On Some Properties of Correlation Functions Used in Optimum Interpolation Schemes

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ABSTRACT

Objective analyses using the so-called method of optimum interpolation incorporates statistical information on the variable(s) by means of the covariance or correlation functions. The concern in this contribution is with some properties of the analytic forms of the correlation functions that are used to model the statistical structure. First, some attention is directed to the question of fitting the various analytic forms (containing adjustable constants) to samples of actual correlations. All but one of the candidate forms were indistinguishable on the basis of the residuals of the statistical fitting procedure. Second, the criterion of positive-definiteness of the correlation function is extended to stipulate that the transform (or spectrum) of the function should possess some features of the spectra of actual variables—the most important one being the spectral decay rate at high wavenumber. Again, all but one of the candidate forms (the same one as above) had transforms that were acceptable. Third, the degree of isotropy of the correlation fields is examined, both for scalar variables (geopotential, temperature) and for the wind field. Finally, the imposition of geostrophy requires some special considerations on the form of the correlation function. For all of these properties a variety of suggested analytic forms are compared and conclusions drawn.

1. Introduction

One method of producing an objective analysis of meteorological variables is the so-called method of optimum interpolation. This method, introduced into the field of meteorology by Gandin (e.g., 1963), is an extension of classical least-square theory originally proposed by Gauss. The advantage of the method is that information about the spatial structure of the variable is incorporated into the procedure through the use of covariance or structure functions. The procedure is indeed “optimum” in the sense of classical least-squares. However, consideration of the assumptions needed to apply the method in practice, or of more advanced topics concerning the dynamics of the atmospheric system that the analysis is representing (Petersen and Middleton, 1963), leave the sense of the word “optimum” obscure.

Some applications of the optimum interpolation procedure have appeared in the meteorological literature: Kluge (1970), Dutt (1972), Bengtsson and Gustavsson (1972) and Belousov et al. (1968), for example. However, there are several subsidiary, rather practical, questions which do not seem to have received attention. And these ought to be resolved before a full-scale operational system producing grid-point analyses for use in dynamical models is put into practice.

There is, for example, the question of whether the objective analysis is to be for single or multiple variables (geopotential, wind, temperature, etc.) and whether the statistical information is to be included jointly (covariablely) or separately. In addition, the imposition of dynamical constraints, such as the geostrophic or a balance relationship, affect the choice of the functional form of the statistical information. In comparing an optimum interpolation scheme with other possible analysis schemes, we would like to have some knowledge of the effect of certain assumptions, such as the homogeneity and isotropy of the statistical fields, on the interpolated field.

In this paper we are concerned with some properties of the statistical information incorporated into an optimum objective analysis in the form of covariance or correlation functions of the variables. Among the considerations which must dictate the choice of the functional form of the covariance or correlation, we concentrate on two which we feel have not received adequate attention to date. These considerations are: first, how well do various possible functions fit sample correlations derived from observed data quantitatively and, second, how well do the functions satisfy constraints placed upon them by the numerical models using the analyses.

Perusal of the literature reveals a wealth of suggested analytic forms for the correlation fields of atmospheric variables; but no critical appraisal of how well such forms fit actual data has been made. Further, although Gandin (1963) points out that the selected correlation...
function must be positive definite, no more stringent constraint has been suggested. In this regard, Merilees (1974) has pointed out that the (spatial) correlation function given by the Gaussian error function used to analyze the geopotential field does not possess an energy spectrum suitable for calculation of vorticity advection by finite difference techniques. Moreover, the geostrophic kinetic energy spectrum of a geopotential field represented by this function does not correspond to that of the atmosphere (see Section 5). Thus, a suggested constraint on the correlation function utilized in optimum interpolation is that its cosine transform, or spectrum, be similar to the observed spectrum of the corresponding meteorological variable.

Another important constraint on the correlation function for isobaric height, \( r_{0h}(s) \), when the geostrophic or balance relationships are to be used, is that the first two derivatives of the function must exist. Buell (1972) has emphasized this requirement and has pointed out that certain candidate correlation functions do not possess finite derivatives at the origin. The necessary and sufficient limitations on the functional form of \( r_{0h}(s) \) imposed by this assumption are developed and discussed in Appendix 1.

Although the problem of fitting a candidate correlation function to a set of sample data has been touched upon by various authors (Gandin, 1963; Buell, 1972), no work yet reported has given adequate attention to the statistical problem of finding the “best” fit of various functions to samples of correlations of meteorological variables. The “fitting” problem is related to the point made above regarding the spectrum of the variable is the Fourier transform of the analytic function used to specify the correlation structure, the greater the fidelity of the function to the sample correlation structure, the more accurate the spectrum, and vice versa.

2. Some mathematical properties of analytic forms of covariance-correlation functions

Before examining the problems of fitting candidate correlation functions to actual data, we summarize some mathematical properties of Fourier transforms. If the correlation function for variable \( h \) (geopotential, for example) is given by \( r_{0h}(s) \), where \( s \) is meridional distance (taken always as positive), and the Fourier transform of \( r_{0h}(s) \) is denoted by \( S_{0h}(k) \), the Fourier transform pair is

\[
\begin{align*}
r_{0h}(s) &= \sum_{k=0}^{\infty} S_{0h}(k) \exp \left( \frac{-i \pi k s}{L} \right) \\
S_{0h}(k) &= \frac{2}{L} \int_{0}^{L/2} r_{0h}(s) \exp \left( -i \frac{\pi k s}{L} \right) ds,
\end{align*}
\]

where \( L \) is the maximum span of \( s \) about the globe and \( k \) the integer wavenumber. We shall denote this correspondence by \( r_{0h}(s) \leftrightarrow S_{0h}(k) \). In practice, \( s \) is restricted to some small interval in comparison to \( L \); and, without difficulty, the transform pair may be considered as if \( S_{0h}(k) \) where a continuous function.

If \( r_{0h}(s) \) is the product of two functions \( f(s) \cdot g(s) \), then the transform pair becomes

\[
\begin{align*}
f(s) &\leftrightarrow F(k) \\
g(s) &\leftrightarrow G(k)
\end{align*}
\]

and

\[
f(s)g(s) \leftrightarrow F(k) \ast G(k),
\]

where \( \ast \) denotes convolution.

A general form suggested for candidate correlation functions is

\[
r(s) = f(s; \alpha, \beta, \omega) \exp \{- (\lambda s)^\gamma \},
\]

in which \( f(s; \alpha, \beta, \omega) \) represents various polynomials in \( s \) and \( \alpha, \beta, \gamma, \lambda, \omega \) are “fitting” constants. Buell (1972), for example, suggests

\[
r(s) = (1 + \sum_{i=1}^{n} \alpha_i s^i) \exp \{- (\lambda s)^\gamma \},
\]

and Gandin (1963) and others have suggested the more general forms

\[
r(s) = \{ \alpha \cos(\omega s) + \beta \} \exp \{- \lambda s^\gamma \}
\]

and

\[
r(s) = \{ \alpha J_0(\alpha s) \} \exp \{- \lambda s^\gamma \}.
\]

In the references cited the rationale followed in developing the suggested analytic forms is not, generally, clear. In the case of (4, 5), however, oscillatory cosine and Bessel functions terms are, while not necessary, desirable because of the tendency for the observed correlations for some variables, i.e., geopotential and temperature, to oscillate about zero. In the spectral domain, this term recognizes that maximum variance is not at the lowest order wavenumbers but at wavenumbers more indicative of maximum baroclinic activity. (See, for example, Gavrilin et al., 1972.)

Quantitatively, we may invoke the modulation theorem (Bracewell, 1965) to specify the behavior of the transforms of (4). Let \( g(s) \) be the exponential decay portion of (4) and \( f(s) \) the cosine term. Then

\[
S(k) = F(k) \ast G(k) = \sum_{k'=\infty}^{\infty} F(k')G(k-k'),
\]

and since

\[
F(k) = \frac{-\delta(k_0 + k') + \alpha}{2} \delta(k_0 - k') + \beta \delta(k'),
\]

then

\[
S(k) = \frac{\alpha}{2} \delta(k-k_0) + \frac{\alpha}{2} \delta(k+k_0) + \beta \delta(k).
\]
where $\delta(\ )$ denotes the Dirac delta function, and $k_0$ the integer wavenumber defined by $\omega L/2\pi$. When $G(k)$ is a monotonically decreasing function, as it is here, then the modulation theorem indicates that the maximum in the spectrum, at $G(k=1)$, is shifted for $S(k)$ toward the wavenumber specified by the argument of the cosine term.

Although the decay of spectral energy (or variance) at large $k$ is primarily determined by the constants $\lambda$ and $\gamma$, the “shift” produced by the cosine term will also affect the decay rate.

Certain of the candidate correlation functions that have appeared in the literature have Fourier transforms that are analytic. These are denoted in Table 1. In general, however, it was necessary to resort to numerical transforms for the purposes of determining the spectral characteristics of the candidate functions. Two different numerical methods were used. Both attempted to circumvent problems associated with making the analytic correlation function fitted to sample data strictly periodic on $L$. One method permits the function $r(s)$ to go to infinity, effectively ignoring its periodic nature. The other method, in the spirit of the estimation of continuous spectra, multiplies $r(s)$ by a function decreasing monotonically to zero, at $L/2$. The methods gave nearly identical results—certainly within the bounds of the uncertainty on actual spectral behavior.

In addition to these considerations, we now review some mathematical properties of “spectral windows” (Jenkins and Watts, 1968). These considerations are relevant for two reasons. First, the procedure of fitting a candidate function to a set of sample correlations has always been characterized by the use of a restricted range of separation distance. Gandin (1963), for example, restricts the correlation fields to a separation of 3000 to 4000 km. Buell (1972) restricts his fitting considerations to a separation distance of 2500 km. Data availability imposes a short distance cut-off of 100–200 km. Thus, the correlation function selected can only be influenced by the correlation structure in a restricted distance range, and the transform of this function, or spectrum, must reflect this restriction. Second, the spectrum of interpolated values produced by an optimum interpolation scheme using the correlation function selected will be determined by the spectrum of the function modified appropriately by the fact that the interpolation procedure is restricted in practice (also) to a restricted range of separation distance. Both of these considerations involve “viewing” the actual spectrum through a spectral window; in the first case determined by the distance range over which the correlation function is estimated or fitted, and in the second case by the range over which it is used. Thus, if we denote the actual spectrum by $S(k)$, the transform of the selected correlation function by $\hat{S}(k)$, and the spectrum of a set of interpolated points using the spectral function as $S'(k)$, then

$$S'(k) = S(k) * W_1(k)$$

and

$$S'(k) = S(k) * W_2(k),$$

where

$$W_1(k) = \left\{ \begin{array}{ll}
0, & 0 \leq s \leq l_1 \\
1, & l_1 \leq s \leq l_2 \\
0, & s > l_2
\end{array} \right.$$

$$W_2(k) = \frac{2}{L} \int_0^{L/3} w_1(s) \exp \left( -i \frac{ks}{L} \right) ds$$

and

$$W_2(k) = \frac{2}{L} \int_0^{L/3} w_2(s) \exp \left( -i \frac{ks}{L} \right) ds$$

where

$$w_1(s) = \sin \left( \frac{\pi k l_2}{L} \right) \sin \left( \frac{\pi k l_1}{L} \right)$$

and

$$w_2(s) = \sin \left( \frac{\pi k l_3}{L} \right).$$

Here, $l_1$ is the least distance, $s$, for which correlation data are available, $l_2$ is the greatest distance at which correlation data are used, and $l_3$ is the greatest distance over which the interpolation is carried. The quantities $k$, $L$, etc., are as defined previously.

We note, to conclude this section, that the form of the spectral window given by $(\sin \pi x)/(\pi x)$ has zeros at integer values of $x$, the first such zero occurring at $x = 1$, or in the above expressions when $k l_1/L = 1$.  

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**Table 1. Candidate analytic correlation functions. Parameters are $\alpha$, $\gamma$, $\lambda$, $\omega$. Zero intercept is $s$, independent (distance) variable is $r$.**

<table>
<thead>
<tr>
<th>Abbr.</th>
<th>Form</th>
<th>Fixed parameters</th>
<th>Analytic spectrum*</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1.0</td>
<td>$[\alpha \cos(\omega r) + r - \alpha \exp(-\lambda \gamma)]$</td>
<td>$\alpha = \gamma = 1.0$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>R1.1</td>
<td>$\gamma = 1.0$</td>
<td>$\gamma = 2.0$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>R1.2</td>
<td>$\gamma = 2.0$</td>
<td>none</td>
<td>$\gamma = 1.0$</td>
</tr>
<tr>
<td>R1.3</td>
<td>none</td>
<td>$\gamma = 0.5$</td>
<td>$\pi$</td>
</tr>
<tr>
<td>R2.0</td>
<td>$[\alpha \cos(\omega r) + r - \alpha \exp(-\lambda \gamma)]$</td>
<td>none</td>
<td>$\gamma = 1.0$</td>
</tr>
<tr>
<td>R2.1</td>
<td></td>
<td>none</td>
<td>$\gamma = 0.5$</td>
</tr>
<tr>
<td>R3.0</td>
<td>$[\alpha \cos(\omega r) + r - \alpha \exp(-\lambda \gamma)]$</td>
<td>none</td>
<td>$\gamma = 1.0$</td>
</tr>
<tr>
<td>R3.1</td>
<td></td>
<td>none</td>
<td>$\gamma = 0.5$</td>
</tr>
<tr>
<td>R4.0</td>
<td>$[\alpha \cos(\omega r) + r - \alpha \exp(-\lambda \gamma)]$</td>
<td>none</td>
<td>$\gamma = 1.0$</td>
</tr>
<tr>
<td>R4.1</td>
<td></td>
<td>none</td>
<td>$\gamma = 0.5$</td>
</tr>
</tbody>
</table>

* See Appendix 2.
3. Statistical fitting techniques for correlation functions

The various parameters in the analytic functions need to be determined by fitting the functions to a sample of observed correlations, \( \hat{r}(s) \). Furthermore, it is desired to have some quantitative measure of goodness-of-fit to allow comparison of the different candidate functions. It is natural to use a mean-square difference, or error variance, statistic defined as

\[
EV = \frac{1}{n} \sum_{j=1}^{n} (r(j) - \hat{r}(j))^2,
\]

where \( j \) includes the set of discrete values for \( s \) for which \( \hat{r} \) is available.

In general the analytic candidate correlation function will be nonlinear in several adjustable constants or parameters. Owing to this nonlinearity, any fitting procedure used must be adapted to minimizing the error variance statistic in a parameter space for which the geometry of this statistic may be highly convoluted. We have investigated three variations on nonlinear least-squares algorithms which are capable of finding the minimum of EV in parameter space. The first of these is a rather straightforward approach using derivatives with respect to the parameters, which may be obtained from the known, analytic form of \( r \). The derivatives are used in an iterative scheme to find the multi-parameter vector which minimizes the error variance. In practice this method proves over-sensitive to the initial guess values for the parameters, and runs into instability problems. Two other methods, not requiring specification of the derivatives in analytic form, were simpler to use and gave satisfactory results.

The Marquardt algorithm for minimizing the error variance (see Marquardt, 1963) uses an optimum melding of the method of steepest descent and the classical Taylor-series, or Gauss, method. The final method, after Zangwill (1967), is a minimization procedure for nonlinear functions, most efficient for quadratic forms. In our case, the local minimum in EV is found; so, in effect, the latter method is performing the same exercise as the Marquardt method.

4. Analysis of isotropic correlation functions for 500 mb height and temperature

In the next two sections we will be concerned with an analysis of certain (scalar) correlation functions. To keep the analysis within bounds we will restrict our considerations to correlation functions which represent isotropic fields; specifically, geopotential and temperature. Thus we have accepted assumptions of homogeneity of the correlation fields and their isotropy. To translate to desired covariance functions, the correlation function may be multiplied by local variances, so that we need not assume homogeneity of the variances.

We extend previous work on suggested correlation functions by examining sample correlation fields to distances as great as 3500 km. The justification for this is the emphasis we wish to place on the spatial characteristics of analyzed fields; we do not imply that the function adopted in optimum interpolation should be used at distances of this order.

We have modified candidate functions suggested in the literature so that they are more adaptable for fitting sample correlations. For example, we have introduced the proper parameterization so that \( r(s=0) \) is not necessarily unity; and we have included an oscillatory term to account for the location of maximum spectral energy away from the lowest-order wavenumbers.

Previous investigators (Buell, 1972; Gandin, 1963) have called attention to the problems encountered when the candidate correlation functions are fitted to data at \( s=0 \). It is important to remember that even though the population correlation function must be identically unity at \( s=0 \), any attempt to estimate this function with instruments containing error will not give \( r(0)=1.0 \). Correspondingly, the evaluation of candidate functions and their behavior at \( s=0 \) should recognize the presence of instrumental error. Thus the zero intercept, \( Z \), is estimated here as

\[
Z = \frac{\sigma_s^2 - \sigma_s^2}{\delta_0^2},
\]

where the measured variances are given by \( \delta_0^2 \) and the instrumental error variances by \( \sigma_s^2 \). Using estimates of rms instrumental error in the U. S. rawinsode system (at 500 mb) of 12 m for geopotential and 0.2° for temperature, we arrive at a \( Z \) of approximately 0.98 for both variables. In general, this value can be expected to be a function of the variable, the season, and the altitude.

The sample correlations were computed over the winter three-month periods for 1966 and 1967. The observations were those originating from all stations within a distance of 3250 km of the Topeka, Kans., rawinsode station—essentially the North American rawinsode station net. Correlations (and covariances) were computed for all station pairs using the local time-average covariance estimates recommended by one of the authors (Thibeaux, 1974). Since these sample correlations are representative of the higher (time) frequency spatial patterns, they are not directly comparable with conventional spatial covariance estimates. However, Thibeaux (1975) gives both covariances (correlations) for comparison.

The distribution of sample points can be seen in Fig. 1a for geopotential height and Fig. 1b for temperature. In addition, the total cloud of points was averaged in 65 km non-overlapping intervals or blocks. These smoothed estimates are shown in Figs. 2a and 2b.
For the set of candidate functions given in Table 1 the results of the fitting algorithm are shown in Table 2. The error variance is entered in the Table, with the "best" values of the parameters.

The following points, concerning the fits of the candidate functions to the 500 mb winter height and temperature correlation fields, are important.

The error variance statistic does not vary greatly for the various candidate functions, for geopotential 3–4% and temperature 13–14%. However, the error variance for the Gaussian-type exponential function $R1.2$ is consistently greater than for the other functions. Qualitatively this can be seen in the poorer fit for small $s$ for this function, compared with a simple exponential,

$R1.1$, or the inverse polynomial form, $R2$. The major difference is clearly in how these functions approach 0.98; that is, the fit in the neighborhood of $s=0$. The fitting algorithm, if left to its own devices with the $R1$, form, chooses a value of $\gamma=1.24$ for the height and a value of 0.96 for the temperature. Neither of these is substantially different from the simple exponentially-damped cosine, $R1.1$.

A great amount of difficulty was experienced in obtaining stable estimates of the parameters for the Gaussian-damped function, $R1.2$. The values returned by the least-squares minimization algorithms proved to be quite sensitive to the starting values. Apparently, local minima were found in the error variance "surface"
TABLE 2. Fitted parameter values for candidate functions, Table 1, for 500 mb height and temperature. Units of s are in 10^9 km.

<table>
<thead>
<tr>
<th>Family</th>
<th>R1.1</th>
<th>R1.2</th>
<th>R1.3</th>
<th>R1.1</th>
<th>R1.2</th>
<th>R1.3</th>
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<tbody>
<tr>
<td>γ</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
<td>1.0</td>
<td>2.0</td>
<td>1.0</td>
</tr>
<tr>
<td>α</td>
<td>0.706</td>
<td>0.783</td>
<td>0.729</td>
<td>0.744</td>
<td>0.707</td>
<td>0.744</td>
</tr>
<tr>
<td>ζ</td>
<td>1.41</td>
<td>1.41</td>
<td>1.38</td>
<td>1.38</td>
<td>1.54</td>
<td>1.40</td>
</tr>
<tr>
<td>λ</td>
<td>0.259</td>
<td>0.192</td>
<td>10^{-4}</td>
<td>0.888</td>
<td>10^{-4}</td>
<td>0.637</td>
</tr>
<tr>
<td>EV</td>
<td>0.0280</td>
<td>0.0285</td>
<td>0.0280</td>
<td>0.0233</td>
<td>0.0267</td>
<td>0.0233</td>
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<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
<td>0.30</td>
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<tr>
<td>α</td>
<td>0.716</td>
<td>0.738</td>
<td>0.694</td>
<td>0.741</td>
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<td>ζ</td>
<td>1.40</td>
<td>1.58</td>
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<td>1.38</td>
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<tr>
<td>λ</td>
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<td>0.848</td>
<td>2.13</td>
<td>1.78</td>
</tr>
<tr>
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<td>0.0280</td>
<td>0.0234</td>
<td>0.0236</td>
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<table>
<thead>
<tr>
<th>Family</th>
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<th>R2.1</th>
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<tbody>
<tr>
<td>γ</td>
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<td>0.65</td>
</tr>
<tr>
<td>α</td>
<td>0.971</td>
<td>0.943</td>
</tr>
<tr>
<td>ζ</td>
<td>?</td>
<td>0.5</td>
</tr>
<tr>
<td>λ</td>
<td>0.215</td>
<td>24.83</td>
</tr>
<tr>
<td>EV</td>
<td>0.0280</td>
<td>0.0281</td>
</tr>
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</table>

5. Spectra derived from correlation functions

Two specific features of observed spectra of atmospheric variables ought to be reproduced by the transform of the correlation function. The first of these is the spectral maximum at wavenumbers k=5-8 caused by maximum baroclinic instability; and the other is a decay rate at large k which matches that of observed spectra.

Charney's theory (1971) predicts that the spectrum for temperature in the free atmosphere should decay at a rate proportional to the inverse third power of the wavenumber, the same behavior expected of the kinetic energy spectra. Such behavior in the atmosphere for which the parameter values were clearly out of range. That is, the synthesized function using those parameter values did not result in a curve which went "through the points." Both fitting algorithms experienced this difficulty.

An appreciation of the latter problem can be gained by referring to Fig. 2b. No variation in the parameters of R1.2 is capable of molding the function to fit simultaneously the initial rapid decrease in the correlations, the rather flat minimum in the range s=1700-2000 km, and the gradual almost linear increase for s>2000 km. We note that four of the observed correlations for s<400 km (Fig. 2) are noticeably lower than those for slightly greater distances. This is due to the anisotropy of the correlation field (Section 6) coupled with a sampling bias in direction for the closest correlations. The fitting procedure was re-run with these points omitted with no significant differences in the parameter values. We conclude that the problems of fitting the Gaussian-damped function R1.2 stem simply from an incorrect form of the function which cannot satisfactorily reproduce the sample correlations.

The goodness-of-fit of the inverse polynomial form R2 is hardly distinguishable from that of the simple exponentially-damped cosine; and, furthermore, does not seem to be very sensitive to the value of γ. The error variance figures are within a single percent, whether γ=0.5 or 0.3.

The principle conclusion drawn from this examination of the statistical fit of the various candidate functions for the isotropic scalar height and temperature fields is that the selection of the function need not be based upon consideration of fit, excepting that the Gaussian function, R1.2, is not satisfactory. Other considerations, such as the form of the spectrum of the function (Section 5) or the requirements on the existence of derivatives at the origin should be of more importance. We note in this regard (Appendix 1) that form R1 possesses the proper derivatives only for values of γ≥2.

![Figure 3a](image-url)  
*Fig. 3a. One-dimensional wavenumber spectra of candidate correlation function family R1 (Table 1) for 500 mb isobaric height, with appropriate values of γ used to label the various spectra. The spectra are shown as continuous curves as an aid in visualizing the decay rates at high wavenumbers. A decay rate of the minus third power of the wavenumber is shown.*

![Figure 3b](image-url)  
*Fig. 3b. Same as Fig. 3c, but for 500 mb temperature.*
has been roughly confirmed by Kao (1970) and Julian and Cline (1974). The behavior of the spectra of geopotential is not as clear. Oddly enough, we could not locate any observed spectra of geopotential in the published literature. Theory would suggest a decay according to the minus fifth power of the wavenumber if the wind and mass fields are in geostrophic balance. However, since 500 mb is near the equivalent barotropic level, the very high correlation between the height and the temperature fields suggests behavior nearer to the minus third power. Lacking any empirical determinations of the decay rate, we will accept as satisfactory any decay rate in the $-3$ to $-5$ range.

The Fourier transforms of a sample candidate function using the "best" values for the parameters given by the least-squares fitting algorithms is shown in Fig. 5. For the $RJ$ family, the spectra are plotted for selected values of $\gamma$. In general, the use of a cosine term in the function produces a spectral peak which is too cusp-like. The zeroth order Bessel function does a better job of producing a smooth spectral peak (this spectrum is not shown; however, reference can be made to Fig. 6). Regarding the decay rates at $k>10$, all versions of the functions are generally satisfactory excluding those damped with the exponential with $\gamma=2$. The decay rate in the range $k=10$ to 20 for this spectrum exceeds a minus five slope; and an examination of the analytic form of the transform of this function reveals that the decay rate becomes even larger at higher wavenumbers. We conclude that, because of the highly unrealistic spectra associated with the Gaussian-type exponential function and because it rather clearly does not do a reasonable job of fitting the observed correlation field, it must be eliminated as a candidate function, for both the height and temperature fields.

The spectra that result from the transform of the fitted correlation functions are approximations to the spectra of the actual meteorological variables that the functions are modeling. In the statistical fitting procedure, and to a large extent in the selection of an analytic form for the correlation function, it is only correlation data in a restricted range of distance separation, say 200 to 3000 km, that are utilized. Therefore, the function that is chosen can only have a transform which is an approximation to the actual spectrum. From Section 2, we see that if the correlations are distributed over the range 200 to 2000 km then the spectral window through which the actual spectrum is viewed will be given by (10) with the appropriate scaling. Figure 4 presents this spectral window. Since the effect on the spectrum of the correlation function caused by obtaining it over this restricted distance range is given by the convolution of the spectral window and the actual spectrum, the proper interpretation of Fig. 4 is made by considering the plot as the right-hand half of a symmetric function which can be centered at any wavenumber, $k$, and with abscissa equal to $k-k'$.

![Spectral Window](image)

**Fig. 4.** Spectral window for a correlation (or covariance) function resolved over a distance of 200 to 3000 km. Abscissa is the one-dimensional integer wavenumber.

The resulting spectrum is given by

$$\hat{S}(k) = \sum_{k'} S(k') W_1 (k-k').$$ (13)

The principle feature of Fig. 4 is the width of the spectral window. The first zero in the $(\sin \pi x)/(\pi x)$ function occurs at $x=1.0$ which, as noted in Section 2, with the appropriate scaling, occurs at roughly $k-k' = L/1 = 28 800/3000 = 9$. Thus, the effect of "viewing" the spectrum through the spectral window resulting from the use of correlation data in the restricted region $S=200$ to 3000 km is to smear out the true spectrum. The smearing (in terms of zonal wavenumber) is rather large: $\pm 9$ for the width to zero and roughly $\pm 6$ for the half-width.

The smearing that occurs when a particular correlation function is used to generate grid point values, using an optimum interpolation scheme, would be even greater. For example, it is doubtful that the interpolation would ever be extended in practice over a distance more than say 800 km. In that case, Eq. (11) stipulates that the half-width of the spectral window would be about $\pm 25$ wavenumbers, which is a substantial fraction of the total resolvable range of wavenumbers in practical grid meshes presently in use.

The considerations given here of the characteristics of the spectral windows inherent in the procedures used by an optimum interpolation scheme suggest that the exact form of the analytic function used to represent the correlation (covariances) structure of atmospheric variables is not overly important, at least as far as the spectrum of the resulting interpolated values is concerned.
6. Anisotropy of geopotential height and temperature correlation fields

The considerable spread of values of the sample correlations (Fig. 1), particularly in the distance range $s=800$–$2700$ km, is not due to sampling error. Taking into account the relative direction between the stations in each pair greatly reduces the variability of the observed correlations. In other words, the scalar correlation fields of isobaric height and temperature are anisotropic.

Figure 5 distinguishes between sample correlations of height and temperature in the zonal and the meridional directions. The stratification of the sample data, Fig. 1, was based on an angular aperture of about $\pm 8^\circ$ from

![Fig. 5a. Correlation function $R4.0$ fitted to 500 mb isobaric height correlations in the zonal direction (solid points and line) and in the meridional direction (open points, dashed line). Abscissa is distance separation in kilometers.](image)

![Fig. 5b. Same as Fig. 5a, but for 500 mb temperature.](image)

Table 3. Results of fitting candidate correlation functions non-isotropic, 500 mb height, winter. Units of $s$ are in $10^3$ km.

<table>
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the east–west and north–south directions. Table 3 contains the error variance and parameter values for the candidate functions for the two directions fitted separately. It is quite evident that the parameters in each of the analytic functions must be quite different in the zonal and meridional directions, simply because the dominant length scale is quite different. The oscillatory behavior of the correlation fields is stronger

![Fig. 6. Wavenumber spectra in zonal (heavy line) and meridional (lighter line) directions. These spectra result from the transform of the functions shown in Fig. 5a.](image)
in the zonal than in the meridional direction. This difference is what would be expected qualitatively from a flow field consisting of wave-like disturbances superimposed on a circumpolar vortex.

The spectra of fitted function $R4.0$ is presented in graphic form in Fig. 6. Spectral peaks are indicated at zonal and meridional wavenumbers 4–5 when the proper respective scaling is used.

Little difference in the spectral decay rates at higher wavenumbers occurs in the two directions. Again the rates are roughly $a = -3$ power for the temperature (not shown) and $a = -5$ power for the geopotential height.

While there is no difficulty, in principle, in accounting for the anisotropy of the statistical fields in an optimum interpolation scheme, it remains to be shown that the increased computer time and associated complexity would be compensated by a more accurate analysis.

7. Analytic correlation functions for the vector wind field

Examination of the problem of arriving at satisfactory analytic functions for the wind field has been delayed to this section because of the special considerations that must apply to a vector rather than a scalar field. The special problem of the meaning of isotropy of a vector field has been discussed in correspondence by Buell (1973) and Alaka and Elvander (1973). Since we wish to examine the degree of reality of the assumptions usually invoked in optimum interpolation schemes applied to wind fields, we prefer not to make assumptions about the isotropy of the wind field. Further, we prefer to use the methodology of Buell (1971, 1973) who points out the advantages of using components of the vector wind along and normal to the direction (vector) $s$; namely, the longitudinal and traverse components, $r_{\mu}$ and $r_{\nu}$, respectively. Such a convention has a number of advantages, not the least of which is that the resulting correlation fields have a simpler morphology.

To arrive at a sample of correlation estimates it was convenient in our case to restrict the sample correlation data to the (conventional) $u$- and $v$-component data in the east–west and north–south directions. Thus, correlations from all station pairs do not appear in the sample. In this stratification, estimates of $r_{\mu}$ are the $u$-component in the E–W and $v$ in the N–S directions, while estimates of $r_{\nu}$ are the $u$-component in the N–S and $v$-component in the E–W directions. Some idea of the anisotropy of the wind field can then be obtained by comparing the first two and the last two samples. The same stratification has been used by Ramanathan et al. (1973) for wind data over the Indian sub-continent.

Table 4 presents the error variance and parameter values for certain of the candidate functions fitted to stratified wind correlations.

An evaluation of the results of fitting the three analytic forms suggests that either the simple exponential $R1.1$ or the $R2$ form with $\gamma$ left to vary is acceptable. A more important consideration would appear to be the problem of the obvious anisotropy of the wind field. The dissimilarity of the two samples of the longitudinal and the transverse correlations can best be seen by referring to Fig. 7. Here, Fig. 7a contains sample correlations of the $u$ (or zonal component) of the wind for station-pairs oriented east–west plotted as dots and for the $v$ (or meridional) component for stations in the north–south direction as open circles. The same angular aperture was used here as for the stratification of the geopotential and temperature correlations in Section 5. The corresponding plots in Fig. 7b are for the transverse correlations.

Rather than show the family of curves resulting from the curve fitting summarized in Table 4, which does not include an attempt to fit an isotropic $r_{\mu}$ and $r_{\nu}$ function, we show instead curves derived from the geostrophic constraint imposed using the candidate function $R2.1$ (Table 2) for geopotential height. These curves are seen to be a good compromise between the two sets of longitudinal and transverse correlations, and therefore, could be used to interpolate the wind field in an objective analysis scheme. If, however, the anisotropy of the wind and height fields is to be accommodated, a more complicated procedure would be needed.

8. Conclusions

In the previous sections we have considered some of the statistical properties of various correlation functions which could be used in optimum interpolation schemes. In Section 3 we were concerned with the modeling of observed correlation fields by the use of analytic
describe the set of sample correlations with one exception: the form containing the Gaussian exponential decay consistently exhibited a larger error of fit than its competitors.

Since the cosine transform of the correlation function, i.e., the wavenumber spectrum, contains equivalent statistical information, we examined the characteristics of the spectra of the fitted functions in Section 4. In general, all candidate forms but one possessed two features exhibited by actual spectra of atmospheric variables: namely, a spectral peak at appropriate wave-numbers and a decay rate at higher wavenumbers consistent with that of observed spectra. Again, the form containing the Gaussian exponential factor did not compare favorably with its competitors because the spectral decay rate was much too rapid. Because the ability of a function to describe an observed correlation field is not independent of how well its transform matches the spectrum of the correlations, it is perhaps not fortuitous that this candidate function was deficient on both counts.

The field of sample correlations used in selecting or fitting a candidate function is normally restricted to separation distances of a few hundred kilometers at the nearest to a few thousand kilometers at the farthest. The procedure of modeling the correlation field by fitting a function on this restricted separation scale of itself limits the spectral information that the chosen function can contain. Similarly, using that function in an optimum interpolation scheme in which the function is used only over a restricted distance range limits the spectral information in the interpolated values. Quantitative considerations of the appropriate spectral windows in the two instances stipulates that only the grossest characteristics of the true spectrum are reproduced by the modeling and use of correlation information.

Finally, simple comparisons of the correlation fields in the zonal and in the meridional directions indicate that neither the geopotential, temperature, or wind field is isotropic. It is not clear, however, that any increased accuracy achieved by using a relatively complicated anisotropic scheme balances the increased effort needed to implement it. Work by one of us (Thiebaux, 1975) indicates that the increase in accuracy resulting from the use of an anisotropic correlation function may be minimal.

The theory of optimum interpolation incorporates statistical information on the structure of the variables through the use of covariance or correlation information. Indeed, given some assumptions about the universality of the statistics, it is this feature which insures the optimality of the scheme, in the least-squares sense. However, it is curious that in modeling the correlation information and applying the scheme, the procedure is not more sensitive to the actual statistical information in the meteorological variables. The inability of the scheme in practice to reproduce the proper spectral
distribution of the variable being interpolated (Section 5) suggests that the practical limitations imposed on how the statistical information of the population is in fact used, limits the sense of the optimality of the theoretical scheme. This raises, then, the fundamental question of just what statistical information is important for incorporation in an objective analysis scheme.

APPENDIX 1

Conditions for the existence of the variances and covariances of the geostrophic wind, at zero distance.

Theorem. If isotropic function $R(s)$ is the autocovariance for the geopotential field, then the covariances for the wind fields, derived from the geostrophic relations $u = - (g / f) (\partial \theta / \partial y)$ and $v = (g / f) (\partial \theta / \partial x)$ together with the second derivatives of $R$, all exist at $s = 0$ if and only if

$$(A) \lim_{s \to 0} \left( \frac{1}{s} \frac{\partial R}{\partial s} \right) \text{ is finite and } (B) \lim_{s \to 0} \left( \frac{1}{s} \frac{\partial^2 R}{\partial s^2} \right) = 0.$$

Proof. Since we are only interested in the behavior of the derivatives in the vicinity of $s = 0$, it is sufficient to work in rectangular coordinates and use the representation $s^2 = (x-x')^2 + (y-y')^2$. Consider first

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial s} \frac{y-y'}{s}$$

and

$$\frac{\partial^2 R}{\partial y^2} = \frac{\partial^2 R}{\partial s^2} \left( \frac{y-y'}{s} \right)^2 + \frac{\partial R}{\partial s} \left( \frac{-s + (y-y')^2}{s^2} \right)$$

$$= \frac{1}{s} \frac{\partial R}{\partial s} \left( \frac{y-y'}{s} \right)^2 + \frac{1}{s} \frac{\partial^2 R}{\partial s^2} + \frac{1}{s} \frac{\partial^2 R}{\partial s^2}.$$

For family $R_1$,

$$R(s) = [\alpha \cos(\omega_0 s) + \beta] \exp(-\lambda s^2), \lim_{s \to 0} \frac{\partial R}{\partial s} = \lim_{s \to 0} \frac{\omega_0}{s} \sin(\omega_0 s)$$

$$-2\lambda [\alpha \cos(\omega_0 s) + \beta] - 4\lambda \alpha \omega_0 s \sin(\omega_0 s)$$

$$+ \alpha (\omega_0^2 + 2\lambda) \cos(\omega_0 s) - 2\lambda \lambda \alpha \cos(\omega_0 s)$$

$$+ 2\lambda \beta (1 - 2\lambda s^2) \exp(-\lambda s^2) = 0.$$

Therefore conditions $A$ and $B$ are satisfied for $\gamma = 2$.

For family $R_2$,

$$R(s) = [\alpha \cos(\omega_0 s) + \beta] \left[ 1 + (\lambda s^2) \right]^{-\gamma}$$

$$\lim_{s \to 0} \left( \frac{1}{s} \frac{\partial R}{\partial s} \right) = - \left\{ \frac{\alpha \omega_0 \sin(\omega_0 s)}{s} \right\}$$

$$+ \left\{ \frac{\alpha \cos(\omega_0 s) + \beta}{s} \right\} \left[ -2\lambda^2 \gamma \right]$$

$$[1 + (\lambda s^2)]^{-\gamma - 1} = - \alpha \omega_0^2 - 2\lambda \gamma (\alpha + \beta).$$

Thus, $A$ is satisfied for all $\gamma$.

$$\lim_{s \to 0} \left( \frac{1}{s} \frac{\partial^2 R}{\partial s^2} \right) = \lim_{s \to 0} \frac{\alpha \omega_0 \left[ 1 + (\lambda s^2) \right] \sin(\omega_0 s)}{s}$$

$$+ 2\lambda^2 \gamma \left[ \alpha \cos(\omega_0 s) + \beta \right] + 4\lambda^2 \alpha \omega_0 s \sin(\omega_0 s)$$

$$- 2\lambda^2 \gamma \left[ \alpha \cos(\omega_0 s) + \beta \right] - \alpha \omega_0^2 \left[ 1 + (\lambda s^2) \right] \cos(\omega_0 s)$$

$$\times \left( 2\lambda^2 \gamma + 1 - \frac{\alpha \cos(\omega_0 s) + \beta}{1 + (\lambda s^2)} \right) [1 + (\lambda s^2)]^{-\gamma - 1} = 0.$$

Therefore conditions $A$ and $B$ are satisfied for all $\gamma$. Similar evaluations indicate that $R_3$ satisfies the conditions if and only if $\gamma \geq 2$, and $R_4$, for all values of $\gamma$.

APPENDIX 2

Analytic forms for the Fourier (cosine) transforms of the correlation functions, Table 1. Note:

$$S(s) = \int_{-\infty}^{\infty} R(s) \exp(-i\omega s) \, ds$$

1) Fourier transform of $R_1.1, R(s) = [\alpha \cos(\omega_0 s) + \beta] \times \exp(-\lambda s^2)$

$$S(\omega) = \alpha \lambda \left\{ \frac{1}{\lambda^2 + (\omega - \omega_0)^2} + \frac{1}{\lambda^2 + (\omega + \omega_0)^2} + \frac{\beta}{\lambda^2 + \omega^2} \right\}$$

2) Fourier transform of $R_1.2, \omega(s) = [\alpha \cos(\omega_0 s) + \beta] \times \exp(-\lambda s^2)$

$$S(\omega) = \frac{\alpha}{2\lambda} \left\{ \exp\left[ - (\omega - \omega_0)^2 / 4\lambda^2 \right] 

+ \exp\left[ - (\omega + \omega_0)^2 / 4\lambda^2 \right] + \frac{2\beta}{\alpha} \exp\left[ - \omega^2 / 4\lambda^2 \right] \right\}$$

3) Fourier transform of $R_2.0, R(s) = [\alpha \cos(\omega_0 s) + \beta] \times [1 + (\lambda s^2)]^{-\gamma}$

$$S(\omega) = \lambda^{-\gamma} (\pi \lambda)^{-\frac{1}{2}} \Gamma(\frac{1}{2}) \left\{ \frac{1}{2\pi} \left[ \frac{(\omega - \omega_0)}{(\omega + \omega_0)} \right]^{\frac{\gamma - 1}{2}} \right\}$$

$$\times K_{(\gamma - 1)} \left( \frac{\omega - \omega_0}{\lambda} \right) + \frac{2\pi}{\lambda^{\gamma - 1}} K_{(\gamma - 1)} \left( \frac{\omega + \omega_0}{\lambda} \right)$$

$$+ 2\beta \omega \left[ K_{(\gamma - 1)} \left( \frac{\omega - \omega_0}{\lambda} \right) - K_{(\gamma - 1)} \left( \frac{\omega + \omega_0}{\lambda} \right) \right]$$

$$+ 2\beta \omega \left[ K_{(\gamma - 1)} \left( \frac{\omega - \omega_0}{\lambda} \right) - K_{(\gamma - 1)} \left( \frac{\omega + \omega_0}{\lambda} \right) \right].$$
where $K_{(v)}(z)$ is the modified Hankel function

$$K_{(v)}(z) = \frac{\pi i}{2} \exp(v \pi i/2) H_{(1)}^{(1)}(iz)$$

with special case

$$K_0(ax) = \int_0^\infty \frac{\cos(xt)}{\sqrt{a^2+t^2}} dt \quad a, x \text{ real and positive.}$$

**REFERENCES**


