A Simple Test of the Initialization of Gravity Modes

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ABSTRACT

The initialization schemes of Machenhauer (1977), Baer and Tribbia (1977), and one requiring the initial second time derivatives of gravity modes to be zero are tested by application to a simple differential equation, which partially simulates the behavior of gravity modes in a forecast model. These initialization schemes are tested to see under what conditions they converge, and they are tested on how well they eliminate future gravity wave oscillations. Preliminary results of initialization experiments with the National Meteorological Center's spectral forecast model are in support of the conclusions derived from analyzing this differential equation.

1. Introduction

It has been known for some time that the initial data for a primitive equation forecast model must be in a well-balanced state, otherwise the resulting forecast will be contaminated with excessive amounts of high-frequency gravity mode oscillations. Currently, a number of modeling centers are using some form of normal mode initialization with operational forecasts models. The first step of normal mode initialization is to calculate the eigenvalues and eigenvectors of appropriately linearized model equations. Such eigenvectors are the model's normal modes, and the eigenvalues give the modal frequencies. The next step in normal mode initialization is to adjust the initial values of the high-frequency gravity modes so that there will be a balanced initial state of the model, so that little high-frequency gravity wave oscillation occurs in the forecast. One of the first attempts at normal mode initialization was by Williamson (1976), who set to zero the amplitudes of all gravity modes of a shallow water model's initial data. Such initialization was not successful as nonlinear forcing quickly generated high-frequency gravity wave activity. Machenhauer (1977) developed a more successful initialization that requires the initial values of gravity modes be adjusted so that their tendencies are zero. This initialization has been applied to various forecast models including application to the Canadian spectral model by Daley (1979) and application to a grid model, Temperton and Williamson (1979). Baer and Tribbia (1977, hereafter referred to as BT) developed an *initialization* based on a two time-scale analysis. This initialization has shown excellent results when applied to simple forecast models by Tribbia (1979) and Ballish (1979). Machenhauer's procedure has gravity modes balanced against the initial nonlinear forcing. Since this nonlinearity changes slightly with time, Machenhauer's procedure will not completely prevent high-frequency gravity wave activity. The BT initialization takes into account the time behavior of such nonlinear forcing and therefore is more likely to prevent the development of high-frequency oscillations.

Although a number of modeling centers are using some form of normal mode initialization, there are a variety of questions concerning both the details of applying such initialization and whether the use of initialization does a complete job of removing unwanted gravity wave noise without altering balanced low-frequency meteorological systems. The gravity modes of a complex model have wide variation in both vertical and horizontal length scales, as well as in frequency. Of this wide variety of gravity modes, one would like to know which modes should be initialized, how well will the use of initialization eliminate future gravity mode oscillation, and when will application of an initialization procedure diverge or cause large changes to the initial conditions.

Application of initialization to a forecast model is an important test; however, it is a complicated problem that is not easily analyzed. A realistic forecast model has a variety of physical processes that can not fully be incorporated into the initialization. For example, convective adjustments and latent heating generate gravity wave activity but could only be partially balanced by initialization. The initialization problem for forecast models is further

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complicated, in that such models allow for motions with a time behavior somewhere between fast and slow. Such motions may nonlinearly affect the high-frequency gravity modes in a manner that initialization can not handle. In addition, initialization theory is based, in part, on having a small Rossby number; unfortunately, in many areas of a model, the Rossby number will exceed unity.

Instead of applying initialization to a complicated model, one can apply initialization to simple differential equations that describe some aspects of the behavior of gravity modes in a model. Such simple equations can be used not only to test the mathematics of an initialization procedure but also to give some insight into the initialization of complicated models.

2. Initialization of a simple gravity mode equation

The simple equation to be examined here is

$$\dot{y} = i\omega y + Ne^{i\nu t} - \bar{u}iky, \quad (1)$$

where y represents a gravity mode of frequency ω, N is a constant, ν is a low-frequency characteristic of synoptic weather systems, $\bar{u}$ is constant advecting wind, and $k = 2\pi/\lambda$ is the mode's wavenumber. The term $Ne^{i\nu t}$ partially simulates Rossby modes interacting nonlinearly to affect y, although in reality the forcing would involve a multitude of frequencies. The term $-\bar{u}iky$ is a simpler modeling of the effects of advection on the mode than what occurs in a forecast model. Finding the proper balanced initial state for (1) is nontrivial, as the forcing of the mode is nonconstant. Furthermore, adjustment of y to balance against the forcing alters $\dot{y}$ through the advective term. Even though (1) is relatively simple, it has two similarities with gravity modes in a model; namely, it has time-dependent nonlinear forcing and nonlinearity which depends on the gravity modes. Although (1) is linear, the initialization and time integration schemes will presume $Ne^{i\nu t}$ and $-\bar{u}iky$ to be nonlinear terms.

Eq. (1) has a general solution

$$y(t) = y_H e^{i(\omega-\bar{u}k)t} + (i\nu - i\omega + \bar{u}ik)^{-1}Ne^{i\nu t}, \quad (2)$$

where $y_H e^{i(\omega-\bar{u}k)t}$ is the homogeneous solution and $(i\nu - i\omega + \bar{u}ik)^{-1}Ne^{i\nu t}$ is the particular solution. The particular solution has low-frequency ν, while the homogeneous solution has high-frequency $\omega - \bar{u}k$. Since high-frequency gravity modes have frequencies an order of magnitude greater than the frequencies associated with synoptic weather systems, ω is assumed to be an order of magnitude greater than either ν or $\bar{u}k$.

With the properties of (1) introduced, the initialization problem can be examined. The value of y at $t = 0$ that leads to no future high-frequency oscillation is

$$y(t = 0) = (i\nu - i\omega + \bar{u}ik)^{-1}N. \quad (3)$$

With this initial value of y, $y_H = 0$.

Machenhauer's initialization requires that y be adjusted until $\dot{y} = 0$ at $t = 0$. This is done iteratively, with $i\omega y$ being adjusted to balance the nonlinear forcing, which changes somewhat with each iteration. For simplicity, we have assumed $y = 0$ before the start of the initialization procedure; this does not affect the final solution which is unique. Let $y = y_1 + y_2 + y_3 + \cdots$, with y_l being the change in y due to the lth iteration. On first iteration, $y_1 = -N/i\omega$ attempts to force $\dot{y}$ to zero, but the advective term $-\bar{u}ik$ prevents $\dot{y}$ from becoming zero. It is easy to show that $y_l = (uk/\omega)^{l-1}y_1$, so that

$$y_1 + y_2 \cdots y_l = \frac{1 - (\bar{u}k/\omega)^l}{1 - \bar{u}k/\omega} y_1. \quad (4)$$

Since uk/ω is assumed to be less than 1, the final solution converges to

$$y\ (t = 0) = -(i\omega - \bar{u}ik)^{-1}N. \quad (5)$$

This initial value of y results in

$$y(t) = y_H(t) + y_p(t) = \frac{\nu Ne^{i(\omega-\bar{u}k)t}}{(i\omega - i\nu - \bar{u}ik)(\omega - \bar{u}k)} + \frac{Ne^{i\nu t}}{i\nu - i\omega + \bar{u}ik}. \quad (6)$$

The latter portion of (6), $y_p(t)$, is the particular solution and also the desired low-frequency solution. The high-frequency homogeneous portion of (6), y_H, has $|y_H| = |\nu/(\omega - \bar{u}k)| \cdot |Y_P|$, which should be an order of magnitude smaller than the correct solution. Although Machenhauer's initialization does not take into account the time behavior of the nonlinear forcing, (6) shows that y_H remains small for all time.

To apply the BT initialization, (1) is written in scaled nondimensional form

$$\dot{y} = i\omega y + \epsilon Ne^{i\nu\tau} - \epsilon\bar{u}iky, \quad (7)$$

where all variables are now assumed to be nondimensional with magnitudes of unit order, except that ϵ is a small number of order 0.1. Time derivatives are expanded as $\partial/\partial t = \partial/\partial t^* + \epsilon\partial/\partial\tau$, where t^* is a fast time and τ slow. We expand y in a power series in ϵ, $y = y^{(0)} + \epsilon y^{(1)} + \epsilon y^{(2)} + \cdots$. The BT procedure adjusts y at $t = 0$ so that there is no fast t^* behavior. This method yields

$$y^{(0)} = 0 \quad \text{and} \quad y^{(1)}(\tau) = -Ne^{i\nu\tau}/i\omega. \quad (8)$$

To lth order, the BT procedure has

$$y^{(l)} = \frac{1}{i\omega}\left(\frac{\partial y}{\partial\tau}\right)^{(l-1)} - \frac{1}{i\omega} NL_y^{(l-1)}. \quad (9)$$

Here we have $NL_y^{(I)} = -\bar{u}iky^{(I-1)}$. We find

$$y^{(2)}(\tau) = \left(\frac{\bar{u}k}{\omega} + \frac{\nu}{\omega}\right)y^{(1)}(\tau), \quad (10)$$

and the general result is

$$y^{(J)} = \left(\frac{\bar{u}k}{\omega} + \frac{\nu}{\omega}\right)y^{(J-1)}.$$

This gives

$$y = \epsilon y^{(1)}[1 - \epsilon(\bar{u}k/\omega + \nu/\omega)]^{-1}$$

which in dimensional form becomes

$$y(t) = -Ne^{i\nu t}(i\omega - \bar{u}ik - i\nu)^{-1}. \quad (11)$$

Thus the BT initialization gives the value of y at $t = 0$ so that no high-frequency oscillations develop in time.

Another initialization procedure of interest that can be applied to (1) requires $\ddot{y} = 0$ at $t = 0$. One can show formally that $\ddot{y} = 0$ gives a result that agrees to first and second order with the *BT* initialization but has error to third and higher order. Although $\ddot{y} = 0$ is correct to second order, we will see that $\ddot{y} = 0$ is more difficult to achieve than $\dot{y} = 0$. Using (1) we find

$$\ddot{y} = (-\omega^2 + 2\bar{u}k\omega - \bar{u}^2k^2)y + (i\omega + i\nu - \bar{u}ik)Ne^{i\nu t}. \quad (12)$$

Requiring $\ddot{y} = 0$ results in

$$y = \frac{-N}{i\omega}\left(\frac{1 + \nu/\omega - \bar{u}k/\omega}{1 - 2\bar{u}k/\omega + u^2k^2/\omega^2}\right). \quad (13)$$

If ν/ω and $\bar{u}k/\omega \sim 0(\epsilon)$, then to second order

$$y \approx (-i\omega + i\nu + \bar{u}ik)^{-1}N, \quad (14)$$

which is the correct solution. Let us solve for y by an iterative procedure, as that is what would be done with a model, where we can not calculate y analytically. We first take $y = 0$ and then adjust y iteratively to make $\ddot{y} = 0$. At first, with $y = 0$, $\ddot{y} = (i\omega + i\nu - \bar{u}ik)Ne^{i\nu t}$, so we set $\Delta y = \omega^{-2}\ddot{y}$

$$\Delta y = y_1 = \omega^{-2}(i\omega + i\nu - \bar{u}ik)N, \quad (15)$$

where $y = y_1 + y_2 + \cdots$, and y_I is Δy from the Ith iteration. Due to terms $(2\bar{u}k\omega - \bar{u}^2k^2)y$, y_1 does not result in $\ddot{y} = 0$, so we set

$$y_2 = \omega^{-2}(2\bar{u}k\omega - \bar{u}^2k^2)y_1. \quad (16)$$

The general result is

$$y_I = \omega^{-2}(2\bar{u}k\omega - \bar{u}^2k^2)y_{I-1}.$$

If $|2\bar{u}k/\omega - \bar{u}^2k^2/\omega^2| < 1$ this procedure converges to

$$y = (1 - (2\bar{u}k/\omega) + \bar{u}^2k^2/\omega^2)^{-1}y_1, \quad (17)$$

which agrees with (13).

A special case of (1) is when $\nu = -\bar{u}k$. This corresponds to a nonlinear forcing of wavenumber k, advected with mean wind $\bar{u}$ but otherwise not changing. Such a disturbance would require some ageostrophic flow for balance, and this ageostrophic flow should be advected along with the disturbance. Machenhauer's scheme again would require y $(t = 0) = -N/[i\omega(1 - \bar{u}k/\omega)]$. The BT method results in $y^{(1)} = -Ne^{i\nu t}/i\omega$, but $y^{(2)} = y^{(3)} \cdots = 0$. That is, $y^{(1)}$ produces a balanced state which is allowed to advect with $\bar{u}$.

Contrary to the BT result, the Machenhauer method keeps iteratively trying to achieve $\dot{y} = 0$ or is effectively trying to prevent the gravitational flow from advecting. This does not result in a stationary solution for y. Instead, (6) shows $y(t) = y_H(t) + y_p(t)$, where y_p is the correct solution, which moves with speed $\bar{u}$, and y_H is relatively small and oscillates with high frequency.

We can now consider the convergence properties of the three different initializations that have been examined. The standard iterative procedure for achieving Machenhauer's constraint diverges if $|\bar{u}k/\omega| \geq 1$, which is equivalent to $|\bar{u}/C| \geq 1$, where $C = \omega/k$ is the phase speed of the gravity mode. This is not a serious problem as high-frequency gravity modes have ω large enough to result in convergence. The iterative initialization procedure to produce $\ddot{y} = 0$ diverges if $|2\bar{u}k/\omega - \bar{u}^2k^2/\omega^2| \geq 1$. This represents a more serious problem as this procedure diverges with a Rossby number, $\bar{u}k/\omega$, as small as 0.41. Specifically, divergence occurs first at $\bar{u}k/\omega = 1 - \sqrt{2}$. The BT initialization procedure does not diverge unless $|\bar{u}k/\omega + \nu/\omega| \geq 1$. It appears less likely for $|\bar{u}k/\omega + \nu/\omega|$ to exceed 1 than for $|\bar{u}k/\omega|$.

Consider the classical case of a Rossby wave imbedded in a mean zonal wind $\bar{u}$. The speed of such a wave is taken to be $C = \bar{u} - \beta/k^2$. Assuming a wave of the form $e^{i(kx+\nu t)}$, $\nu < 0$ indicates motion towards the east. If ν, the frequency of this wave, is the frequency of the nonlinear forcing $Ne^{i\nu t}$ in (1), then we have $\nu = -\bar{u}k + \beta/k$ and $|\bar{u}k/\omega + \nu/\omega| = |\beta/k\omega|$, which is not likely to exceed 1. This is an indication that the BT method is less likely to diverge Machenhauer's. That is, in areas where $\bar{u}$ is large and positive, weather systems tend to move eastward at a speed less than $\bar{u}$. Therefore, ν tends to be opposite in sign of $\bar{u}k$, and $|\bar{u}k/\omega + \nu/\omega|$ is likely to be smaller than $|\bar{u}k/\omega|$. On the other hand, where $\bar{u}$ is large and blowing over mountains, there may be a nonlinear forcing of gravity modes with $\nu \approx 0$. Here it is possible that neither the BT or Machenhauer procedure will converge.

3. Summary

For the simple equation examined here, it is shown that Machenhauer's initialization results in an approximate balance condition, with high frequency oscillations limited to second order amplitude for all time. This initialization converges for the single gravity mode equation if the Rossby number $\bar{u}k/\omega$ is less than 1 and diverges otherwise. This indicates that initialization of internal gravity modes with phase speed C may diverge in areas where $\bar{u} > C$. Although Machenhauer's initialization errors in that it results in gravitational flow that initially does not move with weather systems, the error is small unless the Rossby number approaches 1.

The BT balance for the equation tested here is exact, with no subsequent high frequency motion. This initialization is less likely to diverge than Machenhauer's and can converge with a Rossby number greater than 1, provided that the gravitational flow is not required to move too slowly compared with the advecting winds.

An initialization requiring the second time derivative of gravity modes to be zero is accurate to second order, but higher order terms can cause appreciable error unless the Rossby number is small. For the single gravity mode equation, this initialization diverges with a Rossby number as small as 0.41.

The conclusions derived from this simple equation are supported by the preliminary results of experiments with the National Meteorological Center's spectral forecast model. These experiments are not yet complete and will be discussed in future work. However, indications are that the BT procedure is somewhat better than Machenhauer's, but the difference is not substantial. An initialization requiring $\ddot{y} = 0$ has serious convergence problems compared with Machenhauer's initialization, which requires $\dot{y} = 0$.

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