Decision-Making Models in the Cost-Loss Ratio Situation and Measures of the Value of Probability Forecasts

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ABSTRACT

In this paper we describe and compare two models of the familiar cost-loss ratio situation. This situation involves a decision maker who must decide whether or not to take protective action, with respect to some activity or operation, in the face of uncertainty as to whether or not weather adverse to the activity will occur. The original model, first described by J. C. Thompson, is based in part upon the (implicit) assumption that taking protective action *completely* eliminates the loss associated with the occurrence of adverse weather. In the model formulated in this paper, on the other hand, it is assumed that taking protective action may reduce *or* eliminate this loss. The original model, then, is a special case of this "generalized" model. We show that the decision rule in each model depends upon a cost-loss ratio and that *in both models* this ratio is simply the cost of protection divided by the protectable portion of the loss. Thus the two models are equivalent from a decision-making point of view. This result also implies that the original model is applicable to a wider class of decision-making situations than has generally been recognized heretofore.

We also formulate measures of the value of probability forecasts within the frameworks of these models. First, the expenses (i.e., costs and losses) are translated into utilities, which are assumed to express the decision maker's preferences for the consequences. Then, probabilistic specifications of the utilities are briefly discussed and general expressions are presented for the appropriate measures of value in cost-loss ratio situations with such specifications, namely expected-utility measures. Finally, we formulate the expected-utility measure associated with each model when the relevant utilities are assumed to possess a uniform probability distribution. Both measures are then shown to be *equivalent* (i.e., linearly related) to the Brier, or probability, score, a familiar measure of the accuracy of probability forecasts. These results provide additional support for the use of the probability score as an evaluation measure.

1. Introduction

Models of decision-making situations, in which weather information in general and weather forecasts in particular play a significant role, serve at least two important purposes. First, they prescribe how the forecasts should be used by decision makers (within the contexts of the respective models). Second, the models provide frameworks within which decision makers (and others) can assess the value of the forecasts. In this paper we investigate both the utilization and evaluation of *probability* forecasts within the framework of two models of a particular decision-making situation.

The so-called "cost-loss ratio situation" is a decision-making situation frequently encountered in the meteorological literature. This situation involves a decision maker who must decide whether or not to take protective action, with respect to some activity or operation, in the face of uncertainty as to whether or not weather adverse to the activity will occur. The original model of the cost-loss ratio situation was formulated by Thompson (1952; see also Thompson and Brier, 1955). While this model is a very simple normative model, it appears to provide a realistic description of situations faced by many forecast-sensitive decision makers and, as a result, the model has been used extensively by meteorologists and others in both real and hypothetical decision-making situations (e.g., Thompson, 1952; Kolb and Rapp, 1962; Allen and Lambert, 1971a, b; Kernan, 1975). Moreover, more complex decision-making situations can, in some cases, be investigated using this model, either by simplifying the situations or by generalizing the model (e.g., Nelson and Winter, 1964; Epstein, 1969). Thus decision-making models associated with the cost-loss ratio situation and the use of forecasts within the frameworks of such models are of considerable interest and importance.

Recently, several studies have been conducted in which meteorologists and others have formulated measures of the value of probability forecasts within the frameworks of models of specific decision-making situations (e.g., Murphy, 1966, 1969, 1972, 1974; Epstein, 1969; Stael von Holstein, 1970a, b; Pearl, 1975). In formulating these measures, it has frequently been assumed that the decision maker's utilities (i.e.,

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his preferences for the consequences) can be specified in either a deterministic *or* a probabilistic manner. When these specifications are used, the measures of value obtained are generally referred to as *utility* and *expected-utility* measures, respectively (see, e.g., Murphy, 1972). Several investigators have demonstrated that the expected-utility measures associated with certain decision-making models and with particular probability distributions on the relevant utilities are equivalent (i.e., linearly related) to familiar measures of the accuracy of probability forecasts (e.g., see Murphy, 1974; Pearl, 1975).[2]

Of particular interest, measures of the value of probability forecasts have been formulated within the framework of the original model of the cost-loss ratio situation (Murphy, 1966, 1969). In this regard, Murphy (1966) demonstrated that the expected-utility measure associated with this model, when the cost-loss ratio is assumed to be uniformly distributed, is related to the Brier, or probability, score (Brier, 1950).[3] Subsequently, Murphy (1969) studied expected-utility measures in situations in which the cost-loss ratio is assumed to have a beta distribution.

This paper has two primary purposes: (i) to describe a "generalized" model of the cost-loss ratio situation and compare the original and generalized models; and (ii) to formulate specific measures of the value of probability forecasts within the framework of these models and demonstrate that these measures are equivalent to a familiar measure of the accuracy of such forecasts. The models are discussed in Section 2. First, we describe the cost-loss ratio situation and the original model of this situation. Then, we formulate a generalized model of the cost-loss ratio situation and compare the original and generalized models. Measures of the value of probability forecasts are formulated in Section 3. First, we transform the expenses (i.e., costs and losses) considered in Section 2 into utilities, discuss the specification of the utilities, and present general expressions for the utility and expected-utility measures. Then, expected-utility measures associated with the original and generalized models are formulated under the assumption that the relevant utilities are uniformly distributed and these measures are shown to be equivalent to the probability score. Section 4 consists of a summary and conclusion.

[2] A paper summarizing the results of these investigations was presented at the American Meteorological Society's Fourth Conference on Probability and Statistics in Atmospheric Sciences, Tallahassee, Fla., November 1975. Copies can be obtained from the author of this paper.

[3] Specifically, Murphy (1966) showed that the expected-utility measure associated with this situation, when the utility x is uniformly distributed, is a linear function of the probability score *and* the event or state that occurs (i.e., adverse weather or no adverse weather). An expected-utility measure in this situation which is a linear function of the probability score alone is described in Section 3d.

2. Models of the cost-loss ratio situation

a. The cost-loss ratio situation and the original model

An indicated in Section 1, the cost-loss ratio situation involves a decision maker who must decide whether or not to take protective action in the face of uncertainty as to whether or not adverse weather will occur. Specifically, the decision maker has two possible actions, "protect" and "do not protect," and two weather events can occur, "adverse weather" and "no adverse weather." In the terminology of decision analysis, this situation is a two-action, two-state decision-making situation. We shall denote the actions by a_1 (protect) and a_2 (do not protect) and the events or states by s_1 (adverse weather) and s_2 (no adverse weather). Each action-state pair $\{a_m,s_n\}$ $(m, n=1, 2)$ leads to a different outcome or consequence. If we denote the consequences by $o_{mn}(m, n=1, 2)$, then $o_{11}=$ {protect, adverse weather}, $o_{12}=$ {protect, no adverse weather}, $o_{21}=$ {do not protect, adverse weather}, and $o_{22}=$ {do not protect, no adverse weather}. In this situation, the appropriate weather forecast is a probability vector $\mathbf{r}=(r_1,r_2)(r_n\geqslant 0,\ r_1+r_2=1)$, where r_n is the probability of occurrence of state $s_n(n=1, 2)$.

The payoffs associated with the consequences in the cost-loss ratio situation are generally expressed in terms of monetary expenses. In the original model, the cost of protection is denoted by C and it is (implicitly) assumed that, when protective action is taken, the activity is completely protected against the effects of adverse weather. Thus, if we denote the expense associated with consequence o_{mn} by $e_{mn}(m, n=1, 2)$, then $e_{11}=e_{12}=C$. Further, if the loss which results when protective action is not taken and adverse weather occurs is denoted by L, then $e_{21}=L$. Finally, since no loss results in this latter case when adverse weather does not occur, $e_{22}=0$. The expense matrix $\mathbf{E}=(e_{mn})$ $(m, n=1, 2)$ associated with this model of the cost-loss ratio situation is depicted in Fig. 1a. It can be assumed, without any loss of generality, that the bounds on the values of C and L are such that $0<C<L<\infty$.

We shall assume that the decision maker wants to select the action which minimizes his expected expense.[4] If the expected expense associated with action a_m in this model is denoted by $E_m(m=1, 2)$, then $E_1=r_1C+r_2C=C$ and $E_2=r_1L$ (see Fig. 1a). Thus, the decision maker should select action $a_1(a_2)$ when the probability of adverse weather r_1, is greater (less) than the cost-loss ratio C/L and he is indifferent between the two actions (i.e., $E_1=E_2$) when $r_1=C/L$.[5] These

[4] This criterion is the only appropriate criterion if it is assumed that the expenses e_{mn} actually reflect the decision maker's preferences for the consequences o_{mn} $(m, n=1, 2)$. This assumption is equivalent to assuming that the expenses are linearly related to the decision maker's utilities for the consequences.

[5] Since $0<C<L<\infty$, the range of the cost-loss ratio C/L is the open unit interval (0,1).

(a)

Expense Matrix **E**

		States: s_1 (Adverse weather)	s_2 (No adverse weather)	Expected expense (E)
Actions	a_1 (Protect)	C	C	$E_1=r_1C+r_2C=C$
	a_2 (Do not protect)	L	0	$E_2=r_1L$
Probability forecast	$\mathbf{r}$	r_1	r_2	

(b)

Decision Rule

a_1 if $r_1>C/L$
a_1 or a_2 if $r_1=C/L$
a_2 if $r_1<C/L$

FIG. 1. The original model of the cost-loss ratio situation, including (a) the expense matrix **E** and expected expenses $E_m(m = 1, 2)$ and (b) the decision rule.

relationships define the decision rule associated with the original model of the cost-loss ratio situation. This rule, which is summarized in Fig. 1b, describes the way in which the forecast should be used by the decision maker to minimize his expected expense in this model.

b. A generalized model

In formulating the original model of the cost-loss ratio situation, it was assumed that, when protective action is taken, the activity of concern is not affected by the occurrence of adverse weather. In many decision-making situations, however, taking protective action may reduce, but not eliminate, the loss associated with adverse weather.[6] If it is assumed that taking protective action may reduce *or* eliminate the loss associated with the occurrence of adverse weather, then the expense associated with consequence o_{11} is greater than or equal to the expense associated with consequence o_{12}. We shall denote this additional expense, which represents the unprotectable portion of the loss, by L_2' and the loss which results in this "generalized" model when no protective action is taken and adverse weather occurs by L'. If the expense associated with the consequence o_{mn} in this model is denoted by e_{mn}' $(m, n=1, 2)$, then $e_{11}'=C+L_2'$, $e_{12}'=C$, $e_{21}'=L'$ and $e_{22}'=0$. The matrix $\mathbf{E}'$ is depicted in Fig. 2a. Without any loss of

[6] Consider the situations in which the owner of an orchard must decide whether or not to protect his fruit trees against freezing temperatures or a contractor must decide whether or not to protect his construction site against precipitation. These (and other similar) situations have frequently been modeled using the original cost-loss ratio decision-making model. However, taking protective action in many such situations may reduce but not eliminate the loss associated with the occurrence of adverse weather (in these cases, freezing temperatures and precipitation, respectively). Thus a generalized version of the original model may be more appropriate in these situations.

(a)

Expense Matrix **E**′

		States: s_1 (Adverse weather)	s_2 (No adverse weather)	Expected expense (E')
Actions	a_1 (Protect)	$C+L_2'$	C	$E_1'=r_1(C+L_2')+r_2C=C+r_1L_2$
	a_2 (Do not protect)	L'	0	$E_2'=r_1L'$
Probability forecast	$\mathbf{r}$	r_1	r_2	

(b)

Decision Rule

a_1 if $r_1>C/(L'-L_2')$
a_1 or a_2 if $r_1=C/(L'-L_2')$
a_2 if $r_1<C/(L'-L_2')$

FIG. 2. The generalized model of the cost-loss ratio situation, including (a) the expense matrix **E**′ and expected expenses $E_m'(m = 1, 2)$ and (b) the decision rule.

generality, it can be assumed that $0<C<L'<\infty$ and $0\leq L_2'<L'-C$.

If the expected expense associated with action a_m in this model is denoted by $E_m'(m=1, 2)$, then $E_1'=r_1(C+L_2')+r_2C=C+r_1L_2'$ and $E_2'=r_1L'$ (see Fig. 2a). Thus action $a_1(a_2)$ should be selected when the probability of adverse weather r_1 is greater (less) than the "generalized" cost-loss ratio $C/(L'-L_2')$, and he is indifferent between the two actions when $r_1=C/(L'-L_2')$.[7] This decision rule, which is summarized in Fig. 2b, describes the way in which the decision maker should use the forecast to minimize his expected expense in the generalized model.

c. *Comparison of the original and generalized models*

First, it should be noted that the original model is a special case of the generalized model, in which the unprotectable portion of the loss, L_2', is equal to zero. Moreover, in the original model it is assumed that taking protective action completely eliminates the loss associated with the occurrence of adverse weather. That is, the expense associated with the consequence o_{11}= {protect, adverse weather} is simply the cost of protection C. This assumption, in turn, implies that the loss L, which is associated with the consequence o_{21}= {do not protect, adverse weather}, is the "protectable" loss. That is, it is a loss that can be completely protected against when protective action is taken. Thus the cost-loss ratio C/L in the original model is, in reality, the cost of protection divided by the *protectable* loss. While this fact may have been recognized heretofore, it has not been mentioned explicitly in any of the papers concerned with the cost-loss ratio situation (at least to our knowledge).

In the generalized model, on the other hand, the loss L' associated with the consequence o_{21} is a "total" loss. Specifically, L' can be expressed as the sum of two terms, the *protectable* loss L_1' and the *unprotectable* loss L_2'; that is, $L'=L_1'+L_2'$. Therefore the generalized cost-loss ratio $C/(L'-L_2')$ can be expressed simply as C/L_1'. Thus the cost-loss ratio in the generalized model is also the cost of protection divided by the protectable portion of the loss.

The foregoing indicates that the original and generalized cost-loss ratios are identical (since, by definition, $L=L_1'$). Therefore, the same decision rule applies in both models and as a result the two models can be said to be decision equivalent.[8] In this regard, it is of interest to note that, from a decision-making point of view, only the protectable portion of the loss (i.e., L or L_1') needs to be specified in each model. This result also implies that the original model is applicable to a wider class of decision-making situations than has generally been recognized heretofore.

[7] Since $0<C<L'<\infty$ and $0\leqslant L_2'<L'-C$, the range of the generalized cost-loss ratio $C/(L'-L_2')$ is also the open unit interval (0,1).

[8] Strictly speaking, these models are decision equivalent only if the expenses associated with the consequences are linearly related to the utilities.

(a)

Utility Matrix **U**

		States s_1	States s_2	Expected utility (U)
Actions	a_1	x	x	$U_1=r_1x+r_2x=x$
	a_2	0	1	$U_2=r_2$
Probability forecast	**r**	r_1	r_2	

(b)

Decision Rule

a_1	if $r_1>1-x$
a_1 or a_2	if $r_1=1-x$
a_2	if $r_1<1-x$

FIG. 3. As in Fig. 1 except that the values of the consequences are expressed in terms of utilities [cf. Fig. 1; $x = 1 - (C/L)$]: (a) the utility matrix **U** and expected utilities $U_m(m = 1, 2)$; (b) the decision rule.

3. Measures of the value of probability forecasts

a. *Utilities*

We shall assume 1) that the utilities of the consequences are linearly related to the respective expenses; 2) that the decision maker can identify the most and least preferred consequences o^M and o^L, respectively; and 3) that, since the origin and unit of measure on the utility scale are arbitrary, o^M has a utility of 1 and o^L has a utility of 0. Then, the expense matrix $\mathbf{E}=(e_{mn})$ in the original model can be transformed into the equivalent utility matrix $\mathbf{U}=(u_{mn})$, where

$$u_{mn}=(e_{mn}-e_L)/(e_M-e_L), \quad m, n=1, 2, \qquad (1)$$

in which e_M and e_L are the expenses associated with the consequences o^M and o^L, respectively. Since $o^M=o_{22}$ and $o^L=o_{21}$ in this model, $e_M=0$ and $e_L=L$. Thus Eq. (1) becomes

$$u_{mn}=1-(e_{mn}/L), \quad m, n=1, 2. \qquad (2)$$

Therefore, $u_{11}=u_{12}=1-(C/L)$, $u_{21}=0$ and $u_{22}=1$ (see Fig. 1a). The utility matrix **U** is depicted in Fig. 3a in which $x=1-(C/L)$. Since $0<C<L<\infty$, $0<x<1$.

Now since $e_M'=0$ and $e_L'=L'$, the expense matrix $\mathbf{E}'=(e_{mn}')$ in the generalized model can be transformed into an equivalent utility matrix $\mathbf{U}'=(u_{mn}')$ using (2) by replacing u_{mn}, e_{mn} and L with u_{mn}', e_{mn}' and L', respectively (m, $n=1$, 2). Therefore, $u_{11}'=1-[(C+L_2')/L']$, $u_{12}'=1-(C/L')$, $u_{21}'=0$ and $u_{22}'=1$ (see Fig. 2a). The utility matrix $\mathbf{U}'$ is depicted in Fig. 4a in which $x=1-[(C+L_2')/L']$ and $y=1-(C/L')$. Since $0<C<L'<\infty$ and $0\leqslant L_2'<L'-C$, $0<x\leqslant y<1$.

The decision criterion in this framework, which is equivalent to the criterion of minimizing expected expense in the standard framework, consists of maximizing expected utility. If we denote the expected utility of action a_m by U_m in the original model ($m=1$,

(a)

Utility Matrix U'

		States s_1	s_2	Expected utility (U')
Actions	a_1	x	y	$U'_1=r_1x+r_2y$
	a_2	0	1	$U'_2=r_2$
Probability forecast	$\mathbf{r}$	r_1	r_2	

(b)

Decision Rule

a_1	if $r_1>(1-y)/(1+x-y)$
a_1 or a_2	if $r_1=(1-y)/(1+x-y)$
a_2	if $r_1<(1-y)/(1+x-y)$

FIG. 4. As in Fig. 2 except that the values of the consequences are expressed in terms of utilities {cf. Fig. 2; $x=1-[(C+L_2')/L']$, $y=1-(C/L')$}: (a) the utility matrix $\mathbf{U}'$ and expected utilities $U_m'(m=1,2)$; (b) the decision rule.

2), then $U_1=r_1x+r_2x=x$ and $U_2=r_2$ (see Fig. 3a). Thus the decision maker will prefer action $a_1(a_2)$ to action $a_2(a_1)$ if $r_1>(<)1-x$; this decision rule is summarized in Fig. 3b. Similarly, if we denote the expected utility of action a_m by U'_m in the generalized model ($m=1$, 2), then $U'_1=r_1x+r_2y$ and $U'_2=r_2$ (see Fig. 4a). Thus the decision maker will prefer action $a_1(a_2)$ to action $a_2(a_1)$ if $r_1>(<)$ $(1-y)/(1+x-y)$, and the decision rule in this model is summarized in Fig. 4b.

b. Specification of the utilities

As previously indicated, the relevant utilities can be specified in either a deterministic or a probabilistic manner. We shall assume that the utilities are specified probabilistically, in terms of a continuous distribution $f(\mathbf{U})$ defined on the elements of the utility matrix $\mathbf{U}=(u_{mn})$ (m, $n=1$, 2). Thus the distributions of concern are $f(\mathbf{U})=f(x)$ for $0<x<1$ in the original model and $f(\mathbf{U}')=g(x,y)$ for $0<x\leqslant y<1$ in the generalized model.

The density functions $f(x)$ and $g(x,y)$ can be given at least two different interpretations (see, e.g., Murphy, 1966, 1972, 1974). Specifically, such distributions can be considered to represent either the uncertainties associated with the values of an individual decision maker's utilities or the distribution of the values of the utilities of many different decision makers. For example, a uniform distribution in the original model implies that the value of the utility x [and, since $x=1-(C/L)$, the value of the cost-loss ratio C/L] is equally likely to be in any sub-interval of the unit interval of equal length in the case of one decision maker, or that the values of the utilities (cost-loss ratios) are distributed uniformly over the unit interval in the case of many decision makers. In the generalized model, a uniform distribution implies that the values of the utilities x and y are equally likely to be in any subregion of the triangle defined by the inequality $0<x\leqslant y<1$ of equal area in the case of one decision maker, or that the values of the utilities are distributed uniformly over this triangle in the case of many decision makers.[9]

c. Utility and expected-utility measures: General expressions

The utility measure $U(\mathbf{r},\mathbf{d})$ in two-action, two-state decision-making situations can be expressed as

$$U(\mathbf{r},\mathbf{d})=\sum_{m=1}^{2}\delta_m\Big(\sum_{n=1}^{2}d_nu_{mn}\Big), \tag{3}$$

where

$$\delta_m=\begin{cases}1, & \text{if } \mathbf{r}\in R_m^2\\ 0, & \text{otherwise}\end{cases}$$

and $\mathbf{d}=(d_1,d_2)$ is the observation, in which $d_n=1$ if state s_n occurs and zero otherwise (m, $n=1$, 2) (see Murphy, 1972, 1974). The set R_m^2 is the set of all forecasts for which the decision maker prefers action a_m to action $a_{m'}$ (m, $m'=1,2$; $m\neq m'$). The measure $U(\mathbf{r},\mathbf{d})$ in (3) is a proper scoring rule (see, e.g., Stael von Holstein, 1970b; Murphy, 1974).

The utility measure $U(\mathbf{r},\mathbf{d})$ associated with the original model of the cost-loss ratio situation is (from Fig. 3a)

$$U(\mathbf{r},\mathbf{d})=\delta_1x+\delta_2d_2, \tag{4}$$

where $\delta_1=1$ if $r_1>1-x$ and $\delta_2=1$ if $r_1<1-x$. The utility measure $U'(\mathbf{r},\mathbf{d})$ associated with the generalized model is (from Fig. 4a)

$$U'(\mathbf{r},\mathbf{d})=\delta_1(d_1x+d_2y)+\delta_2d_2, \tag{5}$$

where $\delta_1=1$ if $r_1>(1-y)/(1+x-y)$ and $\delta_2=1$ if $r_1<(1-y)/(1+x-y)$. The measures in (4) and (5) have positive orientations (i.e., a larger score is better), and the range of these measures is the closed unit interval [0, 1]. In particular, $U(\mathbf{r},\mathbf{d})=U'(\mathbf{r},\mathbf{d})=1$ if $\delta_2=1$ and $d_2=1$, while $U(\mathbf{r},\mathbf{d})=U'(\mathbf{r},\mathbf{d})=0$ if $\delta_2=1$ and $d_1=1$.

The expected-utility measure $EU(\mathbf{r},\mathbf{d})$ in these situations can be expressed as

$$EU(\mathbf{r},\mathbf{d})=\int_\Theta U(\mathbf{r},\mathbf{d})f(\mathbf{U})d\mathbf{U},$$

or from Eq. (3)

$$EU(\mathbf{r},\mathbf{d})=\int_\Theta\sum_{m=1}^{2}\delta_m\Big(\sum_{n=1}^{2}d_nu_{mn}\Big)f(\mathbf{U})d\mathbf{U}, \tag{6}$$

[9] Of course, the assumption of a uniform distribution on the relevant utilities may be inappropriate in many cost-loss ratio situations. Nevertheless, we (as meteorologists) are frequently almost completely uncertain about the value(s) of the decision maker's(s') utilities, and a uniform distribution provides a reasonable representation of this state of knowledge (see, e.g., Winkler, 1972).

where $\mathbf{U}=(u_{mn})$ is the utility matrix of concern and Θ is the "region" over which the probability distribution $f(\mathbf{U})$ is defined (m, $n=1, 2$) (see Murphy, 1972, 1974). The measure $EU(\mathbf{r},\mathbf{d})$ in (6) is a proper scoring rule if $f(\mathbf{U})$ is a positive function and a strictly proper scoring rule if $f(\mathbf{U})$ is a strictly positive function (Murphy, 1974).

The expected-utility measure $EU(\mathbf{r},\mathbf{d})$ associated with the original model is from (4) and (6)

$$EU(\mathbf{r},\mathbf{d})=\int_{0<x<1}(\delta_1 x+\delta_2 d_2)f(x)dx, \qquad (7)$$

since $\Theta=\{0<x<1\}$. The expected-utility measure $EU'(\mathbf{r},\mathbf{d})$ associated with the generalized model is from (5) and (6)

$$EU'(\mathbf{r},\mathbf{d})=\iint_{0<x\leq y<1}[\delta_1(d_1x+d_2y)+\delta_2 d_2]g(x,y)dxdy, \qquad (8)$$

since $\Theta=\{0<x\leqslant y<1\}$. The measures in (7) and (8) have positive orientations and the range of these measures is the closed unit interval $[0,1]$.

d. An expected-utility measure associated with the original model

As indicated in Section 1, Murphy (1966) has shown that the expected-utility measure associated with the original model of the cost-loss ratio situation, when the utility x (or the cost-loss ratio C/L) is uniformly distributed, is related to the probability score (see footnote 3). In this subsection, we show that an expected-utility measure associated with this model is *equivalent* (i.e., linearly related) to the probability score when the utility x is assumed to be uniformly distributed. Specifically, we shall assume that

$$f(x)=\begin{cases}1, & 0<x<1\\ 0, & \text{otherwise.}\end{cases} \qquad (9)$$

If we denote the expected-utility measure associated with the utility matrix $\mathbf{U}$ in Fig. 3a by $EU_+(\mathbf{r},\mathbf{d})$, then from (7) and (9)

$$EU_+(\mathbf{r},\mathbf{d})=\int_{0<x<1}(\delta_1 x+\delta_2 d_2)dx. \qquad (10)$$

The specific limits of integration are determined by identifying the subintervals of the interval $\Theta=\{0<x<1\}$ in which the indicator variables $\delta_m (m=1, 2)$ assume a non-zero value. These subintervals are depicted in Fig. 5. Thus, $EU_+(\mathbf{r},\mathbf{d})$ in (10) becomes

$$EU_+(\mathbf{r},\mathbf{d})=\int_{r_2}^{1}xdx+d_2\int_{0}^{r_2}dx. \qquad (11)$$

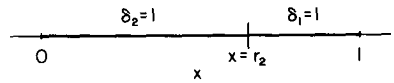

FIG. 5. The interval $\Theta=\{0<x<1\}$ over which the density function $f(x)$ is defined (heavy solid line), and the subintervals of Θ in which the indicator variables $\delta_m (m=1, 2)$ assume a non-zero value.

Performing the integrations in (11), we obtain

$$EU_+(\mathbf{r},\mathbf{d})=(\tfrac{1}{2})(1+d_2)-(\tfrac{1}{2})(r_2-d_2)^2. \qquad (12)$$

The range of $EU_+(\mathbf{r},\mathbf{d})$ in (12) is the closed interval $[\frac{1}{2},1]$ when $d_2=1$ and the closed interval $[0,\frac{1}{2}]$ when $d_2=0$.

In order to obtain an overall expected-utility measure in the original cost-loss ratio situation the range of which is less dependent upon the state that occurs, we shall also formulate the expected-utility measure $EU_-(\mathbf{r},\mathbf{d})$ which corresponds to a utility matrix identical to the matrix $\mathbf{U}$ in Fig. 3a except that the states are interchanged. Then,

$$EU_-(\mathbf{r},\mathbf{d})=(\tfrac{1}{2})(1+d_1)-(\tfrac{1}{2})(r_1-d_1)^2 \qquad (13)$$

(cf. Murphy, 1966, p. 535). Let $EU(\mathbf{r},\mathbf{d})$, the overall expected-utility measure, be the sum of these two measures. Then from (12) and (13)

$$EU(\mathbf{r},\mathbf{d})=(\tfrac{3}{2})-(r_1-d_1)^2, \qquad (14)$$

since $r_2=1-r_1$ and $d_2=1-d_1$. The range of $EU(\mathbf{r},\mathbf{d})$ is the closed interval $[\frac{1}{2},\frac{3}{2}]$ and, since $f(x)$ is strictly positive ($0<x<1$), the measure $EU(\mathbf{r},\mathbf{d})$ is a strictly proper scoring rule.

The Brier, or probability, score $PS(\mathbf{r},\mathbf{d})$ for an individual forecast in two-state situations can be expressed as

$$PS(\mathbf{r},\mathbf{d})=2(r_1-d_1)^2 \qquad (15)$$

(Brier, 1950). Thus, $EU(\mathbf{r},\mathbf{d})$ in (14) can be rewritten as

$$EU(\mathbf{r},\mathbf{d})=(\tfrac{3}{2})-(\tfrac{1}{2})PS(\mathbf{r},\mathbf{d}). \qquad (16)$$

Therefore, the (overall) expected-utility measure associated with the original model of the cost-loss ratio situation, when the utility x is uniformly distributed, is equivalent (i.e., linearly related) to the probability score.

e. An expected-utility measure associated with the generalized model

In this subsection, we formulate an expected-utility measure associated with the generalized model of the cost-loss ratio situation, when the utilities x and y are assumed to be (jointly) uniformly distributed. Specifically, we shall assume that

$$g(x,y)=\begin{cases}2, & 0<x\leqslant y<1\\ 0, & \text{otherwise.}\end{cases} \qquad (17)$$

If we denote the expected-utility measure associated with the utility matrix $\mathbf{U}'$ in Fig. 4a by $EU'_+(\mathbf{r},\mathbf{d})$,

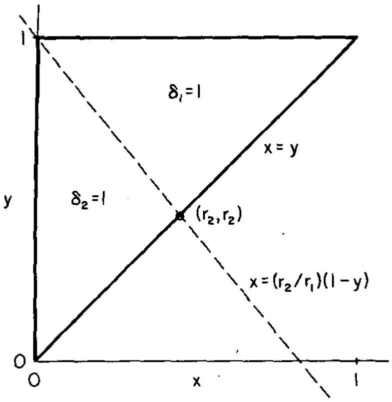

FIG. 6. The region $\Theta = \{0 < x \leqslant y < 1\}$ over which the density function $g(x,y)$ is defined (heavy solid triangle), and the subregions of Θ in which the indicator variables $\delta_m (m = 1, 2)$ assume a nonzero value.

then from (8) and (17)

$$EU'_+(\mathbf{r},\mathbf{d}) = 2 \iint\limits_{0<x<y<1} [\delta_1(d_1x+d_2y)+\delta_2 d_2]dxdy. \quad (18)$$

The specific limits of integration in this situation are determined by identifying the subregions of the triangle $\Theta = \{0<x\leqslant y<1\}$ in which the indicator variables $\delta_m (m=1, 2)$ assume a non-zero value; these subregions are depicted in Fig. 6. Thus, $EU'_+(\mathbf{r},\mathbf{d})$ in (18) becomes

$$EU'_+(\mathbf{r},\mathbf{d}) = 2\int_{r_2}^{1}\int_{(r_2/r_1)(1-y)}^{y} (d_1x+d_2y)dxdy + 2d_2\int_{0}^{r_2}\int_{x}^{1-(r_1/r_2)x} dxdy. \quad (19)$$

Performing the integrations in (19), we obtain

$$EU'_+(\mathbf{r},\mathbf{d}) = 1-(\tfrac{2}{3})d_1-(\tfrac{1}{3})(r_1-d_1)^2. \quad (20)$$

The range of $EU'_+(\mathbf{r},\mathbf{d})$ in (20) is the closed interval $[0,\tfrac{1}{3}]$ when $d_1=1$ and the closed interval $[\tfrac{2}{3},1]$ when $d_1=0$.

In order to obtain an overall expected-utility measure in the generalized cost-loss ratio situation the range of which is less dependent upon the state that occurs, we also formulate the expected-utility measure $EU'_-(\mathbf{r},\mathbf{d})$ which corresponds to a utility matrix identical to the matrix $\mathbf{U}'$ in Fig. 4a except that the states are interchanged. Then,

$$EU'_-(\mathbf{r},\mathbf{d}) = 1-(\tfrac{2}{3})d_2-(\tfrac{1}{3})(r_2-d_2)^2. \quad (21)$$

As in the case of the original model, we let the overall measure $EU'(\mathbf{r},\mathbf{d})$ be the sum of these two measures. Then from (20) and (21)

$$EU'(\mathbf{r},\mathbf{d}) = (\tfrac{4}{3})-(\tfrac{2}{3})(r_1-d_1)^2. \quad (22)$$

The range of $EU'(\mathbf{r},\mathbf{d})$ is the closed interval $[\tfrac{2}{3},\tfrac{4}{3}]$ and, since $g(x,y)$ is strictly positive $(0<x\leqslant y<1)$, the measure $EU'(\mathbf{r},\mathbf{d})$ is a strictly proper scoring rule.

Using (15) we can rewrite (22) as

$$EU'(\mathbf{r},\mathbf{d}) = (\tfrac{4}{3})-(\tfrac{1}{3})PS(\mathbf{r},\mathbf{d}). \quad (23)$$

Therefore, as in the case of the original model, the (overall) expected-utility measure associated with the generalized model of the cost-loss ratio situation, when the utilities x and y are jointly uniformly distributed, is equivalent to the probability score.

4. Summary and conclusion

In this paper we have described and compared two models of the familiar cost-loss ratio situation. This situation involves a decision maker who must decide whether or not to take protective action, with respect to some activity or operation, in the face of uncertainty as to whether or not weather adverse to the activity will occur. The original model, which was first formulated by Thompson (1952), is based in part upon the assumption that taking protective action completely eliminates the loss associated with the occurrence of adverse weather. In the generalized model described in this paper, on the other hand, it is assumed that taking protective action may reduce *or* eliminate this loss. Thus, the original model is a special case of the generalized model in which the unprotectable portion of the loss is equal to zero. We have shown that the decision rule associated with each model depends upon a cost-loss ratio and that this ratio in both models is simply the cost of protection divided by the protectable portion of the loss. Thus, the decision rules in the two models are identical and the models can be said to be decision equivalent. Moreover, this result indicates that from a decision-making point of view only the protectable portion of the loss needs to be specified in these models. Finally, it should also be noted that this result implies that the original model is applicable to a wider class of decision-making situations than has generally been recognized heretofore.

We have also translated the expenses (i.e., costs and losses) associated with the relevant consequences in both models into utilities and have formulated measures of the value of probability forecasts which correspond to each model when the utilities of concern are assumed to possess uniform probability distributions. Such distributions can be considered to represent either the uncertainty associated with the values of an individual decision maker's utilities or the distribution of the values of the utilities of many different decision makers. In formulating these measures of value, or expected-utility measures, we have considered both the utility matrices corresponding to the original and generalized models and their respective mirror images. This approach yields measures the ranges of which are not overly dependent upon the event or state which occurs. The measures were then shown to be equivalent (i.e., linearly related) to the probability score, a familiar measure of the accuracy of probability forecasts. These results provide additional support for the use of the probability score as an evaluation measure, particularly in those situations in which the models considered in this paper *and* uniform probability distributions on the relevant utilities appear to describe the decision makers and/or decision-making situations of concern in a reasonable manner.

Other more general models of the cost-loss ratio situation could, of course, be considered. For example, we could formulate a two-stage model[10] in which the decision maker must first make a *strategic* decision regarding the amount of protection that he will obtain and have available for subsequent use. This decision would then be followed by a *tactical* decision as to whether or not to employ a certain amount of protection on a particular occasion. Such a model might provide a more realistic description of the decision-making situations considered in this paper in connection with the generalized model (see footnote 6). With regard to measures of the value of probability forecasts, we could, of course, formulate other expected-utility measures by assuming different probability distributions on the relevant utilities (see, e.g., Murphy, 1969, 1974). Such measures may be of considerable interest in certain special situations, but they will in general have rather complex forms and are therefore unlikely to be equivalent to any of the familiar evaluation measures.

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[10] Howe and Cochrane (1976) recently formulated such a model and applied it to decision-making situations involving urban snow storms. The use of such a model was independently suggested to the author by J. C. Thompson.

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