=\frac{r(1-\cos\beta_B)}{\overline{\rho J_\eta}^\alpha|_{\beta_B}\sin\beta_B}\times\left[\frac{\delta}{\delta t_\eta}\left(-\frac{1}{g}\overline{\frac{\partial p}{\partial\eta}}^{\alpha\beta_B}\right)+\frac{\delta}{\delta t_\eta}\left(\overline{\frac{\chi}{g}\rho J_\eta}^{\alpha\beta_B}\right)+\frac{\partial}{\partial\eta}\left(\overline{\rho J_\eta\frac{d\eta}{dt}}^{\alpha\beta_B}\right)\right]. \tag{39}$$

In this expression the relationship between the tendency of the area-averaged pressure distribution, the tendency of the hydrostatic imbalance, the vertical redistribution of mass, and the mean mode of transport is explicitly illustrated.

a. The mean mode of lateral transport and the internal mass redistribution in isobaric coordinates

The selection of isobaric coordinates for a budget study eliminates the first term of (39) and shows that the area averaged internal redistribution of mass

$$\frac{\delta}{\delta t_p}\overline{\rho J_p}^{\alpha\beta_B} \text{ becomes } \frac{\delta}{\delta t_p}\left(\overline{\frac{\chi}{g}\rho J_p}^{\alpha\beta_B}\right),$$

i.e., the local change of a covariance between the hydrostatic defect and the mass ρJ_p. By virtue of the magnitude of χ/g and the continual hydrostatic adjustment of the atmosphere, this internal redistribution of mass must be negligible when compared to other terms except possibly in intense small scale convection.

On the assumption of hydrostatic balance, the azimuthally averaged relative normal component reduces to

$$\langle U_\beta\rangle^\alpha|_{\beta_B}=\bar{U}^\alpha_{\beta\chi}|_{\beta_B}=\frac{r(1-\cos\beta_B)}{\sin\beta_B}\frac{\partial}{\partial p}\bar{\omega}^{\alpha\beta_B}, \quad (40)$$

since $\langle W_{\beta_B}\rangle^\alpha$ is zero.

This result shows that the isobaric mean mode of transport is equal to the mean normal component of the irrotational component of motion and that the mean mass circulation in the (β,p) plane is independent of the effects of translation. The comparison of (27) and (40) demonstrates that the mean mode of lateral transport for a vortex viewed in the isobaric framework on the assumption of hydrostatic balance is identical to the mode for a vortex with axisymmetry of the mass distribution. Because the total lateral transport is invariant and the mean modes appear identical, the eddy mode of lateral transport in the isobaric framework will also be like that found in a vortex with axial symmetry of the mass distribution.

The observed facts that large scale flows on the rotating earth are primarily rotational and tend to geostrophic balance, and that smaller scales have in the relative sense much larger irrotational components, are conditions which are independent of any coordinate system. For example, in the vicinity of the relatively small scale vortex of the hurricane the magnitude of the irrotational component is large in comparison with its magnitude in the environment of the larger scale. In an isobaric analysis of the hurricane vortex, (40) would imply that $\bar{\omega}^{\alpha\beta_B}$ will attain its maximum magnitude over a small budget volume where the magnitude of U_{β_χ} is significant on the lateral boundaries. However, when the analysis is conducted for a larger storm volume about the same vortex, the condition that the large scale environment tends to be rotational and that ω is zero at the upper boundary implies that $\bar{\omega}^{\alpha\beta_B}$ should tend to zero on all isobaric surfaces.

The constraint of a zero value of $\bar{\omega}^{\alpha\beta_B}$ over large areas which include the storm and its large scale environment implies that, surrounding the ascending motion near and within the storm vortex, there must be a large region where compensating ω's of opposite sign are found. Such situations are exemplified in the ATS-III satellite picture of an extratropical cyclone over the southeastern U. S. on June, 1970 (Fig. 3) and of hurricanes (Fig. 4). For the extratropical cyclone the strong ascent, which is indicated by the convection embedded in the smaller scales, the vortex center, and along the frontal bands, is surrounded by a much larger cloudless region where ω is positive. In the hurricane, intense ascent is similarly surrounded by clear regions indicating subsidence. Although the concept of compensation in the atmosphere is classical for small scale convection, it is interesting to find within the isobaric framework that compensation is required from the condition that the irrotational component is relatively large in smaller scales, but insignificant in the larger scale environment of a storm vortex. These constraints are basic to the theory of CISK (Charney and Eliassen, 1964; Charney, 1973).

The constraints between the vertical motion and the irrotational component of isobaric motion have another interesting but subtle implication in the dynamics of storms. With respect to the transport of any property through the lateral boundary from a storm's immediate environment into its interior, the mean mode of transport may be the primary means (β_B is small). However, because there is no mean mode at the boundaries of a large storm volume, the transport of properties essential to vortex maintenance from the larger scale environment must be through covariances found in the eddy mode. This feature for the modes of transport is dis-

FIG. 3. ATS-III satellite picture of extratropical cyclone over southeastern United States, 1627 GMT 21 January 1968.

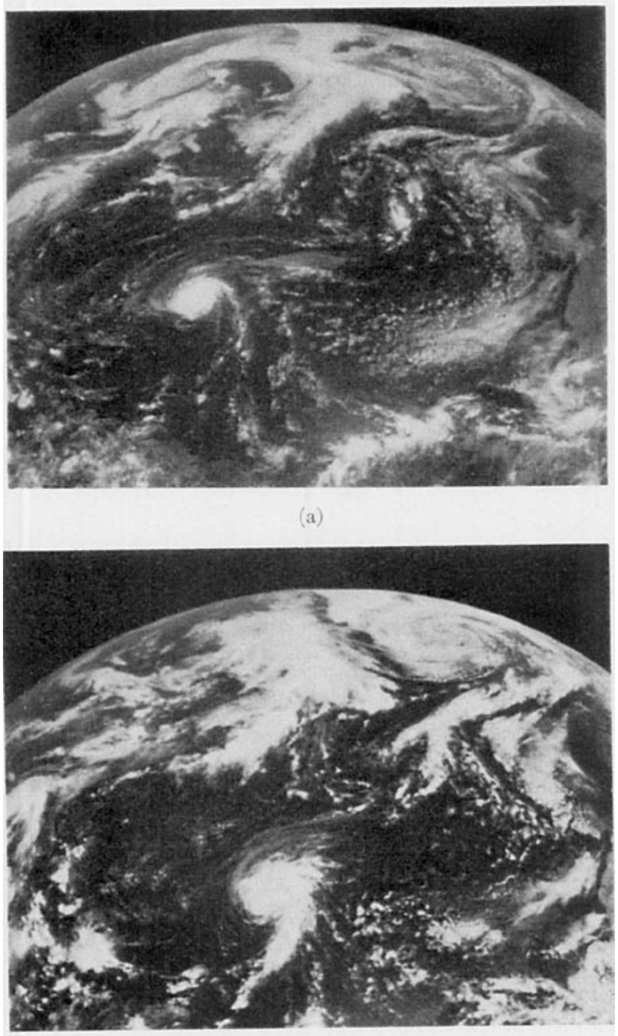

FIG. 4. ATS-III satellite pictures of Hurricane Debbie: (a) 1420 GMT 18 August, and (b) 1407 GMT 19 August 1969.

played in Pfeffer's (1958) results for the hurricane. With respect to a scale decomposition of atmospheric flow within the isobaric framework, the "handover" from the eddy to the mean mode occurring in the transport of any property from the environment to the storm vortex implies a strong interaction between larger and smaller scales. These implications are particularly significant for vortices where their maintenance depends on a supply of the property from the environment, e.g., water vapor or angular momentum.

This problem of scale interaction involved in the "handover" between mean and eddy modes ultimately involves inertial terms in the momentum equation even if the storm is not translating. The problem is further compounded in extratropical latitudes where storm translation is most pronounced. At these latitudes, the effects of translation are also embedded in the inertial terms of the momentum equation expressed in isobaric coordinates and the problem of the separation of the forcing of vortex development in the lower troposphere from the forcing of wave propagation in the upper troposphere becomes confounded.

b. The mean mode of lateral transport and the internal mass redistribution in isentropic coordinates

With the selection of isentropic coordinates for a budget study, (39) becomes

$$\langle (U-W)_\beta\rangle^\alpha|_{\beta_B}=\frac{r(1-\cos\beta_B)}{\overline{\rho J_\theta}^{\alpha}|_{\beta_B}\sin\beta_B}\times\left[\frac{\delta}{\delta t_\theta}\left(-\overline{\frac{1}{g}\frac{\partial p}{\partial\theta}}^{\alpha\beta_B}\right)+\frac{\delta}{\delta t_\theta}\left(\overline{\frac{\chi}{g}\rho J_\theta}^{\alpha\beta_B}\right)+\frac{\partial}{\partial\eta}\left(\overline{\rho J_\theta\frac{d\eta}{dt}}^{\alpha\beta_B}\right)\right], \quad (41)$$

The tendency of the internal mass distribution given by the first two terms in (41) is proportional to changes of the vertical derivative of the area-averaged pressure on the isentropic surface and a covariance of the hydrostatic defect and the mass, ρJ_θ. In a hydrostatic atmosphere the second term vanishes and the internal mass redistribution is determined by the first term. From the definition of the stability measure

$$\sigma=-\frac{1}{g}\frac{\partial\theta}{\partial p}, \quad (42)$$

the local rate of change of the area averaged pressure distribution may be expressed by

$$\frac{\delta}{\delta t}\left(-\overline{\frac{1}{g}\frac{\partial p}{\partial\theta}}^{\alpha\beta_B}\right)=-\overline{\sigma^{-2}\frac{\delta\sigma}{\delta t}}^{\alpha\beta_B}. \quad (43)$$

This result when combined with (41) shows that the azimuthally averaged normal component $\langle (U-W)_{\beta_B}\rangle^\alpha$ is also coupled with internal stability changes within the storm volume. If the flow is adiabatic and hydrostatic and a mean mode of transport exists, a tendency for local stability $\delta\sigma/\delta t$ and for the area averaged pressure $(\delta/\delta t)\bar{p}^{\alpha\beta_B}$ must be realized.

It is also important to realize that in isentropic coordinates the mean mode of transport may involve correlations between the mass ρJ_θ (or stability σ) and the velocity components. The azimuthal mass weighted normal velocity takes the form

$$\langle (U-W)_{\beta_B}\rangle^\alpha=\overline{\rho J_\theta(U_\chi+U_\psi-W)_\beta}^{\alpha}/\overline{\rho J_\theta}^{\alpha}|_{\beta_B}, \quad (44)$$

which expands to

$$\langle (U-W)_{\beta_B}\rangle^\alpha=\bar{U}^{\alpha}_{\beta_B\chi}+\overline{(\rho J_\theta)'(U_\chi{}'+U_\psi{}'-W')_\beta}^{\alpha}/\overline{\rho J_\theta}^{\alpha}|_{\beta_B}. \quad (45)$$

Transport by the rotational and translating components only occurs through a covariance structure, e.g., large ρJ_θ (low stability σ) with inward normal components and low ρJ_θ (high stability σ) with outward normal components.

A third characteristic state of internal balance, that the vertical derivative of the area-averaged pressure $(\partial/\partial\theta)\bar{p}^{\alpha\beta_B}$ is steady, is used in conjunction with hydrostatic conditions in (41) to find

$$\langle (U-W)_{\beta_B}\rangle^{\alpha}=\frac{r(1-\cos\beta_B)}{\overline{\rho J_\theta}^{\alpha}|_{\beta_B}\sin\beta_B}\frac{\partial}{\partial\theta}\overline{\rho J_\theta\frac{d\theta}{dt}}^{\alpha\beta_B} . \quad (46)$$

On these conditions (46) illustrates that the mean inward (or outward) transport is linked to the vertical divergence (convergence) of the mass transport in isentropic coordinates and must involve processes of internal heating. The lateral transport may occur through any or all the components shown in (44) or (45). Alternatively, under the steady hydrostatic conditions posed, a zero value for $\overline{\rho J_\theta(d\theta/dt)}^{\alpha\beta_B}$ is a necessary and sufficient condition for the mean relative normal component to be zero, and there will be no interaction of the vortex with the environment of the storm by the mean mode of transport. While some may question the validity of this inference due to use of the steady state assumption for $(\partial/\partial\theta)\bar{p}^{\alpha\beta_B}$, the results of Bullock and Johnson (1971), Petersen (1971), and Downey (1972) provides support for this assumption in some storms. Such an assumption seems consistent with the utilization of a horizontally invariant static stability measure in quasi-geostrophic theory developed in isobaric coordinates.

One important difference between the isentropic and isobaric schemes is that one is unable to infer the "handover" in scale between the mean and eddy modes of transport for isentropic coordinates. However, such an inference is not required since the unique relationship between vertical mass flux $\overline{\rho J_p\omega}^{\alpha\beta_B}$ and the irrotational component of hydrostatic flows found in isobaric coordinates does not exist in isentropic coordinates. Along the trajectory of a parcel moving isentropically from the vortex's environment to its interior, the relative velocity and thus the transport of any property by the mean mode will shift from the rotational to the irrotational component. This shift is due to the condition that the trajectory encounters a large time and space scale in the environment, where it is able to adjust to geostrophy, but in the reduced time and space scale of the vortex itself it is unable to adjust to geostrophy. The largest departures from geostrophy along the parcel trajectory should be encountered in its vertical displacement with respect to height. In the cold polar air in the rear of the extratropical vortex, the parcel's velocity becomes supergeostrophic as it undergoes adiabatic descent from the upper to lower troposphere. In the warm subtropical air ahead of the vortex center the parcel's velocity becomes subgeostrophic as it translates horizontally and inwardly from the weak pressure gradient region of the environment to the stronger gradient in the vortex center. Those parcels ascending diabatically to the high troposphere in association with condensation and latent heat release are still likely subgeostrophic with respect to the stronger pressure gradient of the upper tropospheric wave and are likely moving towards lower pressure. For trajectories leaving the immediate environment of the vortex the transport will shift from the irrotational to the rotational component as the outward flow adjusts to the geostrophic balance of the environment.

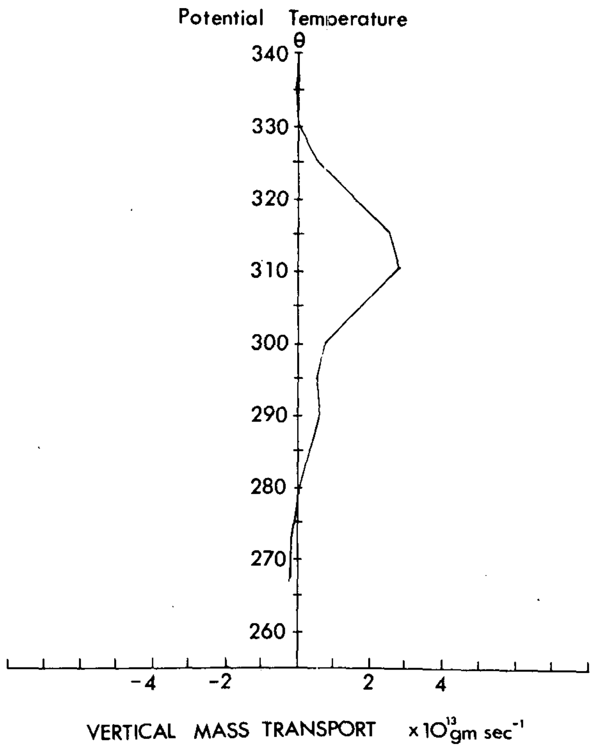

FIG. 5. Mean vertical profile of area averaged vertical mass transport for the period 23/1200 to 24/0000 GMT April 1968 (Downey, 1972).

Figures 5 and 6 present estimates of the area-averaged vertical mass transport $\overline{\rho J_\theta(d\theta/dt)}^{\alpha\beta_B}$ and the average heating rate $\langle d\theta/dt\rangle^{\alpha\beta}$ determined from budget studies. Note the mid-tropospheric maximum that must be associated with the release of latent heat through the processes of condensation and a mass circulation which serves to transport water vapor into the low troposphere of the storm vortex through which the vertical diabatic mass transport is realized. These results were first computed by Petersen (1971) and Downey (1971) in studies of the mass and angular momentum budgets of the Midwest cyclone of 22–24 April 1968. These examples are presented and serve to illustrate the utility of isentropic budget studies for the estimation of heating rates from wind and pressure data in isentropic coordinates. Their role in angular momentum transport and cyclone development will be discussed in our subsequent papers (Johnson and Downey, 1975; Johnson and Downey, 1976).

6. Important atmospheric properties needed for the development and maintenance of a storm vortex

In the development and maintenance of a storm vortex in approximate gradient balance, several important properties and events are needed in the dynamical evolution of the system. Within the framework of mass, momentum, and energy, vortex development involves a net outward transport of mass and a net inward flux of angular momentum about a local storm axis. An increase in kinetic energy and vorticity occurs with the convergence of the angular momentum during vortex development. The source of kinetic energy must come from an internal conversion of available potential energy within the storm volume and/or the inward transport of kinetic energy through the lateral boundaries (Johnson, 1970). Sources of vorticity are manifold, either by tilting, divergence, friction, or vertical advection within the volume or transport through a lateral boundary.

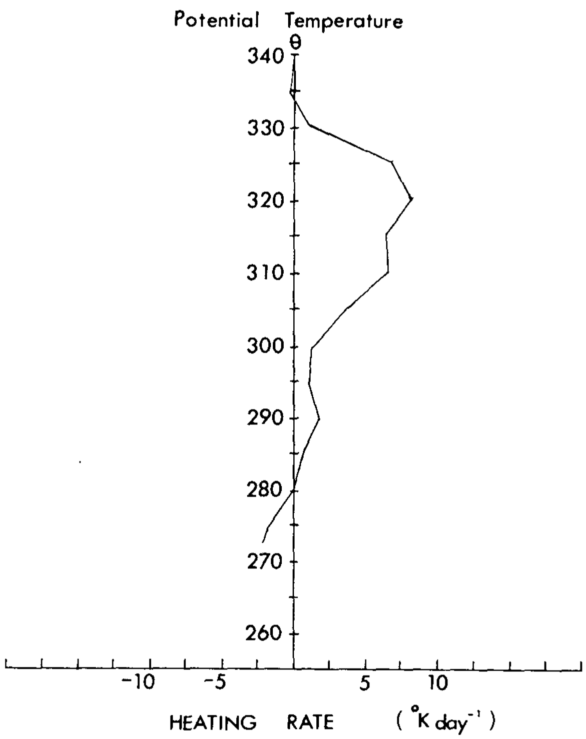

Fig. 6. Mean vertical profile of mass weighted, area-averaged heating rate, $\langle d\theta/dt \rangle$, for the period 23/1200 to 24/0000 GMT April 1968 (Downey, 1972).

Of all the properties, the constraints on the budgets of mass and absolute angular momentum provide special basic insight on dynamical forcing. These constraints are imposed from the conditions that mass is conserved and that increasing absolute angular momentum about a local storm axis can only be realized by an inward transport of the property. Friction, which is the only net force affecting angular momentum within the volume, serves to extract angular momentum at the lower boundary. The requirement of a net outward mass transport in a developing hydrostatic vortex and a net inward transport of absolute angular momentum requires that both inward and outward branches of a mass circulation exist. The inward branch must serve to transport more absolute angular momentum inward than is removed by the outward branch, while more mass is removed by the outward than inward branch. This implies that in a stratified atmosphere there may be systematic inflow within certain isentropic layers and systematic outflow in other layers. If the development is forced through instability associated with adiabatic motions, destabilization will occur in the layer of inflow, and stabilization will occur in the layer of outflow. If the development is coupled to diabatic processes, the inward mass transport may be coupled to the outward mass transport by a vertical branch in the storm vortex associated with diabatic trajectories. A systematic analysis of the modes of transport of properties essential to the vortex and their internal redistribution should serve to ultimately establish important processes which force the storm scale.

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APPENDIX A

Boundary Velocity of the Translating Storm Volume

In Fig. A1, the boundary velocity is $\mathbf{W}_B = \boldsymbol{\omega}_0 \times \mathbf{r}_B$, where $\boldsymbol{\omega}_0$ is the angular velocity of rotation of the axis through the earth's center and perpendicular to $\mathbf{r}_0$.

Likewise the storm center velocity is $\mathbf{W}_0 = \boldsymbol{\omega}_0 \times \mathbf{r}_0$.

Thus $|W_B| = |W_0| \sin\gamma$.

The normal component of W_B is $W_\beta = W_B \sin\eta$, where η is the interior angle of the spherical triangle formed by the great circle arcs from (λ_0, ϕ_0) and (λ_B, ϕ_B) to the

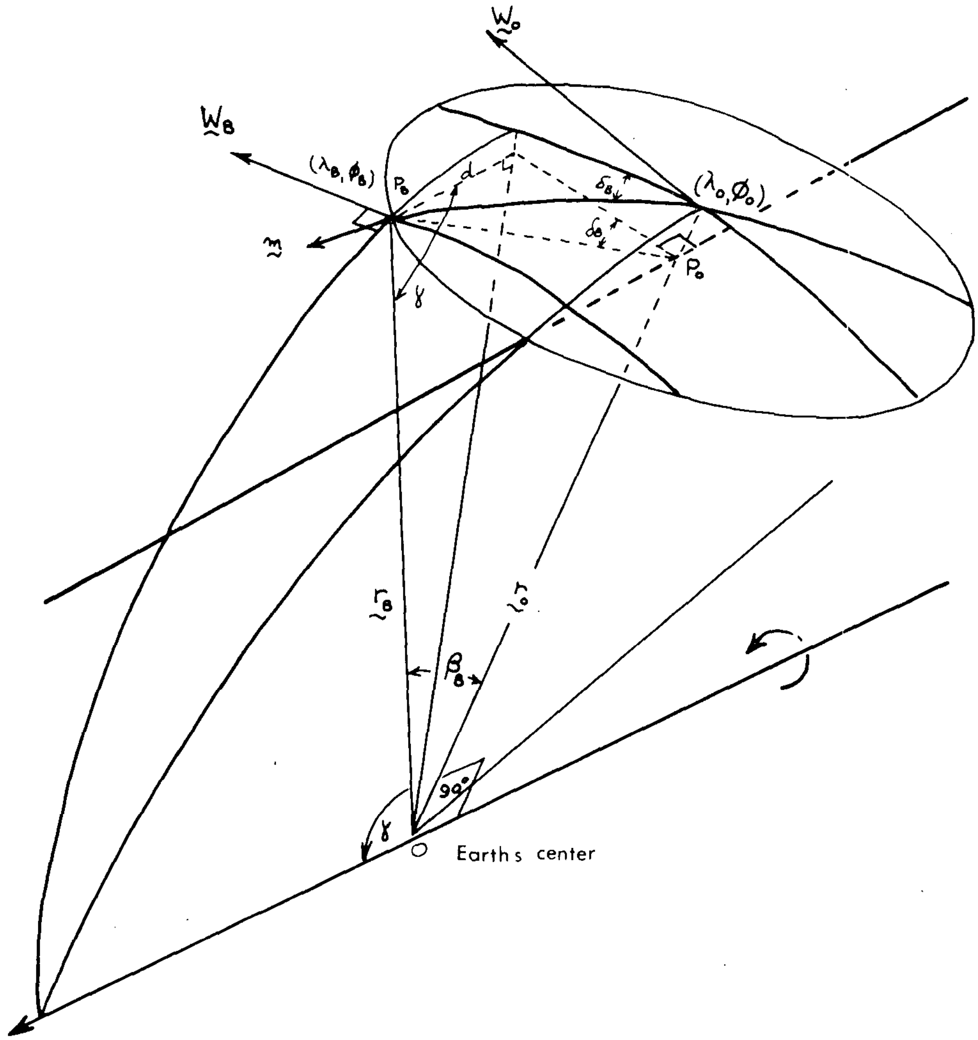

FIG. A1. Geometry related to the boundary velocity of the translating budget volume and the geographical position of the storm boundary.

pole of rotation in Fig. A1. From the law of sines,

$$\sin\eta = \frac{\sin(90^\circ - \delta_B)}{\sin\gamma},$$

and therefore

$$W_\beta = W_0 \cos\delta_B$$
$$= W_0 \cos(\alpha - \alpha_0).$$

APPENDIX B

Geographical Position of the Storm Boundary

From the haversine relation of spherical trigonometry

$$\text{hav}a = 1/2(1-\cos a)$$
$$\doteq \text{hav}(b-c) + \sin b \sin c \,\text{hav}A.$$

In Fig. A1, we let

$$a = 90 - \phi_B$$
$$b = \beta_B$$
$$c = 90 - \phi_0$$
$$A = 90 - \alpha_B.$$

Then

$$\phi_B = 90 - \cos^{-1}\{\cos(\beta_B + \phi_0 - 90^\circ) - \sin\beta_B \sin(90 - \phi_0) \times [1 - \cos(90 - \alpha_B)]\}.$$

From the law of sines for spherical coordinates,

$$\sin(\lambda_0 - \lambda_B) = \frac{\sin\beta_B \sin(90 - \alpha_B)}{\sin(90 - \phi_B)}$$

and

$$\lambda_B = \lambda_0 - \sin^{-1}\left\{\frac{\sin\beta_\delta \sin(90 - \alpha_B)}{\sin(90 - \phi_B)}\right\}.$$

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