

Treatment of Normal and Abnormal Modes

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The National Meteorological Center, in common with other groups, has experienced difficulty in applying the Machenhauer (1976) method of initializing data. If $\gamma(t)$ denotes the complex amplitude of a "fast mode" with frequency ω , the prediction equation for γ has the form

$$\frac{d\gamma}{dt} = -i\omega\gamma + iF_{\gamma} \cdot \text{NL}(\gamma; \alpha, \beta, \dots). \quad (1)$$

Here F_{γ} denotes projection onto the spatial eigenvector associated with γ , and NL is the total field of model tendencies not included in the linear term $-i\omega\gamma$; α, β, \dots denotes other modal amplitudes, some of which are assumed known and not subject to change. Machenhauer's valuable insight was to realize that instead of setting $\gamma(0) = 0$, a value of $\gamma(0)$ giving a better state of dynamic balance could be determined by ignoring $d\gamma/dt$:

$$\gamma(0) = \omega^{-1}F_\gamma \cdot \text{NL}(\gamma; \alpha, \beta, \dots). \quad (2)$$

Machenhauer iterated this equation, with γ in NL initially coming from the input analysis, γ_0 , say. Thus

$$\gamma_{l+1} = \omega^{-1}F_\gamma \cdot \text{NL}(\gamma_l; \alpha, \beta, \dots). \quad (3)$$

Finally, γ_l is inserted in the initial data in place of γ_0 for a forecast.

The difficulty is that this procedure only seems to converge experimentally when γ corresponds to a large equivalent depth h , i.e.,

$$h = c^2/g, \quad (4)$$

corresponding to the external and the internal modes of largest vertical wave length. Why should greater vertical detail be unobtainable, when the standard quasi-geostrophic procedure (e.g., Phillips, 1962, p. 157) will, at least in extratropical latitudes, give any vertical detail desired? This paradox appears to violate the theoretical results of Leith (1980).

One part of the answer appears to be that (2) and (3) should be corrected to the first stage of the method suggested by Baer (1977). This amounts to setting

$$\gamma_0 = 0, \quad (5)$$

followed by only one iteration

$$\gamma_1 = \omega^{-1}F_\gamma \cdot \text{NL}(0; \alpha, \beta, \dots). \quad (6)$$

If higher approximations are desired, the $d\gamma/dt$ term should be included.

The interested reader can readily see what happens by considering the very simple case of waves on a uniform basic current $\bar{\mu}$, with constant Coriolis parameter f . If α denotes the amplitude of the desired slow wave, and γ and δ denote the amplitudes of the eastward and westward moving gravity waves, the modal equations corresponding to (1) are

$$\frac{d\alpha}{dt} = -i(k\bar{\mu})\alpha, \quad (7)$$

$$\frac{d\gamma}{dt} = -i\omega_0\gamma - i(k\bar{\mu})\gamma, \quad (8)$$

$$\frac{d\delta}{dt} = i\omega_0\delta - i(k\bar{\mu})\delta, \quad (9)$$

where k is the wavenumber, and $\omega_0 = (f^2 + k^2c^2)^{1/2}$. The advective terms proportional to $k\bar{\mu}$ can be viewed as a simple form of $F \cdot \text{NL}$. Although simple, they imitate a pervasive nonlinearity in atmospheric flow.

Eqs. (8) and (9) can be manipulated readily to show that the conventional Machenhauer procedure attempts to arrive at the desired solution $\gamma = \delta = 0$, by an iteration process. For (9), this is

$$\delta_{l+1} = \left(\frac{k\bar{\mu}}{\omega_0} \right) \delta_l. \quad (10)$$

This will diverge if $\omega_0 < k\bar{\mu}$. The gravity wave frequency ω_0 can be small for high-order vertical modes (small h) in a global model, where ω_0 is effectively bounded from below by kc .

However, even if

$$\epsilon = k\bar{\mu}/\omega_0 \quad (11)$$

is < 1 , the iterations are undesirable. This can be seen by modifying (9) to

$$\frac{d\delta}{dt} = i\omega_0\delta - ik\bar{\mu}(\delta + s). \quad (12)$$

The fixed term $k\bar{\mu}s$ represents truly nonlinear contributions to $F_\gamma \cdot \text{NL}$. They vary slowly with time and come primarily from the vorticity and temperature tendency in the slow modes after subtraction of the uniform advection represented, for example, by $\bar{\mu}\partial\zeta/\partial x$. The factor $k\bar{\mu}$ normalizes these tendencies so that s is of order one if d/dt is of size $k\bar{\mu}$. The Machenhauer iterative solution of (12) is

$$\delta_l = \epsilon^l \delta_0 + \left(\frac{1 - \epsilon^l}{1 - \epsilon} \right) (\epsilon s). \quad (13)$$

For $\epsilon < 1$, the limit for large l is

$$\delta_l \rightarrow \frac{(\epsilon s)}{1 - \epsilon}. \quad (14)$$

The numerator (ϵs) is the Baer first stage solution. The $-\epsilon$ in the denominator is, according to the Baer theory, an incorrect $O(\epsilon^2)$ effect. This is because the left side of (12) in general is also of size ϵ^2 , i.e.,

$$\frac{d\delta}{dt} \approx \frac{d}{dt} (\epsilon s) = \epsilon \frac{ds}{dt} = O(\epsilon^2).$$

As such it can change drastically the effect of the $-\epsilon$ term in the denominator of (14). The Machenhauer iteration will not get the correct $O(\epsilon^2)$ correction for δ , and, if ϵ is only moderately small (e.g., $1/4$), it will even distort the correct $O(\epsilon)$ value for δ . (This failure may not be obvious if attention is focussed primarily on surface pressure noise, since it depends mostly on the external gravity waves, for which ϵ is indeed small.)

The solution for this part of the problem is to simply replace the Machenhauer method by the first stage (5) and (6) of Baer's method. "Convergence" (or, accuracy) will then depend on much more subtle effects than the forbidding size of $k\bar{\mu}$ relative to ω_0 for small h . These can be expected to include:

1) The baroclinic quasi-geostrophic condition that the Richardson number

$$Ri = gd \ln \theta / dz \div |\partial \mathbf{v} / \partial z|^2$$

be sufficiently large.¹

2) "Smallness" of NL

The last of these is violated most dramatically in regions of intense latent heat release. In such regions a "slow-time" behavior requires large upward velocities to cancel the otherwise large temperature tendency $\partial T / \partial t$. These motions will therefore require "gravity waves" to be included initially, with (5) being replaced by

$$\gamma_0 = \omega^{-1} F_\gamma \cdot NL \text{ (heating)}. \quad (15)$$

This might be accomplished without too much difficulty for large and moderate horizontal wavelengths in low latitudes by using (15) for the Kelvin wave, the mixed Rossby wave, and perhaps the slowest ordinary (paired) gravity waves, for most internal modes. These have their maximum amplitude near the equator.

The use of the Baer first stage may now allow adjustment of enough vertical modes that the paradox posed above after Eq. (4) is removed. Leith's demonstration (1980) that the Baer first stage on an f -plane is equivalent to quasi-geostrophic initialization might thereby be realized in practice as well as theory. However, a peculiar truncation effect arises in the linearized eigensolutions for a spherical harmonic model that is not present on an f -plane. At small positive equivalent depths the eigensolutions of the continuous Laplace tidal equations are concentrated in low latitudes, equatorward of $\theta_l = \pm [c(2n + 1)/2\Omega a]^{1/2}$ (Phillips, 1973, p. 53). Here $n = 1, 2, \dots$ is the latitudinal ordinal number, giving an effective north-south wavelength of $2[c/(\Omega a n)]^{1/2}$ radians of latitude. Its representation in spherical harmonics P_{N-l}^l will require the order $N - l$ to be at least of magnitude

$$N - l \geq \frac{1}{2} \pi (\Omega a n c^{-1})^{1/2}. \quad (16)$$

In the NMC model (Sela, 1980), the sixth vertical

mode has $c = 5.9 \text{ m s}^{-1}$. With $(N - l)_{\max} = 24$, Eq. (16) is attained for $n = 9$. At larger values of n , this harmonic truncation leads to a disappearance of the theoretical frequency separation between latitudinally symmetric (n odd) and asymmetric (n even) modes, suggesting the appearance of eigenfunctions that are concentrated in polar latitudes. Since Longuet-Higgins (1968) has shown that this phenomenon occurs only for *negative* equivalent depths, it seems likely that a single one of these modes at large n is not a good solution of the continuous linear equations of motion. Their collective properties, on the other hand, have some likelihood of being satisfactory since spherical harmonic forecast models do not appear to have problems in high latitudes. The collective contribution of these abnormal modes to a Green's function for divergence and for non-geostrophic vorticity would therefore be worth study.

A fuller version of the first part of this note has been written in Office Note 226 of the National Meteorological Center.

Acknowledgment. Discussion with Dr. B. Ballish has been very helpful in clarifying my thoughts on this subject. In particular, he supplied the frequency information upon which my conjecture about the effectively negative equivalent depth behavior is used. A more thorough discussion by him of an equation similar to (12) will appear in an early issue of this JOURNAL.

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¹ The so-called "ellipticity" problem that arises when $\nabla^2 \phi < -f^2/2$ is probably a factor that only arises in initialization methods that rely too heavily on the geopotential field ϕ as the primary input data. The balance equation is also not a valid relation for low-latitude internal Rossby modes even though these have very small frequencies (Phillips, 1973, p. 55).