Diagnostic Equations in Isobaric Coordinates

G. J. BOER

Canadian Climate Centre, Downsview, Ontario M3H 5T4

(Manuscript received 10 October 1981, in final form 3 September 1982)

ABSTRACT

Diagnostic studies of the general circulation are conducted, almost exclusively, using data and equations where pressure is the vertical coordinate. The complexity of the lower boundary condition in these isobaric coordinates may lead to some ambiguity in the manner of calculating and displaying the pertinent diagnostic statistics and budget terms.

A formalism is developed using the unit and delta functions which permits an unambiguous calculation of the appropriate diagnostic terms. Using this formalism, budget equations are derived which display a number of unfamiliar boundary terms. The more familiar budget equations are seen to be approximations to these more complete equations.

Examples of some basic general circulation statistics involving temperature are used to illustrate the application of the formalism.

1. Introduction

The primitive equations of motion form the basis of most efforts to model and diagnose the large scale behavior of the atmosphere. The use of pressure as the vertical coordinate leads to the isobaric (or pressure coordinate) form of these equations which is, in some ways, simpler than that of the corresponding equations with geometric height as the vertical coordinate. The price paid for this simplification is a more complex lower boundary condition even in the case of a geometrically flat surface. For realistic topography, the lower boundary is not a coordinate surface in either case. The height of the lower boundary is a function of position in the geometric height coordinate system, \( z = z(\lambda, \varphi) \), while in the pressure coordinate system, it is also a function of time, \( p = p_{s}(\lambda, \varphi, t) \).

The complications of the lower boundary condition have led to the widespread use, in numerical models of atmospheric flow, of the “sigma” vertical coordinate \( \sigma = p/p_{s} \), for which both upper and lower boundaries are coordinate surfaces. The resulting equations of motion are more complex than the pressure-coordinate equations, however, and the coordinate surfaces in this system “follow the topography,” i.e., they are not horizontal.

Diagnostic studies of the large-scale behavior of the atmosphere have been based almost exclusively on the use of pressure as the vertical coordinate since the data are routinely collected at standard pressure levels and the equations have a relatively simple form in this coordinate system. Most objective analysis schemes operate with pressure as the vertical coordinate or at least produce analyzed data on constant pressure surfaces. Such objectively analyzed data is being used increasingly for diagnostic studies of atmospheric behavior. General circulation statistics and associated budgets are most often obtained and presented using pressure as the vertical coordinate (Newell et al., 1972, 1974; Oort and Rasmussen, 1971; and many others).

In some ways, the apparent simplicity of the equations in pressure coordinates is illusory. This is particularly true if averages and integrals of the equations and data are required. The statistical and budget calculations carried out in general circulation diagnostic studies are made difficult by the complications of the temporally- and spatially-varying lower boundary condition. For a given pressure, atmospheric variables are defined in a multiply connected domain. Less pedantically, the situation may be characterized by saying that atmospheric variables are undefined where topography pierces the pressure surface. Integration and averaging of various terms is difficult if one of the limits of integration is a function of time and space, since averaging and integration cannot be interchanged.

In what follows, a formalism is presented which allows the equations of motion in pressure coordinates to be written in a form which, at the expense of reintroducing some complexity, automatically includes the consequences of the lower boundary condition under averaging, integration and the decomposition of terms into components. The diagnostic equations of general circulation studies can be handled without approximation (other than that associated with the discrete nature of the data). Various
unfamiliar boundary terms arise in this derivation of the equations. Various approximate versions of the diagnostic equations may in turn be obtained in a straightforward consistent way using the formalism. Examples are given of simple calculations in which the results depend on taking account of the lower boundary condition in the manner developed here.

Holopainen (1970) developed some of these ideas in an Appendix to his study of the energy balance of stationary disturbances. The formalism he used is quite distinct from that given here.

2. The modified equations in isobaric coordinates

a. The primitive equations in isobaric coordinates

The primitive equations of atmospheric motion in isobaric coordinates in the notation used here are

\[
\frac{dV}{dt} + f k \times V + \nabla \phi = F, \quad (1a)
\]

\[
C_p \frac{dT}{dt} = \omega \alpha + Q, \quad (1b)
\]

\[
\frac{\partial \omega}{\partial p} + \nabla \cdot V = 0, \quad (1c)
\]

\[
\frac{\partial \phi}{\partial p} = -\alpha, \quad (1d)
\]

\[
dX/dt = S, \quad (1e)
\]

where \(X = q\) is the specific humidity or other constituent and \(S\) is a source-sink term.

The domain of definition is \(0 \leq \lambda \leq 2\pi, -\pi/2 \leq \varphi \leq \pi/2, 0 \leq p \leq p_0(\lambda, \varphi, t)\). The usual kinematic boundary conditions are assumed with

\[
\omega = 0, \quad p = 0, \quad (2a)
\]

\[
\omega_s = \frac{\partial p_s}{\partial t} + V_s \cdot \nabla p_s, \quad p = p_s, \quad (2b)
\]

where the subscript \(s\) implies that the terms are evaluated at \(p = p_s\).

The element of mass in this coordinate system is

\[
dm = g^{-1} a^2 \cos \varphi d\lambda d\varphi dp
\]

and the integral over the mass of the atmosphere of a variable is

\[
\int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \int_0^{p_0} X g^{-1} a^2 \cos \varphi d\lambda d\varphi dp,
\]

where one of the limits of integration, \(p_0(\lambda, \varphi, t)\), is a function of the other independent variables.

The fact that the position of the lower boundary is a function of the independent variables leads to considerable difficulty when averaging the equations in time or space. Obviously averaging and integration cannot be interchanged and the evaluation of the time average of an expression such as (3) cannot be done by first averaging \(X\) in time and space and subsequently integrating in the vertical. The dependence of the limit of integration on the independent variables is removed by using the features of the unit function.

b. The unit and impulse functions

Danese (1965) and many others discuss the properties of the unit or Heaviside function

\[
\beta(\xi) = \begin{cases} 1, & \xi > 0 \\
0, & \xi < 0 \end{cases}
\]

and the impulse or Dirac delta function \(\delta(\xi)\). The mathematical details of the definition and properties of these functions are not considered here. Certain operational properties of these functions are found to be useful. They are:

\[
a) \frac{d\beta}{d\xi} = \delta(\xi) \\
b) \delta(\xi) = 0, \quad \xi \neq 0 \\
c) \xi \delta(\xi) = 0 \\
d) \delta(-\xi) = \delta(\xi) \\
e) \int_{-\infty}^{\infty} f(z - \xi) \delta(\xi) d\xi = f(z) \\
f) f(z - \xi) \delta(\xi) = f(z) \delta(\xi)
\]

where \(f\) is a continuous function. These relationships together with the usual rules of calculus will be sufficient for the approach adopted here.

While \(\beta\) is a piecewise continuous function, \(\delta\) is a "singular" function or, as it is sometimes said, not a proper function at all. The properties of this "function" have been used in physical problems for some time, of course, and have an appropriate mathematical basis. Generally the delta function occurs only in the intermediate stages of a problem. If it appears in a result it is usually in the form of an integral in which \(\delta\) is multiplied by a well behaved function so that what is required to be known of the \(\delta\) function is that

\[
\int f(\xi) \delta(\xi) d\xi = f(0)
\]

when the integration includes the essential point at \(\xi = 0\).

In application to the atmospheric equations in pressure coordinates, \(\xi\) is taken to be \(\xi = p_s - p\) so that

\[
\beta(p_s - p) = \beta(\lambda, \varphi, p, t) = \begin{cases} 1, & p < p_s \\
0, & p > p_s \end{cases}
\]

Using the chain rule,

\[
\frac{\partial \beta}{\partial t} = \frac{d\beta}{d\xi} \frac{d\xi}{dt} = \delta \frac{dp_s}{dt}
\]

\[
\nabla \beta = \delta \nabla p_s
\]

\[
\frac{\partial \beta}{\partial p} = -\delta
\]
and
\[
dx = \frac{\partial p_s}{\partial t} dt + \frac{\partial p_s}{\partial \lambda} d\lambda + \frac{\partial p_s}{\partial \phi} d\phi - dp,
\]
which is useful when evaluating integrals involving \( \delta(\xi) \) in a particular coordinate direction.

The boundary condition (2b) leads to
\[
\frac{d\beta}{dt} = \delta \left( \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{p}_s - \omega \right) = 0, \tag{7}
\]
so that the statement of the boundary condition (2b) is equivalent to the statement \( d\beta/dt = 0 \). Obviously a different boundary condition would lead to a different result, and a different equation for \( d\beta/dt \). The kinematic boundary condition (2b) is built into subsequent calculations by the use of (7).

c. The equations in an extended domain of definition

The equations of motion in isobaric coordinates are defined for \( 0 \leq p \leq p_s \). It is useful to extend the domain of definition to the region \( 0 \leq p \leq p_0 \) where \( p_0 \) is some fixed relatively large value of pressure for which \( p_0 > p_s \) always.

Variables are defined in the extended domain as
\[
\hat{X}_e = \begin{cases} X, & p \leq p_s \\ \hat{X}, & p > p_s, \end{cases}
\]
where \( \hat{X} \) is some interpolated/extrapolated value when \( p > p_s \). Other definitions are possible but this is the approach often used in practice to produce simply connected fields of variables on pressure surfaces.

The equations of motion (1) apply to the extended domain of definition under these circumstances if they are multiplied by \( \beta \). Obviously the various terms in the equations have their usual values for \( p \leq p_s \), while there is no contribution from the “subterranean” region \( p > p_s \) because of the multiplication by \( \beta \) which is zero for \( p > p_s \).

The equations may be rewritten so that most of the variables are in the form \( \beta X_e \) which is convenient for subsequent averaging. Dropping the subscripts on the variables defined over the extended domain of definition and using some of the relationships (4–7) allows the equations to be written as
\[
\frac{d\beta X}{dt} + fk \times \beta \mathbf{V} + \beta \nabla \phi = \beta \mathbf{F}, \tag{8a}
\]
\[
C_p \frac{d\beta T}{dt} = \beta \omega + \beta \mathbf{Q}, \tag{8b}
\]
\[
\frac{\partial \beta}{\partial t} + \nabla \cdot \beta \mathbf{V} + \frac{\partial \beta}{\partial p} \beta \omega = 0, \tag{8c}
\]
\[
\frac{d}{dp} \beta(\phi - \phi_s) = -\beta \alpha, \tag{8d}
\]
\[
\frac{d\beta X}{dt} = \beta S. \tag{8e}
\]

Moreover,
\[
\frac{d\beta X}{dt} = \frac{\partial \beta X}{\partial t} + \nabla \cdot \beta \mathbf{V} + \frac{\partial \beta X}{\partial p} \beta \omega,
\]
so that the “flux” and advective forms of the equations have an appropriate representation.

The boundary conditions (2) are now simply
\[
\begin{cases}
\omega = 0, & p = 0 \\
\beta = 0, & p = p_0
\end{cases}
\tag{9}
\]
As was noted above, the kinematic lower boundary condition (2b) is now built into the modified equations (8–9).

d. Useful features of this representation

Eqs. (8) together with the boundary conditions (9) form the basis for the diagnostic budget equations in pressure coordinates. This form of the equations has a number of useful features. The \( \beta \) term carries information on the whereabouts (i.e., the pressure) of the surface.

One of the useful features of this formulation is that averaging and integration are interchangeable because the limit of integration is no longer a function of the other independent variables. That is,
\[
\overline{\int_0^{p_0} X dp} = \int_0^{p_0} \overline{\beta X dp} = \int_0^{p_0} \overline{\beta X dp}, \tag{10}
\]
where the overbar implies a time average, for instance.

Another useful feature of the formulation is that boundary terms arise automatically and the use of Leibnitz’s rule is avoided. For instance, the surface pressure equation may be obtained in the usual fashion from the continuity equation (1c) by integrating and applying Leibnitz’s rule and the boundary condition (2b). The reformulated continuity equation (8c) may be written as
\[
\frac{dp_s}{dt} + \nabla \cdot \beta \mathbf{V} + \frac{\partial}{\partial p} \beta \omega = 0,
\]
which, upon integration, gives the surface pressure equation directly as
\[
\frac{dp_s}{dt} + \nabla \cdot \int_0^{p_0} \beta \mathbf{V} dp = 0,
\]
or using (10), as
\[
\frac{dp_s}{dt} + \nabla \cdot \int_0^{p_0} \mathbf{V} dp = 0.
\]

Another example, also using the continuity equation, involves the diagnostic expression for \( \omega \) in terms
of the divergence, and the rate of change of surface pressure. Integrating the usual form of the continuity equation (1c) from \( p = 0 \) to some value \( p \) is not very straightforward where topography pierces the pressure surface. Integrating Eq. (8c), however, gives

\[
\beta \omega(p) = -\nabla \cdot \int_0^p \beta \nabla \, dp - \int_0^p \delta \frac{\partial p_s}{\partial t} \, dp.
\]

The second term on the right-hand side is rewritten using (4, 6) as

\[
-\int_0^p \delta \frac{\partial p_s}{\partial t} \, dp = \int_0^p \frac{\partial \beta}{\partial \phi} \frac{\partial p_s}{\partial \phi} \, dp
\]

\[
= \int_0^p \frac{\partial}{\partial \phi} \left( \beta \frac{\partial p_s}{\partial t} \right) \, dp = (\beta - 1) \frac{\partial p_s}{\partial t}
\]

so that

\[
\beta \omega = -\nabla \cdot \int_0^p \beta \nabla \, dp - (1 - \beta) \frac{\partial p_s}{\partial t}.
\]

This equation explicitly accounts for changes of surface pressure over that part of the topography which pierces the pressure surface. Averaging over a horizontal surface

\[
\langle X \rangle = (4\pi)^{-1} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} X \cos \phi \, d\phi \, d\lambda
\]

gives

\[
\langle \beta \omega \rangle = -\left( (1 - \beta) \frac{\partial p_s}{\partial t} \right).
\] (11)

On a pressure surface which is everywhere above the topography, \( \beta = 1 \) and \( \langle \omega \rangle = 0 \) as expected. On a pressure surface pierced by topography, however, the average \( \omega \) need not be zero if the surface pressure over that part of the topography is changing.

These examples are meant to show the usefulness of this representation of the equations of motion in pressure coordinates where boundary terms are obtained easily and automatically and where vertical integration and averaging are interchangeable.

e. Application to other coordinate systems

It should be apparent that this formalism is directly applicable to the equations written in terms of other vertical coordinates such as geometric height or potential temperature. For geometric height as the vertical coordinate, equations (5, 6) become

\[
\beta = \beta(z - z_s) = \begin{cases} 1, & z > z_s \\ 0, & z < z_s \end{cases}
\]

and

\[
\frac{\partial \beta}{\partial t} = 0,
\]

\[
\nabla \beta = -\delta \nabla z_s,
\]

\[
\frac{\partial \beta}{\partial z} = \delta.
\]

Similar equations apply in potential temperature coordinates. The transformation of the equations of motion from one coordinate system to another in this approach will include the appropriate representation of \( \beta \).

3. Budget equations

General circulation diagnostic studies are largely concerned with the terms which arise in various budget equations under averaging. In particular, the budgets of mass, angular momentum, moisture, total energy, and kinetic and available potential energy are studied under a variety of averaged and integrated conditions.

a. Mass

Conservation of mass is represented by the continuity equation (8c) and various integrated and averaged forms of this equation.

b. Angular momentum

The equation for angular momentum

\[
M = a \cos \phi (\Omega a \cos \phi + u)
\]

is written as

\[
\frac{\partial}{\partial t} \beta M + \nabla \cdot \beta M \mathbf{V} + \frac{\delta}{\partial p} \beta M \omega + \frac{\partial}{\partial \phi} \beta \phi = a \cos \phi \beta F_\lambda.
\] (12)

The term \( \beta \phi \partial \phi / \partial \lambda \) represents the torque due to pressure gradients.

For vertically integrated budgets, familiar terms arise as

\[
\int_0^{\rho_0} \beta \frac{\partial \phi}{\partial \lambda} \, dp = \int_0^{\rho_0} \left( \frac{\partial}{\partial \phi} \beta \phi - \delta \frac{\partial p_s}{\partial \phi} \right) \, dp
\]

\[
= \frac{\partial}{\partial \lambda} \int_0^{\rho_0} \beta \phi \, dp - \phi_s \frac{\partial p_s}{\partial \lambda}.
\]

c. Moisture or other constituent equation

The constituent equation (8e) represents a quantity which is transported by the atmosphere from source to sink region. The moisture budget equation is typical and is written in the form

\[
\frac{\partial \beta q}{\partial t} + \nabla \cdot \beta q \mathbf{V} + \frac{\delta}{\partial p} \beta q \omega = \beta S,
\] (13)

where \( q \) is the specific humidity and \( S \) the source and sink term for the water substance.
d. Kinetic energy

The kinetic energy equation is obtained by taking the dot product of the horizontal momentum equations with \( \mathbf{V} \) to get

\[
\frac{d}{dt} \beta \mathbf{V} \cdot \mathbf{V} + \beta \mathbf{V} \cdot \nabla \phi = \beta \mathbf{V} \cdot \mathbf{F}. \tag{14a}
\]

For \( k = \mathbf{V} \cdot \mathbf{V}/2 \), this may also be written as

\[
\frac{\partial}{\partial t} \beta k + \nabla \cdot \beta \mathbf{k} \mathbf{V} + \frac{\partial}{\partial p} \beta k \omega + \beta \mathbf{V} \cdot \nabla \phi = \beta \mathbf{V} \cdot \mathbf{F}, \tag{14b}
\]

where \( \beta \mathbf{V} \cdot \nabla \phi \) represents a source–sink term for kinetic energy via conversions from other forms of energy.

e. Potential and internal energy

The kinetic energy equation (14) for the horizontal velocity is a direct transformation of the corresponding equation in \( z \) coordinates. The associated potential energy \( (\phi) \) equation is

\[
\frac{d}{dt} \beta \phi - \beta \mathbf{V} \cdot \nabla \phi = g \beta w = 0, \tag{15}
\]

where \( w \) is the vertical velocity in geometric coordinates. The internal energy \( (C_v T) \) equation is

\[
\frac{d}{dt} \beta C_v T + \frac{\partial}{\partial p} \beta p \alpha T - \beta \omega \alpha = \beta Q. \tag{16}
\]

Using (8c, d), the relationship

\[
\beta \mathbf{V} \cdot \nabla \phi = \nabla \cdot \beta \phi \mathbf{V} + \frac{\partial}{\partial p} \beta \phi \omega + \beta \omega \alpha + \phi \frac{\partial \beta}{\partial t}, \tag{17}
\]

together with (15), allows (16) to be written as

\[
\frac{d}{dt} \beta C_v T + \nabla \cdot \beta p \alpha \mathbf{V} + \frac{\partial}{\partial p} \left[ \beta p \alpha \omega - \frac{\partial}{\partial t} \beta p (\phi - \phi_3) \right] - \beta \mathbf{V} \cdot \nabla \phi + \beta g w = \beta Q. \tag{18}
\]

Eqs. (14, 15, 18) constitute a set of energy equations the terms of which are the transforms of the associated \( z \) coordinate equations. The terms \( -\beta \mathbf{V} \cdot \nabla \phi \) and \( \beta g w \) represent conversions between the internal energy and the kinetic and potential energy respectively.

It would be possible to consider these equations as they stand but this is not done for several reasons. Firstly the transformation term between potential and internal energy is written using the geometric vertical velocity which is not natural to the system. Secondly, the internal and potential energies of a vertical column of a hydrostatic atmosphere are related so that internal and potential energy are usefully considered together.

The internal plus potential energy equation is

\[
\frac{d}{dt} \beta (C_v T + \phi) + \nabla \cdot \beta p \alpha \mathbf{V} + \frac{\partial}{\partial p} \left[ \beta p \alpha \omega \right. - \frac{\partial}{\partial t} \left. \beta p (\phi - \phi_3) \right] - \beta \mathbf{V} \cdot \nabla \phi = \beta Q. \tag{19}
\]

Using \( \alpha = RT, C_p = C_v + R \) and the hydrostatic equation in the form

\[
\beta \phi = \beta \phi_3 + \frac{\partial}{\partial p} \beta p (\phi - \phi_3) + \beta RT,
\]

Eq. (19) becomes

\[
\frac{\partial}{\partial t} \left[ \beta C_p T + \frac{\partial}{\partial p} \beta p (\phi - \phi_3) + \beta \phi_3 \right] + \nabla \cdot \beta (C_p T + \phi) \mathbf{V} + \frac{\partial}{\partial p} \left\{ \beta (C_p T + \phi) \omega \right. \frac{\partial}{\partial t} \left. \beta p (\phi - \phi_3) \right\} - \beta \mathbf{V} \cdot \nabla \phi = \beta Q.
\]

The "usual" approach is to use Eqs. (8b) and (17) to get

\[
\frac{\partial}{\partial t} \left( \beta C_p T + \beta \phi_3 \right) + \nabla \cdot \beta (C_p T + \phi) \mathbf{V} + \frac{\partial}{\partial p} \left( \beta (C_p T + \phi) \omega \right. \frac{\partial}{\partial t} \left. \beta p (\phi - \phi_3) \right) - \beta \mathbf{V} \cdot \nabla \phi = \beta Q, \tag{20}
\]

which is the equation for enthalpy, \( C_p T \). On integration in the vertical, the internal plus potential energy or "total potential energy" of a column is

\[
g^{-1} \int_0^\infty \beta (C_v T + \phi) dp = g^{-1} \int_0^\infty \left\{ \beta (C_v T + RT) + \frac{\partial}{\partial p} \beta p \phi + \beta p \phi_3 \right\} dp = g^{-1} \int_0^\infty \beta C_v T dp + p \phi_3 dg, \tag{21}
\]

so that on integration, Eq. (20) becomes the total potential energy equation.

f. Total energy

The total energy equation is obtained by combining Eqs. (14), (15) and (18) to get

\[
\frac{d}{dt} \beta (\frac{1}{2} \mathbf{V} \cdot \mathbf{V} + C_v T + \phi) + \nabla \cdot \beta p \alpha \mathbf{V} + \frac{\partial}{\partial p} \left[ \beta p \alpha \omega - \frac{\partial}{\partial t} \beta p (\phi - \phi_3) \right] = \beta (\mathbf{V} \cdot \mathbf{F} + \beta Q).
\]

For \( E = (\frac{1}{2} \mathbf{V} \cdot \mathbf{V} + C_v T + \phi) \), this can also be written
as
\[
\frac{\partial}{\partial t} \beta E + \nabla \cdot (E + p\alpha) \mathbf{V} + \frac{\partial}{\partial p} \left\{ \beta (E + p\alpha) \omega - \frac{\partial \beta}{\partial t} (\phi - \phi_0) \right\} = \beta (\mathbf{V} \cdot \mathbf{F} + Q).
\]

For general circulation diagnostic purposes, however, the equation is usually written in the form
\[
\frac{\partial}{\partial t} (\beta C_p T + \beta \phi_0) + \nabla \cdot (\beta C_p T + \phi) \mathbf{V} + \frac{\partial}{\partial p} \beta (C_p T + \phi) = \beta (\mathbf{V} \cdot \mathbf{F} + Q),
\]
which is obtained by neglecting kinetic energy in comparison with the internal and potential energies. The resulting approximate equation is used to investigate the total energy budget.

g. Available potential energy

The definition of available potential energy given by Lorenz (1967, p. 104) is formulated in isentropic coordinates and “neglects topography.” This formula, termed “exact” by Lorenz, must now be replaced by that of Taylor (1979). Taylor defines available potential energy, again in isentropic coordinates, in a manner which includes the presence of topography.

The formalism developed in previous sections is designed to facilitate the development of the diagnostic equations in pressure (and other) coordinates in the presence of topography. The exact available potential energy equations in pressure coordinates are a natural outgrowth of this approach. Together with the kinetic energy equations (14) they form the basis for energy cycle calculations.

The development of the energy cycle equations in isobaric coordinates, including the presence of topography, will be presented elsewhere. The results are an extension of those of Boer (1975). It is found that the decomposition of the energy equations into mean and eddy parts gives rise to various unfamiliar terms.

The form of the available potential energy equations most often used in practice are the Lorenz approximate equations in pressure coordinates (Lorenz, 1967, p. 108; Holopainen, 1970; Peixoto and Oort, 1974; and others). Of course, available potential energy is not properly defined if “subterranean” values of temperature may be assigned arbitrarily and enter the calculations. As an indication of some of the consequences of applying the formalism developed here to quantities depending on temperature, mean and eddy components of available potential energy defined as
\[
A = \int \frac{1}{2} C_p \gamma (T - \langle T \rangle)^2 \, dm
\]
\[
\gamma = \left( \frac{\theta}{T} \right) \left( \frac{-k}{p} \frac{\partial \langle \theta \rangle}{\partial p} \right)^{-1}
\]
will be evaluated in a later section.

4. Averaging operations

a. Definitions

In diagnostic studies, various averaged and integrated forms of the budget equations of Section 3 are considered. The notation used for various averaging processes is
\[
\bar{X} = \frac{1}{T} \int_{t-\tau/2}^{t+\tau/2} X \, dt
\]
\[
[X] = \frac{1}{2\pi} \int_0^{2\pi} X \, d\lambda
\]
\[
\langle X \rangle = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} X \cos \phi \, d\phi \, d\lambda
\]
The overbar may also be interpreted as a probability or ensemble average of a number of independent realizations of the random process although such averages are not often available in practice. Under certain ergodicity assumptions, the time average and probability average can converge.

The averages (24) are well defined when \( X \) is continuous in time and space. In this case, the associated deviations from these averages are defined as
\[
X' = X - \bar{X}
\]
\[
X^* = X - [X]
\]
\[
X^+ = X - \langle X \rangle
\]

These relationships permit the decomposition of averaged quadratic terms into “mean” and “eddy” components in the standard manner for general circulation diagnostic calculations. Typically
\[
\bar{XY} = \bar{X} \bar{Y} + \bar{X}^* \bar{Y}^*
\]
\[
[X Y] = [X] [Y] + [X^* Y^*] + [X Y^*]
\]
\[
\langle XY \rangle = \langle X \rangle \langle Y \rangle + \langle X^* \rangle \langle Y^* \rangle + \langle X^* Y^* \rangle + \langle X Y^* \rangle
\]
although other combinations of terms are possible.

b. Averaging in the presence of topography

The mean and covariance expressions (24, 26) are not clearly defined for an atmospheric variable on a pressure surface pierced by topography. It is necessary to adopt some suitable definition for averages and covariances which will properly take this into account.
The ideas of section (2) are used. Atmospheric variables on a pressure surface are replaced by the "extended" variables which are everywhere defined. The new variable $\beta$ serves to delineate the topography.

Means and covariances are well defined for the extended variables and are themselves used in some diagnostic studies. This is not desirable however since "subterrana" values of the variables are treated as if they are real rather than as convenient extensions of the fields for data handling and display purposes.

Consider the value of some meteorological variable $X$ at a particular point $(\lambda, \varphi, p)$ in pressure coordinate space. If this point is near the surface, it is quite possible that for certain time intervals $p(\lambda, \varphi)$ will be less than $p$, the pressure of the coordinate surface at that point. Then $X$ is undefined for a portion of the time interval. Interpolated/extrapolated subterranean values $\bar{X}$ are used to fill in these gaps so that a new continuous variable may be dealt with. This is the "extended" variable. The value of $\beta$ denotes those portions of the record for which $p < p$. An appropriate definition for the mean of $X$ or the covariance of $X$ with $Y$ at a point under time averaging should not contain fictitious data. The average value of $X$ when it exists (i.e., the average of the superterranean values) is an obvious statistic of interest. Such an average is given by $\bar{\bar{X}}$, $\beta \neq 0$. A "representative" average incorporating this statistic is defined as

$$\bar{\bar{X}} = \frac{\beta X}{\beta}, \quad \beta \neq 0$$

This statistic is defined over the entire domain including subterranean points ($\beta = 0$ means the point is always below the surface) where it is just the average of the "estimated" values. This means that $\bar{\bar{X}}$ is a continuous variable which is useful in data handling and display. Of course, the value of $\bar{\bar{X}}$ where $\beta = 0$ does not enter into the budget calculations.

Terms in the budget equations are typically of the form $\beta X$ and it follows that the appropriate time average is

$$\bar{\bar{\beta X}} = \bar{\beta X}$$

The decomposition of $X$ into mean and eddy components and the evaluation of variance and covariance terms follows naturally from this definition of the representative mean in the form:

$$X = \bar{\bar{X}} + X'$$

This definition of $X'$ is now different from that of (25). Of course, the two definitions are the same at points away from the surface where $\beta = 1$ and $\bar{\bar{X}} = \bar{X}$. Defined in this way, $X'$ is continuous in space and time.

It follows that

$$X' = X - \bar{\bar{X}}$$

and that

$$\bar{\bar{\beta X}} = \bar{\beta \bar{X}} - \bar{\beta X} = 0$$

This manner of decomposing variance/covariance terms into mean and eddy parts is appropriate for the form of the budget equations which were derived in Section 3.

Unless $\beta = 0$, the mean term $\bar{\bar{X}} \bar{\bar{Y}}$ is the product of representative values and contains no fictitious contribution from subterranean points. The percentage of time the point is above ground or alternatively the probability of finding the point above ground is given by $\bar{\bar{X}}$. The resulting term $\bar{\bar{\beta X}} \bar{\bar{Y}}$ can be interpreted as the proper mean values of $X$ and $Y$ multiplied by the chance of finding the point above ground.

Similar remarks apply to the eddy term $\beta X' Y' \bar{\bar{Y}} = \bar{\beta X} Y$. When $\beta \neq 0$, the deviations are from the proper representative mean rather than from a mean involving fictitious values. The term can also be written as a representative value multiplied by $\bar{\bar{X}}$. Again the representative values are everywhere defined in the region of interest.

Similar operations are defined for zonal averaging. The representative zonal average is

$$[X]_R = \begin{cases} [\beta X]/[\beta], & \beta \neq 0 \\ [X], & \beta = 0 \end{cases}$$

while the deviation from the zonal average is redefined as

$$X^* = X - [X]_R.$$ 

Again it follows that

$$[\beta X^*] = [\beta X] - [\beta][X]_R = 0$$

and that

$$\bar{\bar{\beta XY}} = [\beta][X]_R[Y]_R + [\beta X^* Y^*] = [\beta][X]_R[Y]_R + [X^* Y^*]_R.$$ 

Using this approach, a variety of other time-, zonally- and area-averaged statistics may be developed. Those in most common use in diagnostic studies are summarized in Table 1. It must be noted that these definitions, and others like them which may be developed for specific calculations, must be treated with care. The various representative averages, for example, are not simple averages of one another, thus the time average of $[X]_R$ is not the zonally and time averaged representative value $[X]_R$. 

Unauthenticated | Downloaded 10/21/23 03:45 PM UTC
Table 1. Summary of statistical terms.

<table>
<thead>
<tr>
<th>Kind of averaging</th>
<th>Representative mean</th>
<th>Decomposition of $X$</th>
<th>Deviation from mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>$X^R = \frac{\beta X}{\beta}$, $\beta \neq 0$</td>
<td>$X = X^R + X^*$</td>
<td>$X^* = X - X^R$</td>
</tr>
<tr>
<td>Zonal</td>
<td>$[X]_T = \left{ \beta X / \beta \right}$, $\beta \neq 0$</td>
<td>$X = [X]_T + X^*$</td>
<td>$X^* = X - [X]_T$</td>
</tr>
<tr>
<td>Global</td>
<td>$\langle X \rangle _T = \left{ \langle \beta X \rangle / \langle \beta \rangle \right}$, $\langle \beta \rangle \neq 0$</td>
<td>$X = \langle X \rangle _T + X^*$</td>
<td>$X^* = X - \langle X \rangle _T$</td>
</tr>
<tr>
<td>Time and zonal</td>
<td>$[\bar{X}]_T = \left{ \beta \bar{X} / \beta \right}$, $\beta \neq 0$</td>
<td>$X = [\bar{X}]_T + X^* + X^*$</td>
<td>$X^* = X - [\bar{X}]_T$</td>
</tr>
<tr>
<td>Time and global</td>
<td>$\langle \bar{X} \rangle _T = \left{ \langle \beta \bar{X} \rangle / \langle \beta \rangle \right}$, $\langle \beta \rangle \neq 0$</td>
<td>$X = \langle \bar{X} \rangle _T + X^*$</td>
<td>$X^* = X - \langle \bar{X} \rangle _T$</td>
</tr>
<tr>
<td>Time, zonal, global</td>
<td>$X = \langle \bar{X} \rangle _T + [\bar{X}]^* + X^* + X^*$</td>
<td>$[\bar{X}]^* = [\bar{X}]_T - \langle \bar{X} \rangle _T$</td>
<td></td>
</tr>
</tbody>
</table>

In general, the representative values are those to be displayed and which give "typical" or expected kinds of averaged values for the variables. The representative value multiplied by $\beta$ and averaged is the appropriate term for the budget calculation.

\textit{c. An example of zonally- and time-averaged statistics}

Temperature data from the FGGE III-a data set for January 1979, are used for this example. Grid point data on the ten pressure levels: 1000, 850, 700, 500, 400, 300, 250, 200, 150 and 100 mb are used.

Since at least the 1000, 850 and 750 mb levels are pierced by topography, the FGGE data as supplied have already been extrapolated/interpolated so as to fill in "holes" in the fields. The data are obtained therefore in terms of the extended variable $T$.

The value of $\beta$ is calculated on the pressure surfaces for which data are available by using the equivalent formula

\[
\beta(\lambda, \varphi, p, t) = \begin{cases} 1, & \phi(\lambda, \varphi, p, t) > \phi(\lambda, \varphi) \\ 0, & \text{otherwise.} \end{cases}
\]

The January 1979 monthly mean value of $\beta$ at 1000 mb is shown in Fig. 1. Apparently $\beta$ is generally small over land and large over the oceans. The consequences of the low pressure band around Antarctica and of the Aleutian low are clearly seen in low values

**Fig. 1.** January 1979 value of $\beta$ at 1000 mb obtained from FGGE III-a data. Not all contours are drawn.
Table 1. (Continued)

\[ \bar{X} = \beta X \]
\[ \bar{X} = 0 \]
\[ \beta XY = \beta X^* Y^* + \bar{X} \bar{Y} = \beta X^* Y^* + \bar{X} \bar{Y} \]

\[ [\beta X] = [\beta] [x]_R \]
\[ [\beta X^*] = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]
\[ \langle \beta X \rangle = \langle \beta \rangle \langle x \rangle \]
\[ \langle \beta X^* \rangle = 0 \]

of \( \beta \) over these oceanic regions where the 1000 mb surface is below sea level for an appreciable part of the time. Apparently the real atmosphere at 1000 mb is largely confined to areas over the oceans. This can be expected to have some implications for average values of low-level temperature and moisture content. Maps of \( \beta \) at 850 and 700 mb may also be obtained but are less dramatic than that at 1000 mb since the topography pierces the pressure surfaces only in the regions of the major mountain ranges.

The zonally- and time-averaged value of \( \beta \) is displayed in Fig. 2. It gives the percentage of the latitude band which is above ground at the various pressure levels. A not insignificant portion of the 500–1000 mb region is affected. The consequences of properly including the topography in diagnostic calculations
Fig. 3. Zonally- and time-averaged temperatures (K): (a) $\bar{T}$ using given extrapolated values; (b) $\bar{T}_R$, the representative temperature; (c) $\bar{T}$, the contribution to an integral. Note that the contour interval is not uniform for values less than 200 K.
may be expected to be non-trivial for atmospheric variables such as moisture and temperature which have largest values in the lower layers of the atmosphere.

The zonally and time averaged values of the given temperatures $[T]$ are shown in Fig. 3a. The term $[T]_R = [\beta T]/[\beta]$ is displayed in Fig. 3b while $[\beta T]$ is displayed in Fig. 3c. The averaged values of the extrapolated temperatures $[T]$ contain imaginary temperatures from subterranean points. The physical meaning of this statistic is therefore ambiguous. It is not the zonally and time averaged temperature of real atmospheric data in the manner of $[T]_R$.

The contribution to the mean January internal energy of the atmosphere from an annular column is

$$I = g^{-1} \int_0^{p_0} \int_0^{2\pi} \bar{\beta} C_\alpha T d\phi d\lambda$$

$$= (2\pi g)^{-1} \int_0^{p_0} C_\alpha [\beta T] dp,$$

so that the cross-section $C_\alpha [\beta T]$ may be termed the internal energy density for the zonally- and time-averaged atmosphere. The internal energy density as represented by $[\beta T]$ is certainly quite different from that implied by $[T]_R$ and gives a very different picture of the distribution of internal energy for the zonally-averaged atmosphere.

Variance/covariance terms behave similarly. Information on the standing eddy variance of mean temperature is contained in the terms $[T^{*2}]_R = [\beta T^{*2}]/[\beta]$ and $[\beta T^{*2}]$ shown in Fig. 4b, c. The corresponding term using the given data without involving beta, $[(T - \bar{T})^2]$, is shown in Fig. 4a. This term might be taken as an approximation to $[T^{*2}]_R$, although there is a noticeable difference in the two cross-sections. The proper term for calculation is $[\beta T^{*2}]$ which will be recognized as being related to the standing eddy available potential energy.

Similar calculations for the transient eddy variance terms show similar kinds of results, although they are not quite so dramatic since the largest values of this term are not found near the surface as is the case for the standing eddy variance term.

Even such a highly averaged term as the global mean temperature can have an appreciable difference in value depending on the nature of the averaging used. Fig. 5 shows the globally averaged representative temperature $\langle T \rangle_R = \langle \beta T \rangle/\langle \beta \rangle$ and the temperature obtained by averaging the given values $\langle T \rangle$. The representative temperature $\langle T \rangle_R$ is about 3°C warmer than $\langle T \rangle$ at 1000 mb. This is a consequence of the real atmosphere being confined primarily to oceanic regions at 1000 mb in January, as was noted previously. This also depends, of course, on how 1000 mb temperatures are specified at subterranean points. The average stability of the atmosphere as measured
FIG. 4. Standing eddy variance of temperature (K); (a) using extrapolated temperatures; (b) $[T^2]_5$, the representative value; (c) $[\beta T^2]$, the contribution to an integral.
by the vertical distribution of $\langle T \rangle_x$ or $\langle \bar{T} \rangle$ might be expected to differ.

5. Averaged diagnostic equations

In Section 3 the diagnostic budget equations were developed using the formalism involving $\beta$. In Section 4 the appropriate averaging operations involving real atmospheric data were developed. In this section, examples of time-averaged and time- and zonally-averaged budget equations based on this approach are presented. The diagnostic equations obtained in this way should be formally complete and correct. It will be seen that "unfamiliar" terms arise in some of the budgets.
a. Mass

The continuity equation under time-averaging is written as
\[ \frac{\partial \bar{\beta}}{\partial t} + \nabla \cdot \bar{\beta} \mathbf{V} + \frac{\partial}{\partial p} \bar{\beta} \omega = 0 \]
or equivalently as
\[ \frac{\partial \bar{\beta}}{\partial t} + \nabla \cdot \bar{\beta} \mathbf{V}^R + \frac{\partial}{\partial p} \bar{\beta} \omega^R = 0. \tag{27} \]
Under time and zonal averaging the continuity equation becomes
\[ \frac{\partial}{\partial t} [\bar{\beta}] + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} [\bar{\beta}][\bar{v}]_R \cos \varphi \]
\[ + \frac{\partial}{\partial p} [\bar{\beta}][\bar{\omega}]_R = 0. \tag{28} \]
The associated surface pressure equations are
\[ \frac{\partial \bar{p}_s}{\partial t} = -\nabla \cdot \int_0^{\rho_0} \bar{\beta} \mathbf{V}^R dp \]
\[ \frac{\partial [\bar{\rho}_s]}{\partial t} = -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \int_0^{\rho_0} [\bar{\beta}][\bar{v}]_R dp \cos \varphi. \]
The time averaged mass budget is characterized by the mean horizontal and vertical velocities. Since the time averaged pressure change term is typically small, to good approximation,
\[ \nabla \cdot \int_0^{\rho_0} \bar{\beta} \mathbf{V}^R \mathbf{g}^{-1} dp = 0, \tag{29} \]
so that the vertically integrated mass flow may be characterized by a stream function
\[ \int_0^{\rho_0} \bar{\beta} \mathbf{V}^R \mathbf{g}^{-1} dp = \mathbf{K} \times \nabla \psi_H. \]

Often in calculations with real data, relationship (29), while manifestly true in light of a lack of local pressure trend, will not be true of the data. The data are often adjusted so as to satisfy this relationship. Clearly the presence of topography must be properly included in such calculations.

In the zonally- and time-averaged case, the mass balance is characterized by the meridional mass stream function
\[ \psi = 2 \pi a g^{-1} \cos \varphi \int_0^{\rho_0} [\bar{\beta} \bar{v}] dp, \]
which may be defined if \((\partial / \partial t) [\bar{\rho}_s] \) is small.

b. Constituent equations

The general constituent equation may serve as a prototype to illustrate some of the consequences of applying the formalism developed above to various averaged and integrated forms of the budget equations.

The time-averaged equation
\[ \frac{\partial}{\partial t} \bar{\beta} \bar{X} + \nabla \cdot \bar{\beta} \mathbf{V} \bar{X} + \frac{\partial}{\partial p} \bar{\beta} \mathbf{X} \omega = \bar{S} \]
may be rewritten in a number of ways. Decomposing the covariance terms into mean and eddy parts gives
\[ \frac{\partial}{\partial t} \bar{\beta} \bar{X}^R + \nabla \cdot (\bar{\beta} \bar{X}^R \mathbf{V}^R + \bar{\beta} \mathbf{X}^R \mathbf{V}^R) \]
\[ + \frac{\partial}{\partial p} (\bar{\beta} \bar{X}^R \omega^R + \bar{\beta} \mathbf{X}^R \omega) = \bar{\beta} \mathbf{S}^R. \tag{30a} \]
The various terms in the equation can be evaluated in this form. When the terms are evaluated from data on a grid, however, derivatives of terms with rather sharp gradients are encountered where \(\bar{\beta}\) has its most rapid variation (see Fig. 1). The equation may be rearranged to separate out calculations involving relatively smoothly-varying representative values and those involving the gradients of \(\bar{\beta}\). Making use of the time-averaged continuity equation, the equation for the representative value of \(X\) may be written as
\[ \frac{\partial \bar{X}^R}{\partial t} + \mathbf{V}^R \cdot \nabla \bar{X}^R + \bar{\omega} \frac{\partial}{\partial p} \bar{X}^R + \nabla \cdot \bar{X} \mathbf{V}^R + \frac{\partial}{\partial p} \bar{X} \omega^R \]
\[ + \bar{X} \mathbf{V}^R \cdot \nabla \ln \bar{\beta} + \bar{X} \omega \frac{\partial \ln \bar{\beta}}{\partial p} = \bar{S}^R, \tag{30b} \]
which holds for \(\bar{\beta} \neq 0\).

This time-averaged equation is just the “usual” budget equation at points where \(\bar{\beta} = 1\), that is at those points always above the topography. For these points the terms involving derivatives of \(\bar{\beta}\) drop out. There is no budget possible at points always below the topography where \(\bar{\beta} = 0\). For the remaining points, where \(0 < \bar{\beta} < 1\), Eq. (30b) is appropriate. The terms involving derivatives of \(\bar{\beta}\) are not always considered in diagnostic budget calculations. These terms will be non-zero in the relatively restricted regions where \(\bar{\beta}\) is varying, i.e., near the boundaries where the topography pierces the pressure surface.

Although Eq. (30b) holds only for \(\bar{\beta} \neq 0\), the various representative terms are defined everywhere and are continuous so that maps of the statistics and terms in the equations can be easily plotted. Values for which \(\bar{\beta} = 0\) do not contain real information, however, and should be clearly delineated.

The zonally- and time-averaged budget equation may be written in a similar form as
\[ \frac{\partial}{\partial t} [\bar{\beta}][\bar{X}]_R + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} [\bar{\beta}][\bar{X}]_R [\bar{v}]_R + [\bar{X}^* \bar{v}^*]_R \]
\[ + [\bar{X}^\ell \bar{v}^\ell]_R \cos \varphi + \frac{\partial}{\partial p} [\bar{\beta}][\bar{X}]_R [\bar{\omega}]_R \]
\[ + [\bar{X}^* \bar{\omega}^*]_R + [\bar{X} \omega]_R = [\bar{\beta}][\bar{S}]_R, \tag{31a} \]
or as
\[
\frac{\partial \{ \bar{X}_R \}}{\partial t} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \{ \bar{X}_R \} + \frac{\partial}{\partial p} \{ \bar{X}_R \} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \{ \bar{X}_R \} \cos \varphi
\]
\[+ \frac{\partial}{\partial p} \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \] (31b)

which holds for $[\bar{X}] \neq 0$.

As may be seen from Fig. 2, the condition $0 < [\bar{X}] < 1$ obtains over a not inconsiderable portion of the time and zonally averaged atmosphere. For this region the terms in equations (31b) involving the derivatives of $[\bar{X}]$ will not be zero.

Other aspects of the prototype budget equation may be considered. For atmospheric equations of this general form it is often possible to approximate the source/sink term as the divergence of the vertical flux of $X$, perhaps in several forms. If this is the case, $S = \gamma \beta \bar{F}/\partial p$ or

\[\beta S = g \frac{\partial}{\partial p} \beta (F - F_T)\]

and the time averaged equation becomes
\[
\frac{\partial}{\partial t} \{ \bar{X}_R \} + \nabla \cdot \{ \beta \bar{X}_R \} + \frac{\partial}{\partial p} \beta \{ X \bar{F}_R - g(F - F_T) \} = 0, \quad (32)
\]

which may also be written in a form similar to (30b).

Eq. (32) suggests the possibility of locally evaluating the vertical transport terms as a residual, although the data will not often support such an evaluation. The vertically integrated form of the equations is
\[
g^{-1} \nabla \cdot \int_0^h \{ \beta \bar{Y}_R \} dp = F_T - F_T, \quad (33)
\]

where $F_T$ is the vertical flux of $X$, if any, at the top of the atmosphere. It has been assumed that the time change term is small. The integrated flux vector may be decomposed into its rotational and divergent parts and the streamfunction and velocity potential calculated if desired. An equation of this form is suitable for a variety of budgets. For moisture balance for instance,
\[
g^{-1} \nabla \cdot \int_0^h \{ \beta \bar{q}_R \} dp = \bar{E} - \bar{P}.
\]

For the zonally- and time-averaged atmosphere the equation obtained by zonally averaging (32) is
\[
\frac{\partial}{\partial t} \{ \bar{X}_R \} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \{ \bar{X}_R \} \cos \varphi
\]
\[+ \frac{\partial}{\partial p} \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \]
\[+ \{ \bar{X}_R \} \cos \varphi \] (31b)

A meridional streamfunction for $X$ may be defined in a manner similar to that for the mass streamfunction if it is reinterpreted as giving the transport of $X$ in various forms (Hantel, 1974). The vertically integrated equation corresponding to (33) is
\[
g^{-1} \frac{\partial}{\partial p} \{ \bar{X}_R \} \cos \varphi dp = [F_T] - [F_T].
\]

This general treatment of time-averaged and time- and zonally-averaged equations may be applied to the angular momentum and total energy budgets as well as to the moisture or other constituent equations.

c. Available potential energy

For the examples given here, the Lorenz approximate formula for available potential energy in the form
\[
A = \int \frac{1}{2} C_p \gamma \beta (\bar{T} - \langle \bar{T} \rangle_R)^2 dm, \quad (34)
\]

where
\[
\gamma = \left( \frac{\theta}{T} \right) \left( \frac{\partial \theta}{\partial p} \right)^{-1}
\]

is adopted. In this form, subterranean temperatures are not used to evaluate the available potential energy nor the stability term $\gamma$. This form of $A$ and $\gamma$ must be adopted if the available potential energy equations are to be derived using the methods of Section 4 for the decomposition of terms into mean and eddy parts.

The time and zonally averaged available potential energy components are obtained by substituting $T = \langle \bar{T} \rangle_R + \langle \bar{T} \rangle + \bar{T} + \bar{T}$ into Eq. (34) to get
\[
A_{Z} = \int \frac{1}{2} C_p \gamma \beta \{ \bar{T} \}^{1/2} dm,
\]
\[
A_{S} = \int \frac{1}{2} C_p \gamma \beta \{ \bar{T} \}^{2} dm,
\]
\[
A_{T} = \int \frac{1}{2} C_p \gamma \beta \bar{T}^{2} dm.
\]

6. Evaluation of terms from data

In Section 4 some basic statistics for temperature were presented to indicate some aspects of the approach developed here for diagnostic calculations. The available potential energy expressions make use of the various temperature statistics and provide a further illustration of some of the consequences of applying the new formalism.

One consequence of the current approach was foreseen in Section 4 where it was noted that in the
lower layers the atmosphere is confined more and more to oceanic regions and that the representative temperature $\langle \hat{T} \rangle_R$ was higher than the corresponding term $\langle \hat{T} \rangle$ which includes extrapolated temperatures (Fig. 5). The differences in the globally averaged temperature implies a difference in the stability parameter $\gamma$ as shown in Fig. 6.

As anticipated, the stability term $\gamma$ is larger at lower levels when calculated as a function of $\langle \hat{T} \rangle_R$ rather than $\langle \hat{T} \rangle$. It is this stability term multiplied by the temperature variance terms which gives the various components of available potential energy.

This difference in $\gamma$ might be expected to be at least as noticeable if calculated only for the Northern Hemisphere in January. The value of $\gamma$ calculated by Peixoto and Oort (1974) for climatological mean Northern Hemisphere temperatures is also shown in Fig. 6.

The integral of the zonal available potential energy term calculated by the methods proposed here and by taking the extrapolated/interpolated data as given is shown in Fig. 7. There are differences primarily in the polar regions of each hemisphere. In the Southern Hemisphere the presence of Antarctica affects the results, while in the Northern Hemisphere the effects of calculating both $\gamma$ and $[\hat{T}]^*$ in different ways is seen. Despite the local differences in the integral of what is essentially the same term, the integrated value is rather similar in this case since the differences are confined to a relatively small portion of the atmosphere.

The standing eddy available potential energy is obtained from the standing eddy temperature variance of Fig. 5 after multiplying by $\frac{1}{2}C_p\gamma$. The transient eddy contribution is obtained in a similar manner. The integrand of the sum of the two, the eddy available potential energy, is shown in Fig. 8. Once again the integrands differ, as might be expected. The different values of $\gamma$ in the two approaches are such as to compensate for the different calculation of the temperature variance or the differences would be even greater. Thus $\gamma = \gamma(\langle T \rangle_R)$ is larger than $\gamma = \gamma(\langle \hat{T} \rangle)$ in the lower layers but multiplies $[\beta\hat{T}^*]^2$ which is smaller than the corresponding variance calculated with $\beta = 1$ (Fig. 4a, b).

The integrated values differ by less than 10% for this global calculation. This should not be considered a trivial amount, however, since such a globally integrated calculation has reduced all the available temperature information to a single number. Such a number would be expected to be rather stable and it must be remembered that this difference arises not because of uncertainties in the data but rather in the manner in which the appropriate statistics are calculated. Rather larger differences occur locally in the evaluation of the various statistics.

There seems no reason why the methods developed here should not be used routinely for diagnostic calculations. This will become more important as less heavily averaged and integrated budgets are attempted. At the very least, each diagnostic calculation should be accompanied by an explanation of which, if any, approximation to the diagnostic statistics and equations has been used.

**Fig. 6.** Lorenz stability parameter $\gamma$ for January 1979 using the representative temperature $\gamma = \gamma(\langle T \rangle_R)$, the extrapolated temperatures $\gamma = \gamma(\langle \hat{T} \rangle)$, and as given by Peixoto and Oort (1974) for climatological mean Northern Hemisphere temperatures.
FIG. 7. Integrand of zonal available potential energy (10^7 J kg^-1), (a) using formulas developed in text, (b) using extrapolated temperatures as given.
Fig. 8. Integrand of eddy available potential energy (standing plus transient) (10 J kg$^{-1}$), (a) using formulas developed in text, (b) using extrapolated temperatures as given.
7. Approximate formulas

The formalism developed here requires that a new diagnostic variable $\beta$ be obtained at each data point in time and space and that mean and covariance terms involving this parameter be calculated and stored. For example, the calculation of the representative mean $\overline{X}$ requires $\overline{X}$, $\beta$ and the covariance $\beta X$ be obtained. Covariance terms $\beta X' Y'$ become triple products. In the budget equations of Section 5 a variety of terms arise which involve the covariance of $\beta$ with other variables.

While there is no overriding reason why the “exact” forms of such statistics and equations cannot be used for modern diagnostic calculations, it may nevertheless be possible to develop and to justify various approximate approaches. The consequences of a particular simplifying assumption for the calculation of basic statistics can be seen by reference to Table 1 while the implications for budget calculations are obtained by making the appropriate substitution into the budget equations of Section 5.

Writing

$$\beta = \beta + \beta' = [\beta] + \beta* + \beta'$$

suggests approximations obtained by replacing $\beta$ with certain of its averages. There are other possibilities which may be obtained under even more severe averaging of $\beta$. The approximation leading to the simplest statistics and equations is $\beta = 1$ which might be called the flat-earth approximation (in pressure coordinates).

One of the more plausible approximations is to replace $\beta$ by $\overline{\beta}$, thereby omitting higher-order terms of the form $\overline{\beta X'}$. Such terms will be non-zero only in the regions where $0 < \overline{\beta} < 1$ (see Fig. 1). For this approximation

$$\overline{X} = \overline{\beta X} = \overline{X} + \overline{\beta' X'} / \beta \approx \overline{X},$$

$$X' = X - \overline{X} \approx X - \overline{X},$$

$$\overline{\beta X' Y'} = \overline{\beta X' Y'} + (\overline{\beta' X' Y'} - \overline{\beta' X' Y'}) / \beta$$

$$\approx \overline{\beta X' Y'}.$$

It follows that $\overline{X' Y'} = \overline{X' Y'}$ and so on.

The time-averaged statistics are just those obtained by treating the extended variables as the data. The resulting statistics enter the calculations multiplied by $\overline{\beta}$ which “masks out” subterranean values. The associated modifications to the zonally- and time-averaged and time- and globally-averaged statistics are obtained from Table 1 by replacing $\beta$ by $\overline{\beta}$ in the appropriate formulas.

This approximation avoids the calculation of temporal covariances of $\beta$ with other variables but retains information on the effects of topography through the time averaged field $\beta$. The implications for the various averaged budget equations follows directly by substituting the approximate values of the statistical terms into them. The diagnostic equations retain their form, in general, and include, as previously, terms involving derivatives of $\beta$.

Considerable simplification of the time- and zonally-averaged statistics and equations results by replacing $\beta$ by $[\beta]$. The resulting mean and covariance terms obtained in this way will be just those calculated from the extrapolated data but multiplied by $[\beta]$ to take some account of the effect of topography. This is however a more drastic approximation than previously.

A different approximation which avoids calculation of $\beta$ at each time level but which retains information about the surface pressure through its average is obtained by replacing $\beta$ with

$$\hat{\beta} = \begin{cases} 1, & p < \bar{p}, \\ 0, & p > \bar{p}. \end{cases}$$

The new variable $\hat{\beta}$ is now discontinuous, as opposed to $\beta$ which is continuous. This approach is followed by Oort and Rasmussen (1971) for example.

These examples suggest the most obvious approximations to the complete calculations and indicate some of the consequences to the budget equations of such approximations. The utility of the formalism developed here is indicated by the relative ease with which consistent approximate formulas are obtained.

It will be of considerable interest to establish from model output and from real data just how important the various neglected terms may be, and in turn what level of approximation should be generally adopted for diagnostic calculations.

8. Concluding remarks

The methods developed here provide a formalism for diagnostic calculations in pressure (and in other) coordinate systems. Such methods are applicable to the analysis of data from both real and model atmospheres. There seems to be no fundamental reason why these methods should not be used. Model output and, increasingly, observed atmospheric data, are accurate enough to warrant considerable care in the calculation of basic diagnostic statistics and balances. One of the arguments for adopting this formalism concerns the desirability of a uniform approach to diagnostic calculations for comparison purposes. As was suggested in Sections 4 and 6, the values of even the most basic statistics depend on the method of calculation.

This uncertainty points out the need for a standard approach to diagnostic calculation of common statistics and budget terms. Certainly budget calculations using the same data should not differ by an appreciable amount due only to the supposedly unimportant details of the calculations.
The methods developed here are currently being applied to the output of a general circulation model (an improved version of the model reported in Boer and McFarlane, 1979) and to FGGE and other data. The suitability of the various approximations is being investigated, as is the importance to the budgets of the boundary-related terms which arise using the formalism.

Acknowledgments. Lynda Smith has been most helpful in the preparation of the manuscript. John Henderson has been elemental in devising the analysis package used for the diagnostic calculations.

REFERENCES


