Horizontal Advection Schemes of a Staggered Grid—An Enstrophy and Energy-Conserving Model

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ABSTRACT

For use in a model on the semi-staggered E (in the Arakawa notation) grid, a number of conserving schemes for the horizontal advection are developed and analyzed. For the rotation terms of the momentum advection, the second-order enstrophy and energy-conserving scheme of Janjić (1977) is generalized to conserve energy in case of divergent flow. A family of analogs of the Arakawa (1966) fourth-order scheme is obtained following a transformation of its component Jacobians. For the kinetic energy advection terms, a fourth- (or approximately fourth) order scheme is developed which maintains the total kinetic energy and, in addition, makes no contribution to the change in the finite-difference vorticity. For the resulting both second- and fourth-order momentum advection scheme, a modification is pointed out which avoids the non-cancellation of terms considered recently by Hollingsworth and Källberg (1979), and shown to lead to a linear instability of a zonally uniform inertia-gravity wave. Finally, a second-order as well as a fourth-order (or approximately so) advection scheme for temperature (and moisture) advection is given, preserving the total energy (and moisture) inside the integration region.

1. Introduction

Horizontal advection schemes to be discussed in this paper have been developed as a part of ongoing efforts at the Geophysical Fluid Dynamics Laboratory (GFDL), Princeton, to construct a grid-point model which would be computationally efficient and reliable, maintaining a number of dynamically important conservation and other properties of the original differential equations, and which would be suitable for short as well as medium-range weather forecasting and, perhaps, eventually be used for long-range forecasting. This GFDL model is being developed as an outgrowth of the model that has been originally formulated for operational and research use at the Federal Hydrometeorological Institute and Belgrade University (HIBU), Yugoslavia, (Janjić, 1977; Mesinger, 1977), and which for some time has been used for operational numerical weather prediction by the Yugoslav Hydrometeorological Service.

The HIBU model, in its 1977 version, is a model on the E (Arakawa notation) grid using longitude-latitude horizontal coordinates and the sigma vertical coordinate, with a second-order horizontal momentum advection scheme conserving enstrophy and kinetic energy for the rotational component of the wind. Subsequently, among other changes, the split time differencing of the model has been modified so as to calculate the horizontal advection only once in several steps of the remaining terms of the equations. Thus, the fraction of time needed for the horizontal advection is small, and an extra effort spent on horizontal advection will result in only a minor increase in the overall computational cost of the model. In this respect, the time differencing is the same as that described by Gadd (1978a). In the present scheme, however, the space differencing for the advection step is determined by enstrophy and energy conservation, and a pure time-differencing scheme is needed to complete the method, whereas in the modified Lax-Wendroff scheme chosen by Gadd the space and time differencing must be considered together. For most experiments with the present schemes, the Heun scheme (e.g., Mesinger and Arakawa, 1976) has been used, because of the impressive storage economy achieved by a two as compared to a three time-level model.

A number of extensions of the horizontal advection schemes of this semi-staggered grid model have been made and will be described in the continuation of this paper; these are as follows:

- The second-order enstrophy and energy conserving scheme for the rotation terms of the momentum advection is generalized for the case of divergent flow.

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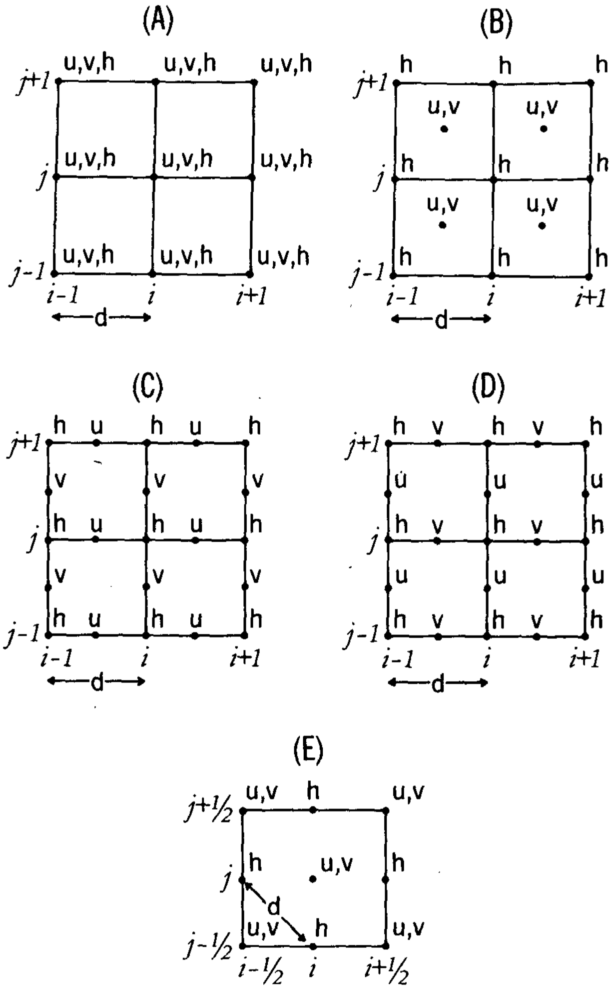

FIG. 1. Five types of horizontal lattices (after Arakawa and Lamb, 1977), shown with help of the usual shallow-water notation.

- The schemes for the rotation and for the kinetic energy terms are both extended to fourth- (or approximately fourth) order accuracy.
- A modification of these schemes is pointed out which avoids the non-cancellation of terms shown by Hollingsworth and Källberg (1979) to lead to a linear instability of a zonally uniform inertia-gravity wave.
- A second- and also a fourth- (or approximately fourth) order temperature (and moisture) horizontal advection scheme is developed, conserving the total energy (and moisture) inside a closed integration domain.
- Generalizations of these schemes to the case of the spherical geometry, and to that of the fully three-dimensional flow, are included.

In describing the schemes, different terms or groups of terms will be discussed in the order in which they are most often written within the full set of the atmospheric prognostic equations. Typically, a subsection on the second-order schemes will be given first, followed by a subsection on the fourth-order schemes. Prior to the discussion of schemes, the relevant terms of the full set of the governing differential equations will be displayed, and the reasons for the choice of the grid shall be summarized.

2. The governing equations

Using the pressure at the top of the model atmosphere and at the surface, p_T and p_S, we define the sigma coordinate as

$$\sigma \equiv \frac{p - p_T}{\pi}, \quad \pi \equiv p_S - p_T. \tag{2.1}$$

The horizontal advection terms of the equations of motion are written in their vector invariant form

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= v\zeta - \frac{1}{a \cos\varphi} \frac{\partial}{\partial \lambda} \frac{1}{2} \mathrm{v}^2 \\ \frac{\partial v}{\partial t} &= -u\zeta - \frac{1}{a} \frac{\partial}{\partial \varphi} \frac{1}{2} \mathrm{v}^2 \end{aligned}\right\}, \tag{2.2}$$

Here the vorticity ζ is defined by

$$\zeta \equiv \frac{1}{a \cos\varphi}\left(\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \varphi} u \cos\varphi\right). \tag{2.3}$$

Explanation of other symbols, here and mostly also later on, will be omitted, since an attempt is made to follow the most customary usage (e.g., Arakawa and Lamb, 1977).

The pressure-tendency equation

$$\frac{\partial \pi}{\partial t} = -\int_0^1 \frac{1}{a \cos\varphi}\left(\frac{\partial}{\partial \lambda} \pi u + \frac{\partial}{\partial \varphi} \pi v \cos\varphi\right) d\sigma \tag{2.4}$$

is a result of the vertical integration of the continuity equation

$$\frac{\partial \pi}{\partial t} + \frac{1}{a \cos\varphi}\left(\frac{\partial}{\partial \lambda} \pi u + \frac{\partial}{\partial \varphi} \pi v \cos\varphi\right) + \frac{\partial}{\partial \sigma} \pi \dot{\sigma} = 0. \tag{2.5}$$

Horizontal temperature (or moisture) advection terms to be considered here are of the form

$$\frac{\partial T}{\partial t} = -\frac{u}{a \cos\varphi} \frac{\partial T}{\partial \lambda} - \frac{v}{a} \frac{\partial T}{\partial \varphi}. \tag{2.6}$$

3. Choice of the finite-difference grid

Five different possibilities, as shown in Fig. 1, have and/or are being used for the horizontal arrangement of variables over grid points in various models. As already stated, the schemes to be described are designed for the E grid. Note, however, that the E and B grids are to a large degree equivalent, as one is obtained from the other by a rotation of the coordinate axes (see, e.g., Mesinger and Arakawa, 1976).

In a number of papers, properties of some or all of these five grids have been systematically compared. Invariably, unsatisfactory aspects of the non-staggered A grid have been revealed. Winninghoff and Arakawa (e.g., Arakawa and Lamb, 1977) have found that compared to the semi-staggered grids B and/or E, and the fully staggered grid C, it poorly simulates the geostrophic adjustment process.

Looking for the vorticity equation analogs, one readily notes that on the non-staggered grid the second-order vorticity analog of highest local accuracy permits a spurious vorticity production by the pressure gradient force. The next most accurate second-order analog on the non-staggered grid, on the other hand, is of poor accuracy compared to vorticity analogs on staggered grids. This situation is clearly reflected in errors of Rossby wave phase speeds obtained using various grids (Mesinger, 1979a). An alternate statement of the problem is that this less accurate vorticity analog (calculated over two-grid intervals), in fact, implies "a redundant calculation generating multiple-computational modes of solution" (Miyakoda *et al.*, 1979).

One more unfavorable property of the A grid was recently pointed out by Ničković (1979). With straightforward differencing and a simple zonal wind profile, use of the A grid was shown to lead to a false shearing instability for the shortest admissible wavelengths.

Accordingly, we feel that convincing reasons exist not to use the non-staggered grid. Of the staggered grids, the grid D appears not to be acceptable, as it is quite poor in simulating the geostrophic adjustment process. Best simulation of the geostrophic adjustment process of the earth's atmosphere Arakawa and Lamb (1977) find is obtained with the fully-staggered scheme C, "except for abnormal situations" of small static stability, such that the radius of deformation is made to be of the order of the grid size or less. In addition, the C grid enables construction of advection and continuity equation schemes conserving both energy and *potential* enstrophy (Sadourny, 1975a,b; Burridge and Haseler, 1977), even with no requirement for a nondivergent mass flux (Arakawa and Lamb, 1980).

The averaging of the Coriolis force on the C grid, however, does seem to create disadvantages. One is the problem with the geostrophic adjustment in cases of very low stability. Another may be a spurious two-grid-interval wave ($\delta_x u = 0$, $\delta_y v = 0$, $\delta_x \bar{v}^{xy} = \delta_y \bar{u}^{xy}$) permitted within the gravity-inertia terms as a stationary solution in the velocity field (Janjić, personal communication). In data assimilation, an attempt to achieve local geostrophic balancing (Hayden, 1973) would apparently meet with some difficulties. Namely, performed in a straightforward way, a geostrophic wind correction balances only one-eighth of the imbalance introduced by a single grid-point insertion of the mass field data. Finally, C-type staggering of variables makes inapplicable the averaging technique proposed by Janjić (1977) to prevent a mountain-induced inconsistency in elevations of the pressure gradient and the Coriolis force at a single grid point in the sigma system.

It may be that some of these inadequacies were the reason for the truncation error difficulty encountered by Gadd (1978a) in attempting to use the C grid for an operational United Kingdom Meteorological Office model.

For a number of these considerations, we have decided to use the E grid. A minor factor influencing that decision also was the fact that for a limited area and with geographic coordinates an E grid model has all of the prognostic variables located along the boundary latitude circles and meridians; a property that may simplify definition of the time-dependent boundary conditions for the model. Incidentally, this is also "the simplest arrangement of points" used by Richardson (1922) in his pioneering numerical modeling work.

As for the vertical grid, the present GFDL model is a layer model: values of the horizontal velocity components, temperature and moisture are defined inside its sigma layers. This feature of the model is of relevance when considering generalizations of conserving schemes to the case of the fully three-dimensional flow.

4. Momentum advection: The rotation terms

a. The second-order scheme

For the E grid, an analog of the Arakawa (1966) second-order enstrophy and energy-conserving scheme for the rotation terms of the equation of motion

$$\frac{\partial u}{\partial t} = v\zeta, \quad \frac{\partial v}{\partial t} = -u\zeta, \tag{4.1}$$

has been found by Janjić (1977). It can be written as

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \tfrac{2}{3} v \tilde{\zeta}^{xy} + \tfrac{1}{3}\overline{\bar{v}^y \zeta}^x \\ \frac{\partial v}{\partial t} &= -\tfrac{2}{3} u \tilde{\zeta}^{xy} - \tfrac{1}{3}\overline{\bar{u}^x \zeta}^y \end{aligned}\right\}. \tag{4.2}$$

Here, and for the time being, a square grid is considered only; the tilde sign denotes arithmetic averaging over four nearest grid points, i.e.,

$$\tilde{\zeta}^{xy} \equiv \tfrac{1}{2}(\tilde{\zeta}^x + \tilde{\zeta}^y), \tag{4.3}$$

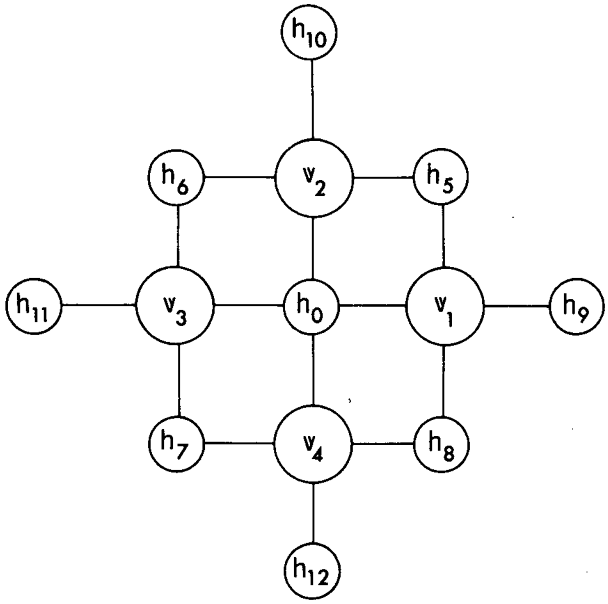

FIG. 2. Stencil used to define finite-difference Laplacians.

and the vorticity for the finite-difference case is redefined by

$$\zeta \equiv \delta_x v - \delta_y u. \qquad (4.4)$$

Eq. (4.2) is an exact analog of the Arakawa (1966 notation) Jacobian $\mathcal{J}_1$, except for this definition of vorticity. Note that, in terms of streamfunction, with centered second-order differencing and using the indexing of Fig. 2, the definition (4.4) at point 0 reduces to

$$\zeta = \nabla_+{}^2\psi \equiv \frac{1}{2d^2}(\psi_9 + \psi_{10} + \psi_{11} + \psi_{12} - 4\psi_0),$$

and not to

$$\nabla_\times{}^2\psi \equiv \frac{1}{d^2}(\psi_5 + \psi_6 + \psi_7 + \psi_8 - 4\psi_0),$$

which would be required for an equivalence with the Arakawa second-order scheme also in streamfunction formulation. However, with primitive equations and the E grid, it seems difficult to conceive of a definition of vorticity other than (4.4); see also (Mesinger, 1979a). After accepting (4.4), we note that (4.2) conserves enstrophy and energy with respect to the rotation terms, and for the two-dimensional flow (Janjić, 1977).

In generalizing (4.2) to the case of three-dimensional, divergent flow, we want to have a scheme which will maintain the property of the rotation terms (4.1) to make no contribution to the change of the total kinetic energy. To this end we consider a fluid described by the shallow-water equations with the kinetic energy given by

$$K = \int_S h\rho \tfrac{1}{2}\mathbf{v}^2 dS. \qquad (4.5)$$

With the E grid and a square geometry, we define the analog of (4.5) as

$$K \equiv \sum h^* \rho \tfrac{1}{2}\mathbf{v}^2 \Delta S. \qquad (4.6)$$

Here the summation is done over all $\mathbf{v}$ points and h^* denotes the value of h calculated at $\mathbf{v}$ points—as yet not further specified. A scheme which will guarantee that the rotation terms make no contribution to (4.6) is obtained by replacing the velocity components in (4.2) with the mass flux components

$$U \equiv h^* u, \quad V \equiv h^* v, \qquad (4.7)$$

and the vorticity analog with the potential vorticity analog

$$Z \equiv \zeta/h. \qquad (4.8)$$

This can be verified by a straightforward calculation of the time change of (4.6). Contributions of the ⅔ terms of (4.2) to the time change of (4.6) then cancel at each grid point, while the contributions of the ⅓ terms cancel between neighboring grid points, regardless of the definition of h^*.

A different generalization of (4.2) is possible, which also guarantees conservation of kinetic energy (4.6) with respect to the rotation terms. However, from the standpoint of potential vorticity advection, replacement of the velocity components and the vorticity in (4.2) by (4.7) and (4.8), respectively, is attractive, and is used in the present model.

For vorticity, kinetic energy and enstrophy conservation with spherical geometry, along with

$$\Delta S = 2a^2 \Delta\lambda \Delta\varphi \cos\varphi,$$

V components of the u momentum equation terms are weighted proportionally to $\cos\varphi$ (as, e.g., in Burridge and Haseler, 1977); and for generalization to the fully three-dimensional flow, h is replaced by π. Thus, the final generalization of (4.2) is

$$\frac{\partial u}{\partial t} = \frac{1}{\cos\varphi}\left(\tfrac{2}{3} V \cos\varphi\, \tilde{\tilde{Z}}^{\lambda\varphi} + \tfrac{1}{3}\overline{\overline{V\cos\varphi}^{\varphi} Z}^{\lambda}\right), \qquad (4.9)$$

$$\frac{\partial v}{\partial t} = -\tfrac{2}{3} U \tilde{\tilde{Z}}^{\lambda\varphi} - \tfrac{1}{3}\overline{\tilde{U}^{\lambda} Z}^{\varphi},$$

$$(U, V) \equiv \pi^*(u, v), \qquad (4.10)$$

$$Z \equiv \frac{1}{\pi a \cos\varphi}(\delta_\lambda v - \delta_\varphi u \cos\varphi). \qquad (4.11)$$

b. The fourth-order scheme

A number of numerical models are presently employing fourth-order accuracy schemes, especially for horizontal advection terms. A specific benefit that has been reported is an increase in the speed of propagation of atmospheric systems (e.g., Mihok and Kaitala, 1978); i.e., a reduction in the phase speed error. The fourth-order schemes of these models have been obtained using the standard "4/3 minus 1/3" procedure.

Here, however, we construct a fourth-order scheme, for the rotation terms, by a transformation of the Arakawa (1966) fourth-order Jacobian. This Jacobian, following Arakawa (1966) notation, can be written as

$$2\mathcal{J}_1 - \mathcal{J}_2, \tag{4.12}$$

where $\mathcal{J}_1$ is the Arakawa second-order Jacobian, and

$$\mathcal{J}_2 = \tfrac{1}{3}(\mathcal{J}^{\times\times} + \mathcal{J}^{\times+} + \mathcal{J}^{+\times}). \tag{4.13}$$

The superscripts of the component Jacobians here on the right side denote the location of points from which the values of $\Delta\zeta$ and ψ are to be taken, relative to an χ axis of a non-staggered grid (e.g., Arakawa χ axis on Fig. 3). For example, since in terms of a contour integral around the circumference of an area ΔS the Jacobian can be defined as

$$J(a,b) \equiv \lim_{\Delta S\to 0} \oint a db,$$

we can write, at point 0 in the notation of Fig. 3,

$$\mathcal{J}^{\times\times}(\zeta,\psi) = \frac{1}{8d^2}[(\zeta_{10} + \zeta_9)(\psi_{10} - \psi_9) + (\zeta_{11} + \zeta_{10})(\psi_{11} - \psi_{10}) + (\zeta_{12} + \zeta_{11})(\psi_{12} - \psi_{11}) + (\zeta_9 + \zeta_{12})(\psi_9 - \psi_{12})]. \tag{4.14}$$

Similarly, expressions for $\mathcal{J}^{\times+}$ and $\mathcal{J}^{+\times}$ can be written, with the + sign denoting values at points 21, 22, 23 and 24.

Now to construct an analog of (4.12) on the E grid, with an analog of $\mathcal{J}_1$ known, an analog of $\mathcal{J}_2$ is needed. We shall find such an analog by transforming each of the three component Jacobians on the right side of (4.13) to the E grid.

First, note that in view of the definition of vorticity and (4.1), we have, at the central point of the stencil in Fig. 3,

$$\frac{\partial\zeta}{\partial t} = \mathcal{J}(\zeta,\psi) = \frac{1}{\sqrt{2}d} \times \{-[u\zeta]_1 + [u\zeta]_3 - [v\zeta]_2 + [v\zeta]_4\}. \tag{4.15}$$

Here brackets are used to denote difference analogs, as yet unspecified, of the rotation terms inside the brackets. We want to determine these analogs, for each of the three Jacobians, in such a way as to have (4.15) give the known grid-point expression of that Jacobian, e.g., (4.14) in case of the Jacobian $\mathcal{J}^{\times\times}$.

To arrive at these analogs which correspond to the Jacobian $\mathcal{J}^{\times\times}$, we transform (4.14) so that, hopefully, terms on the right side of (4.15) can be recognized. Consider first the streamfunction factors of (4.14); the Arakawa (1966) scheme velocity components being nondivergent, note that we can write, e.g.,

$$\psi_{10} - \psi_9 = -\sqrt{2}d(\bar{u}^x + \bar{v}^y)_5.$$

As for the vorticity factors,

$$\zeta_{10} + \zeta_9 \equiv 2(\bar{\zeta}^{2y'})_5.$$

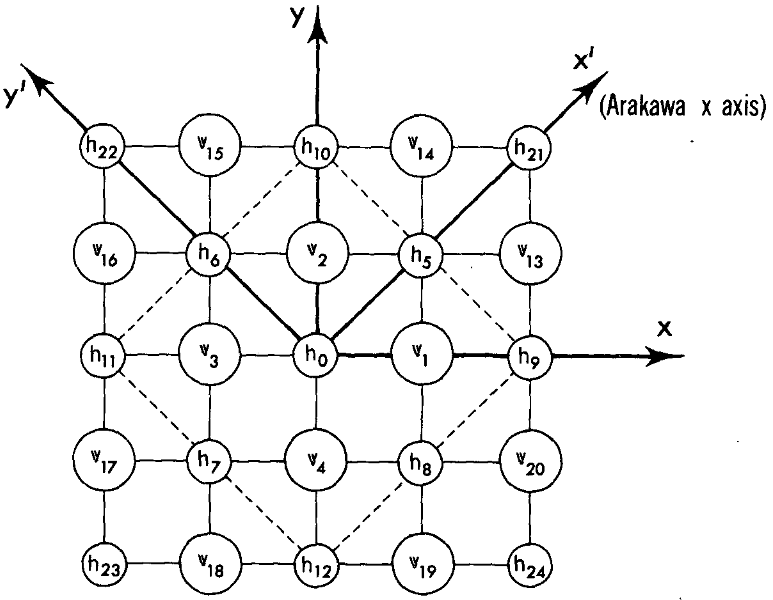

FIG. 3. Stencil used to transform the Arakawa (1966) fourth-order Jacobian.

Proceeding in this way, we can rewrite (4.14) as

$$\mathscr{J}^{\times\times}(\zeta,\psi) = \frac{1}{\sqrt{2}d}\{-\tfrac{1}{2}[(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}]_5 + \tfrac{1}{2}[(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}]_6 + \tfrac{1}{2}[(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}]_7 - \tfrac{1}{2}[(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}]_8\}.$$

This can be brought into a form more resembling (4.15) by further rewriting it as

$$\mathscr{J}^{\times\times}(\Delta\zeta,\psi) = \frac{1}{\sqrt{2}d}\{-\tfrac{1}{4}[(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}]_8 - \tfrac{1}{4}[(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}]_5 + \tfrac{1}{4}[(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}]_6 - \tfrac{1}{4}[(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}]_5 + \cdots\}.$$

It is now not difficult to see that the right side here will be equivalent to that of (4.15) if in (4.15) we define

$$\left.\begin{aligned} [v\zeta]^{\times\times} &= \tfrac{1}{2}\overline{[-\overline{(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}} + \overline{(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}}]}^{x} \\ -[u\zeta]^{\times\times} &= -\tfrac{1}{2}\overline{[\overline{(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2x'}} + \overline{(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2y'}}]}^{y} \end{aligned}\right\}. \qquad (4.16)$$

These, therefore, are the desired analogs for the Jacobian $\mathscr{J}^{\times\times}$.

Similarly, we obtain

$$\left.\begin{aligned} [v\zeta]^{\times +} &= \overline{\bar{v}^x\zeta}^{y} \\ -[u\zeta]^{\times +} &= -\overline{\bar{u}^y\zeta}^{x} \end{aligned}\right\} \qquad (4.17)$$

and

$$\left.\begin{aligned} [v\zeta]^{+\times} &= \tfrac{1}{2}\overline{[-\overline{(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2y'}} + \overline{(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2x'}}]}^{x} \\ -[u\zeta]^{+\times} &= -\tfrac{1}{2}\overline{[\overline{(\bar{u}^x - \bar{v}^y)\bar{\zeta}^{2y'}} + \overline{(\bar{u}^x + \bar{v}^y)\bar{\zeta}^{2x'}}]}^{y} \end{aligned}\right\} \qquad (4.18)$$

as the analogs for the Jacobians $\mathscr{J}^{\times +}$, and $\mathscr{J}^{+\times}$, respectively.

Averaging the expressions (4.16), (4.17) and (4.18) now gives as an E grid analog of the Jacobian $\mathscr{J}_2$ the scheme

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \tfrac{2}{3}\overline{\bar{v}^y\bar{\zeta}^{xy}}^{x} + \tfrac{1}{3}\overline{\bar{v}^x\zeta}^{y} \\ \frac{\partial v}{\partial t} &= -\tfrac{2}{3}\overline{\bar{u}^x\bar{\zeta}^{xy}}^{y} - \tfrac{1}{3}\overline{\bar{u}^y\zeta}^{x} \end{aligned}\right\} \qquad (4.19)$$

The linear combination of (4.2) and (4.19), as prescribed by (4.12), is thus an E grid analog of the Arakawa (1966) enstrophy and energy conserving fourth-order advection scheme.

The Arakawa (1966) second-order scheme has occasionally been criticized for having the linear phase speed error greater than that obtained using the standard centered second-order momentum advection scheme. Indeed, if we consider the simple linearized case

$$u = U + u'(x,t), \quad v = v'(x,t), \quad U = \text{constant},$$

we find that the second-order scheme (4.2) results in the vorticity advection equation

$$\frac{\partial\zeta}{\partial t} = -U(\tfrac{2}{3}\delta_x\zeta + \tfrac{1}{3}\delta_x\bar{\zeta}^{x}).$$

This gives the phase speed

$$c^* = U\left(\frac{2}{3}\frac{\sin kd/\sqrt{2}}{kd/\sqrt{2}} + \frac{1}{3}\frac{\sin\sqrt{2}kd}{\sqrt{2}kd}\right),$$

which is less and therefore also less accurate than

$$U\frac{\sin kd/\sqrt{2}}{kd/\sqrt{2}},$$

obtained using the standard centered second-order scheme.

The fourth-order scheme (4.12), (4.2) and (4.19), however, results in the vorticity equation

$$\frac{\partial\zeta}{\partial t} = -U(\tfrac{4}{3}\delta_x\zeta - \tfrac{1}{3}\delta_x\bar{\zeta}^{x}),$$

which gives the phase speed

$$c^{**} = U\left(\frac{4}{3}\frac{\sin kd/\sqrt{2}}{kd/\sqrt{2}} - \frac{1}{3}\frac{\sin\sqrt{2}kd}{\sqrt{2}kd}\right)$$

This phase speed is the same as that obtained using the standard "4/3 minus 1/3" fourth-order advection scheme. Thus, contrary to the case of the second-order scheme, from the point of view of phase speed accuracy, the Arakawa approach here is seen to be associated with no disadvantage compared with the standard centered scheme.

Note, however, that neither the analog (4.2) nor the analog (4.19) are the only possible E grid analogs of the considered Arakawa Jacobians. In the case of $\mathscr{J}_2$, it is possible to add to the right sides of (4.19) contributions as follows: to $(4.19)_1$, at point 2:

$$a\tfrac{1}{24}[(v_{14} - v_1)(\zeta_{10} + \zeta_9 - \zeta_{21} - \zeta_0) + (v_3 - v_{15})(\zeta_0 + \zeta_{22} - \zeta_{11} - \zeta_{10})]; \qquad (4.19)_1$$

to $(4.19)_2$, at point 1:

$$-a\tfrac{1}{24}[(u_{13} - u_2)(\zeta_9 + \zeta_{10} - \zeta_{21} - \zeta_0) + (u_4 - u_{20})(\zeta_0 + \zeta_{24} - \zeta_{12} - \zeta_9)]; \qquad (4.19)_2$$

and contributions of the same type to both Eqs. (4.19) also at other points. Adding (4.20) does not change the vorticity equation analog of (4.19) when the velocity components are nondivergent, irrespective of the value chosen for a. Thus, all schemes obtained by adding (4.20) to (4.19) are still analogs of the Arakawa Jacobian $\mathcal{J}_2$ on the E grid. For the time being, we use

$$a = 1. \tag{4.21}$$

This choice results in a scheme in which, within the analog of $\mathcal{J}_2$, each velocity component multiplies only three, rather than four, values of vorticity.

An interesting property of (4.19) to (4.21) is that it involves variables of only one of the two C-type subgrids which form an E grid. Note also that viewed on that subgrid, it is identical to the $\mathcal{J}_1$ scheme found for the C grid by Grammeltvedt (1969), and generalized later to the three-dimensional case, and/or rediscovered, by Sadourny (Burridge and Haseler, 1977).

Generalization of the fourth-order rotation terms scheme to the spherical geometry, and to the three-dimensional case, can be performed in the same way as that of the second-order scheme. Thus, there is no need to include this generalization here.

5. The kinetic energy terms

a. The second-order scheme

We now proceed to the next pair of terms on the right side of (2.2), the kinetic energy advection terms. We shall again first disregard the effect of spherical geometry; thus, we are looking for an analog of

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\frac{1}{2}\mathbf{v}^2, \quad \frac{\partial v}{\partial t} = -\frac{\partial}{\partial y}\frac{1}{2}\mathbf{v}^2. \tag{5.1}$$

A straightforward approximation to (5.1) on the E grid is

$$\frac{\partial u}{\partial t} = -\delta_x k, \quad \frac{\partial v}{\partial t} = -\delta_y k, \tag{5.2}$$

where k is an approximation to $\frac{1}{2}\mathbf{v}^2$, calculated at h points. Note that with (5.2) there will be no false production of vorticity due to kinetic energy advection terms, since they will cancel in differencing required by (4.4), regardless of the definition of k.

Another important property to be required is that the kinetic energy advection terms make no contribution to the time change in the total kinetic energy. Indeed, in the differential case, if we combine (5.1) and, for example, the shallow-water version of the continuity equation (2.5)

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}hu - \frac{\partial}{\partial y}hv, \tag{5.3}$$

we readily demonstrate that the kinetic energy advection terms are not associated with a change in the total kinetic energy (4.5).

For verification of this property a definition of h^* in (4.6) is needed. A simple choice is

$$h^* \equiv \tilde{h}^{xy}. \tag{5.4}$$

A change in K can now be calculated for a given definition of k and choice of the scheme for the continuity equation. Suppose

$$k \equiv \tfrac{1}{2}\widetilde{\mathbf{v}^2}^{xy} \tag{5.5}$$

(Janjić, 1977) and

$$\frac{\partial h}{\partial t} = -\delta_x h^* u - \delta_y h^* v. \tag{5.6}$$

Then

$$\frac{\partial K}{\partial t} \sim -\sum\, [h^*(u\delta_x\widetilde{\mathbf{v}^2}^{xy} + v\delta_y\widetilde{\mathbf{v}^2}^{xy}) + \mathbf{v}^2(\delta_x\widetilde{h^*u}^{xy} + \delta_y\widetilde{h^*v}^{xy})], \tag{5.7}$$

and inspection of terms on the right side obtained at a given $\mathbf{v}$ grid point shows that, for each of the terms that do not cancel at that point, there always appears at a surrounding point an identical term with an opposite sign; thus, all terms cancel in summation required by (5.7).

For kinetic energy conservation with spherical geometry, in

$$\frac{\partial u}{\partial t} = -\frac{1}{a\cos\varphi}\delta_\lambda \frac{1}{2}\widetilde{\mathbf{v}^2}^{\lambda\varphi},$$

$$\frac{\partial v}{\partial t} = -\frac{1}{a}\delta_\varphi \frac{1}{2}\widetilde{\mathbf{v}^2}^{\lambda\varphi}, \tag{5.8}$$

$$\frac{\partial h}{\partial t} = -\frac{1}{a\cos\varphi}(\delta_\lambda h^* u + \delta_\varphi h^* v\cos\varphi), \tag{5.9}$$

as well as in (4.6), the definition (5.4) is replaced by

$$h^* \equiv \frac{1}{\cos\varphi}\widetilde{h\cos\varphi}^{\lambda\varphi}. \tag{5.10}$$

Incidentally, this is the weighting appropriate also on geometric considerations.

Note that (5.9) does guarantee conservation of total mass

$$M \equiv \sum 2a^2\Delta\lambda\Delta\varphi\cos\varphi\, h\rho, \tag{5.11}$$

with summation performed over all h points.

For the generalization to the fully three-dimensional flow, h is again replaced by π, and the continuity equation is generalized to obtain analogs of its three-dimensional forms, (2.4) and (2.5).

b. The fourth-order scheme

In a higher accuracy scheme for the kinetic energy advection terms, we keep the approximation (5.2), with the same definition of k in the u and in the v momentum equation terms. However, this definition will be made more general than (5.5); an obvious next step is to consider the linear combination

$$k \equiv \tfrac{1}{2}(a\widetilde{\mathbf{v}^2}^{xy} + b\widetilde{\mathbf{v}^2}^{2x2y}), \tag{5.12}$$

where

$$\widetilde{\mathbf{v}^2}^{2x2y} \equiv \tfrac{1}{2}\left(\overline{\overline{\mathbf{v}^2}^{2x}}^{y} + \overline{\overline{\mathbf{v}^2}^{2y}}^{x}\right), \tag{5.13}$$

denotes an eight-point average of $\mathbf{v}^2$.

In addition to consistency, it is now possible to impose an additional requirement on the constants a and b in (5.12). The requirement that the scheme (5.2) be of the fourth-order accuracy for $\mathbf{v}^2 = \mathbf{v}^2(x)$ and $\mathbf{v}^2 = \mathbf{v}^2(y)$ gives the values

$$a = 17/12, \quad b = -5/12. \tag{5.14}$$

If, on the other hand, it is required that the fourth-order accuracy be accomplished in cases when $\mathbf{v}^2$ is changing along the diagonal grid lines, that is, for $\mathbf{v}^2 = \mathbf{v}^2(x')$ and $\mathbf{v}^2 = \mathbf{v}^2(y')$, one obtains the standard result

$$a = 4/3, \quad b = -1/3. \tag{5.15}$$

This, not surprisingly, could also have been foreseen by an inspection of (5.12).

Of course, it is also possible to impose a strict fourth-order accuracy on (5.2) by a still more general definition of k. However, the present author does not believe that a meaningful benefit, if any, should be expected from such an effort, and it has not been attempted. The motivation behind the present development of fourth-order schemes is an improved advective phase speed; there is no reason to believe that for this purpose strictly fourth order schemes should be particularly valuable (see, e.g., also Gadd, 1978a).

The difference between the values (5.14) and (5.15) is slight, and it may not matter much which one of these two possibilities is chosen. One could argue that since the effective resolution for functions of x' and of y', on the E grid, is poorer than that for functions of x and of y, one should give preference to the choice (5.15). Thus, (5.15) is chosen for the fourth-order scheme for the kinetic energy terms in the present model.

The definition (5.12), with $a \neq 1$, of course, disrupts the cancellation of terms resulting from (5.2) and from the second-order continuity equation (5.6) in the change in the total kinetic energy (5.7). Thus, if a strict conservation of the kinetic energy is to be preserved, an appropriate modification of the continuity equation has to be sought. A logical candidate to replace (5.6) then is the analog

$$\frac{\partial h}{\partial t} = -c\boldsymbol{\nabla}\cdot\mathbf{v} - d\overline{\boldsymbol{\nabla}\cdot\mathbf{v}}^{xy}, \tag{5.16}$$

where $\boldsymbol{\nabla}\cdot\mathbf{v} \equiv \delta_x h^* u + \delta_y h^* v$ and, for consistency, $c + d = 1$.

An investigation of terms resulting from (5.12) and (5.16) in the equation for the change in the total kinetic energy shows that it is indeed possible to define the remaining free constant in (5.16) so as to accomplish the cancellation needed for conservation of the kinetic energy. This happens for

$$c = 2a - 1. \tag{5.17}$$

However, (5.17) associated with (5.16) would, for reasons of linear stability, bring about a major loss in the economy of the model, and again the benefit, if any, is not likely to be very meaningful. It has been demonstrated by Sadourny (1975a,b) that strict conservation of kinetic energy within the momentum advection terms does not appear to be nearly as important as strict conservation of enstrophy; besides, a test made with the present model failed to show any visible effect, in a short-range forecast, of the change from the continuity equation (5.6) to the much more expensive form (5.16) and (5.17). [On the other hand, in the same forecast the fourth-order scheme (5.12) and (5.15) did show a small but a clearly visible benefit over the second-order scheme (5.5).] Therefore, strict energy conservation is not enforced within the fourth-order kinetic energy terms of the present staggered grid GFDL model; i.e., the continuity equation is kept in its second-order form (5.6).

Generalization to the spherical geometry, and to the fully three-dimensional flow, is performed in the same way as that of the second-order scheme.

6. The Hollingsworth-Källberg instability

Recently a problem has been discovered in real data integrations using the global model of the European Centre for Medium Range Weather Forecasts, and an enstrophy- and energy-conserving scheme. Within a few days in a high-resolution integration a dramatic decrease in the intensity of the jet streams was noticed, associated with a large increase in short-wave kinetic energy.

An analysis and a numerical experiment performed by Hollingsworth and Källberg (1979) seems to offer conclusive evidence that the source of the problem was a non-conservation of momentum in the linearized equations. In the differential case, when the v momentum equation

$$\frac{\partial v}{\partial t} = -u\zeta - \frac{\partial}{\partial y}\tfrac{1}{2}\mathbf{v}^2 \tag{6.1}$$

is linearized for the basic state

$$\bar{u} = \text{constant}, \quad \bar{v} = 0, \tag{6.2}$$

and a perturbation

$$u', \zeta' = u', \zeta'(y), \quad v' = 0, \tag{6.3}$$

one obtains

$$\frac{\partial v'}{\partial t} = -\bar{u}\zeta' - \frac{\partial}{\partial y}\bar{u}u'. \tag{6.4}$$

Since in the considered case

$$\zeta' = -\frac{\partial u'}{\partial y},$$

the two terms on the right side of (6.4) will cancel, and (6.3) is permitted as a stationary solution of the linearized equations. The enstrophy- and energy-conserving scheme which developed the trouble did not maintain this cancellation property. When the integrations were repeated using a closely related enstrophy conserving scheme, which did guarantee the cancellation in the considered linearized case, reasonable results were obtained. Renner, moreover, has found a way to maintain the cancellation property without sacrificing the energy conservation, by a redefinition of the mass fluxes of the scheme (personal communication).

Using the second-order schemes of the preceding two sections instead of (6.4) we obtain

$$\frac{\partial v'}{\partial t} = -\tfrac{2}{3}\bar{u}\,\overline{\tilde{\zeta}'}^{xy} - \tfrac{1}{3}\bar{u}\,\overline{\overline{\zeta'}}^{y} - \delta_y \bar{u}\,\overline{\tilde{u}'}^{xy}. \tag{6.5}$$

Substituting

$$\zeta' = -\delta_y u'$$

we find that

$$\frac{\partial v'}{\partial t} = -\bar{u}\left(\tfrac{2}{3}\overline{\overline{\zeta'}}^{y} + \tfrac{1}{3}\overline{\overline{\zeta'}}^{x} - \overline{\tilde{\zeta}'}^{xy}\right). \tag{6.6}$$

Thus, these schemes of the preceding two sections also do not maintain the cancellation property of (6.4).

Similarly, as done by Renner, instructed by (6.6), we can modify the considered schemes so as to restore the momentum conservation in the analog of (6.4). Namely, we see that the cancellation would have been accomplished had the kinetic energy advection term been calculated as

$$-\delta_y \tfrac{1}{2}\left(\tfrac{2}{3}\overline{u^2}^{y} + \tfrac{1}{3}\overline{u^2}^{x} + \tfrac{2}{3}\overline{v^2}^{x} + \tfrac{1}{3}\overline{v^2}^{y}\right).$$

The coefficients of the v kinetic energy terms here are chosen for symmetry reasons.

Thus, to remove the Hollingsworth-Källberg non-cancellation in the linearized equations we can redefine the second-order kinetic energy at the h points as

$$k \equiv \tfrac{1}{2}\left(\tfrac{1}{3}\overline{u^2}^{\lambda} + \tfrac{2}{3}\overline{u^2}^{\varphi} + \tfrac{2}{3}\overline{v^2}^{\lambda} + \tfrac{1}{3}\overline{v^2}^{\varphi}\right). \tag{6.7}$$

If, concurrently, the definition of kinetic energy (4.6) is replaced by

$$K \equiv \sum \rho \tfrac{1}{2}(h_u{}^* u^2 + h_v{}^* v^2) 2\Delta\lambda\Delta\varphi \cos\varphi, \tag{6.8}$$

where

$$h_u{}^* \equiv \frac{1}{3}\bar{h}^{\lambda} + \frac{2}{3\cos\varphi}\overline{h\cos\varphi}^{\varphi},$$

$$h_v{}^* \equiv \frac{2}{3}\bar{h}^{\lambda} + \frac{1}{3\cos\varphi}\overline{h\cos\varphi}^{\varphi}, \tag{6.9}$$

and the continuity equation scheme (5.9) is replaced by

$$\frac{\partial h}{\partial t} = -\frac{1}{a\cos\varphi}(\delta_\lambda h_u{}^* u + \delta_\varphi h_v{}^* v \cos\varphi), \tag{6.10}$$

then the kinetic energy conservation property of the horizontal advection scheme will not be disrupted.

In the fourth-order schemes, the cancellation of terms in the analogue of (6.4) will be accomplished if (6.7) is replaced by

$$k \equiv \tfrac{1}{2}\left(\tfrac{4}{3}\overline{u^2}^{\varphi} - \tfrac{1}{3}\overline{u^2}^{\lambda 2\varphi} + \tfrac{4}{3}\overline{v^2}^{\lambda} - \tfrac{1}{3}\overline{v^2}^{\varphi 2\lambda}\right), \tag{6.11}$$

and (6.9) by

$$h_u{}^* \equiv \frac{1}{\cos\varphi}\left(\tfrac{4}{3}\overline{h\cos\varphi}^{\varphi} - \tfrac{1}{3}\overline{h\cos\varphi}^{\lambda 2\varphi}\right),$$

$$h_v{}^* \equiv \frac{4}{3}\bar{h}^{\lambda} - \frac{1}{3\cos\varphi}\overline{h\cos\varphi}^{\varphi 2\lambda}. \tag{6.12}$$

As before, for generalization to the fully three-dimensional flow h is replaced by π; thus, instead of (4.10), then we will have

$$U \equiv \pi_u{}^* u, \quad V \equiv \pi_v{}^* v. \tag{6.13}$$

Recently it has been demonstrated by Lazić (personal communication) that the Hollingsworth-Källberg instability does not occur with split calculation of the advection and of the gravity-inertia terms. Thus, the modification described here should be needed for stability only in models calculating these terms using a non-split time differencing.

7. Horizontal temperature and/or moisture advection

a. The second-order scheme

Let us look for an analog of

$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y}, \tag{7.1}$$

using in this section T to represent either temperature or the mixing ratio of water vapor. An important aspect of (7.1) is its energy (or moisture) conservation property; for example, combining (7.1) with the shallow-water continuity equation (5.3) one readily demonstrates that the integral

$$\int_S h\rho T dS \tag{7.2}$$

is conserved for a closed domain S. In fact, with individual mass elements of fluid conserving their values of T, the mass integral of an arbitrary power of T is conserved; of course, this is also easily shown.

Having these properties in mind and using the notation introduced by the preceding two figures, an attractive analog of (7.1) on the E grid appears to be

$$\frac{\partial T}{\partial t} = -\frac{1}{8h_0}\left[(U_1+U_2)\frac{T_5-T_0}{\Delta x} + (U_2+U_3)\frac{T_0-T_6}{\Delta x} + (U_4+U_3)\frac{T_0-T_7}{\Delta x} + (U_1+U_4)\frac{T_8-T_0}{\Delta x}\right.$$
$$\left. + (V_1+V_2)\frac{T_5-T_0}{\Delta y} + (V_2+V_3)\frac{T_6-T_0}{\Delta y} + (V_4+V_3)\frac{T_0-T_7}{\Delta y} + (V_1+V_4)\frac{T_0-T_8}{\Delta y}\right]. \tag{7.3}$$

To check the consistency and order of accuracy of (7.3) the standard Taylor series expansion can be performed; the first four of the eight terms on its right side then give

$$-\frac{1}{h}\Big[UT_x + \tfrac{1}{6}U(T_{xxx}p^2 + 3T_{xyy}q^2) + \tfrac{1}{4}U_x(T_{xx}p^2 + T_{yy}q^2) + \tfrac{1}{2}U_yT_{xy}q^2 + \tfrac{1}{4}(U_{xx}T_xp^2 + U_{yy}T_xq^2) + \text{higher order terms}\Big]. \quad (7.4)$$

Here, for brevity, p and q stand for Δx and Δy, respectively, and subscripts are used to denote derivatives. Since an analogous expression can be written for the remaining four terms, it is seen that (7.3) is indeed a consistent analog to (7.1), and of the second-order accuracy.

As for the conservation properties of (7.3), consideration in conjunction with the scheme for the continuity equation is required. This, in the square grid case, is the scheme (5.6); or, restated in terms of the present notation

$$\frac{\partial h}{\partial t} = -\frac{U_1 - U_3}{2\Delta x} - \frac{V_2 - V_4}{2\Delta y}. \quad (7.5)$$

Calculation of the time change of the analog of (7.2)

$$\sum h\rho T\Delta S, \quad (7.6)$$

and inspection of terms of that time change obtained at a given h (or T) grid point shows that, again, for each of the terms that do not cancel at that point, there always appears at a surrounding point an identical term with an opposite sign. Thus, all terms cancel in summation, and the scheme (7.3) in association with (7.5) is seen to guarantee conservation of the internal energy (or moisture) inside a closed domain and with respect to the horizontal advection terms.

Calculation of the time change of

$$\sum h\rho T^2\Delta S, \quad (7.7)$$

shows that the cancellation of terms, necessary to maintain (7.7), is also accomplished. Thus, the scheme (7.3) and (7.5) maintains, within the horizontal advection terms, also the second moment of T. This represents one more constraint on the possibility of generation of the small-scale noise in the T field, within the terms considered here.

In this respect, an additional favorable property of the schemes (7.3) and (7.5) is that in both cases, the cancellation of terms is completed between nearest neighbors among the T grid points; rather than, as would have happened with some other schemes, between points that are not nearest neighbors to each other. In fact, a conserving temperature advection scheme which did not have this nearest neighbor cancellation property has been tested in earlier stages of our study, and has been found to be associated with a fairly intensive generation of the scale noise in the temperature field.

In generalizing (7.3) to spherical geometry, in order to maintain its conservation properties, Δx is replaced by $a\cos\varphi\Delta\lambda$ calculated at the central point, rather than at the midpoints relative to the terms approximating $U\partial T/\partial x$. Otherwise, as before, V mass flux components are weighted proportionally to $\cos\varphi$; and, also the (5.10) type averaging is performed in calculating both of the mass flux components. For the generalization to the fully three-dimensional flow, h is replaced by π. Thus, using for brevity again the standard difference notation for the averaging and differencing along the x', y' grid axes, defined now as curvilinear coordinate lines going through the same grid points as in the square grid case, one obtains

$$\left.\begin{aligned}
\frac{\partial T}{\partial t} &= -\frac{1}{2\pi_0 a\cos\varphi}\left[\overline{\left(\bar{U}^{y'}\frac{\Delta_{x'}T}{\Delta\lambda} + \overline{V\cos\varphi}^{y'}\frac{\Delta_{x'}T}{\Delta\varphi}\right)}^{x'} + \overline{\left(\bar{U}^{x'}\frac{\Delta_{-y'}T}{\Delta\lambda} + \overline{V\cos\varphi}^{x'}\frac{\Delta_{y'}T}{\Delta\varphi}\right)}^{y'}\right]\\
(U,V) &\equiv \pi^*(u,v), \qquad \pi^* \equiv \frac{1}{\cos\varphi}\widetilde{\pi\cos\varphi}^{\lambda\varphi}
\end{aligned}\right\}. \quad (7.8)$$

b. The fourth-order scheme

In looking for a scheme which would have a smaller phase speed error than the second-order scheme (7.3), one can construct a more general scheme by forming a linear combination of the terms present in (7.3) and terms of the same type but calculated one grid distance further away from the central point. In other words, one can consider the scheme

$$\begin{aligned}
\frac{\partial T}{\partial t} = -\frac{1}{2h_0}\Bigg\{\alpha\Bigg[&\overline{\left(\bar{U}^{y'}\frac{\Delta_{x'}T}{\Delta x} + \bar{V}^{y'}\frac{\Delta_{x'}T}{\Delta y}\right)}^{x'} + \overline{\left(\bar{U}^{x'}\frac{\Delta_{-y'}T}{\Delta x} + \bar{V}^{x'}\frac{\Delta_{y'}T}{\Delta y}\right)}^{y'}\Bigg]\\
+ \beta\Bigg[&\overline{\left(\bar{U}^{y'}\frac{\Delta_{x'}T}{\Delta x} + \bar{V}^{y'}\frac{\Delta_{x'}T}{\Delta y}\right)}^{3x'} + \overline{\left(\bar{U}^{x'}\frac{\Delta_{-y'}T}{\Delta x} + \bar{V}^{x'}\frac{\Delta_{y'}T}{\Delta y}\right)}^{3y'}\Bigg]\Bigg\}, \quad (7.9)
\end{aligned}$$

where, for consistency,

$$\alpha + \beta = 1,$$

and, once more, the subscript notation of (7.3) is replaced by the difference notation.

Inspection of (7.9) for the linear case $U, V =$ constant shows that the values

$$\alpha = 7/6, \quad \beta = -1/6, \tag{7.10}$$

are needed to have (7.9), in that case, reduce to a simple 4/3 minus 1/3 type fourth-order advection scheme.

These values, not surprisingly, can also be obtained by an analysis of the truncation error of (7.9). The Taylor series expansion of the new terms approximating $U\partial T/\partial x$ in (7.9), of the type as those that have resulted in (7.4) when evaluating the truncation error of (7.3), gives the expression

$$\begin{aligned} -\frac{1}{h}[UT_x &+ \tfrac{7}{6}U(T_{xxx}p^2 + 3T_{xyy}q^2) \\ &+ \tfrac{9}{4}U_x(T_{xx}p^2 + T_{yy}q^2) + \tfrac{9}{2}U_yT_{xy}q^2 \\ &+ \tfrac{5}{4}(U_{xx}T_xp^2 + U_{yy}T_xq^2) + 2U_{xy}T_yq^2 \\ &+ \text{higher order terms}]. \end{aligned} \tag{7.11}$$

Thus, obviously, factors (7.10) are seen to be needed in (7.9) to eliminate in case $U =$ constant the second-order terms in its truncation error.

Comparison of (7.9) and (7.3), already by inspection shows that the property of the conservation of the time change of (7.6) has not been disrupted, irrespective of the value chosen for α. Calculation of the time change of (7.7), however, shows that, unfortunately, conservation of the second moment of T is not guaranteed by the fourth-order scheme (7.9) and (7.10).

Generalization of the fourth-order scheme (7.9) and (7.10) to the sperical geometry, and to the fully three-dimensional flow, is performed in the same way as that of the second-order scheme.

8. Summary and concluding remarks

Schemes for the horizontal advection of momentum and temperature (and/or moisture) on the E grid have been constructed satisfying simultaneously a number of requirements obviously desirable or considered desirable by various modeling groups; among those the fourth- (or approximately fourth) order accuracy and conservation of enstrophy and energy. Details of these schemes have been given and properties discussed in the preceding sections. These advection schemes have been built into a limited-area model and extensively tested. Some results have already been published in informal publications (Mesinger *et al.*, 1979; Mesinger, 1979b); it is planned to publish other results in a subsequent paper or papers.

This E grid model also has been further developed to a full weather prediction model by provision of one of the available GFDL "physics" packages, in which detailed calculation of the radiation, hydrologic cycle and boundary-layer processes are incorporated. Furthermore, schemes have been developed to handle the polar boundary condition in such a way as to maintain at the polar and subpolar points all the conservation properties of differencing schemes at points away from pole. These schemes have been implemented to obtain a hemispheric and also a global version of the model. For efficiency, in these versions polar filtering is performed of the gravity wave and the horizontal advection terms poleward of chosen latitude circles. A paper describing the technical details of these polar boundary schemes is in preparation.

As a result of the enstrophy conservation property and the absence or careful handling of computational modes, these model versions have little need for lateral diffusion and similar artificial smoothing devices. Their fourth-order advection feature as, for example, reported also by Campana (1979) has proven to be of a noticeable benefit especially in the medium and low-resolution integrations. The split time differencing, with long advection steps, enables the use of more complex advection schemes at a most modest computational cost. Thus, the fourth-order version of the present enstrophy-conserving schemes would seem to be particularly suitable for the extended range, medium and low-resolution integrations, such as required for the medium and long-range forecasts and for the climate modeling experiments.

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REFERENCES

Arakawa, A., 1966: Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I. *J. Comput. Phys.*, **1**, 119–143.

——, and V. R. Lamb, 1977: Computational design of the basic dynamical processes of the UCLA general circulation model. *Methods in Computational Physics*, Vol. 17, *General Circulation Models of the Atmosphere*, J. Chang, Ed., Academic Press, 173–265.

——, and ——, 1981: A potential enstrophy and energy conserving scheme for the shallow water equations. *Mon. Wea. Rev.*, **109**, 18–36.

Burridge, D. M., and J. Haseler, 1977: A model for medium range weather forecasting—adiabatic formulation. Tech. Rep. No. 4, European Centre for Medium Range Weather Forecasts, 46 pp.

Campana, K. A., 1979: Higher order finite-differencing experiments with a semi-implicit model at the National Meteorological Center. *Mon. Wea. Rev.*, **107**, 363–376.

Gadd, A. J., 1978a: A split explicit integration scheme for numerical weather prediction. *Quart. J. Roy. Meteor. Soc.*, **104**, 569–582.

——, 1978b: A numerical advection scheme with small phase speed errors. *Quart. J. Roy. Meteor. Soc.*, **104**, 583–594.

Grammeltvedt, A., 1969: A survey of finite-difference schemes for the primitive equations for a barotropic fluid. *Mon. Wea. Rev.*, **97**, 384–404.

Hayden, C. M., 1973: Experiments in the four-dimensional assimilation of Nimbus 4 SIRS data. *J. Appl. Meteor.*, **12**, 425–436.

Hollingsworth, A., and P. Källberg, 1979: Spurious energy conversions in an energy/enstrophy conserving finite difference scheme. Int. Rep. No. 22, European Centre for Medium Range Weather Forecasts, 32 pp.

Janjić, Z. I., 1977: Pressure gradient force and advection scheme used for forecasting with steep and small scale topography. *Contrib. Atmos. Phys.*, **50**, 186–199.

Mesinger, F., 1977: Forward-backward scheme, and its use in a limited area model. *Contrib. Atmos. Phys.*, **50**, 200–210.

——, 1979a: Dependence of vorticity analogue and the Rossby wave phase speed on the choice of horizontal grid. *Bull. Serb. Acad. Sci. Arts*, **64**, 5–15.

——, 1979b: Numerical simulation of Genoa cyclogenesis. *Workshop on Mountains and Numerical Weather Prediction*, European Centre for Medium Range Weather Forecasts, 209–231.

——, and A. Arakawa, 1976: Finite difference schemes used in atmospheric models, Vol. I. GARP Publ. Ser., No. 17, WMO, Geneva, 64 pp.

——, R. F. Strickler, J. Chludzinski and J. Sirutis, 1979: Numerical simulation of Genoa cyclogenesis, including the incipient stage and the upper-level cut-off. Annex H to the WGNE Report of the XV Session of the Joint Organizing Committee, WMO, Geneva, 6–13.

Mihok, W. F., and J. E. Kaitala, 1978: Reply. *Mon. Wea. Rev.*, **106**, 1023–1027.

Miyakoda, K., G. D. Hembree and R. F. Strickler, 1979: Cumulative results of extended forecast experiments. II: Model performance for summer cases. *Mon. Wea. Rev.*, **107**, 395–420.

Ničković, S., 1979: The effect of horizontal differencing on the shearing instability mechanism. *Contrib. Atmos. Phys.*, **52**, 126–135.

Richardson, L. F., 1922: *Weather Prediction by Numerical Process*. Cambridge University Press [reprint, with a new introduction by Sydney Chapman, Dover, 1965], 236 pp.

Sadourny, R., 1975a: The dynamics of finite-difference models of the shallow-water equations. *J. Atmos. Sci.*, **32**, 680–689.

——, 1975b: Compressible model flows on the sphere. *J. Atmos. Sci.*, **32**, 2103–2110.