NOTES AND CORRESPONDENCE

A Generalized Class of Exact Time-Dependent Solutions of the Vorticity Equation for Nondivergent Barotropic Flow

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ABSTRACT

This note deals with a new class of solutions of the nondivergent barotropic vorticity equation. In general, these solutions require a complete representation in spherical harmonics and are therefore good comparison solutions for testing the accuracy of spectral methods of numerical integration.

1. Introduction

In testing any new and untried method of numerically integrating the hydrodynamical equations for large-scale atmospheric flows or in comparing alternative methods, one must have some standard against which to judge the accuracy of each procedure. A common practice has been to compare numerical integrations (which are otherwise identical, but with increasingly finer resolution) with a very high-resolution integration. The tacit assumption is that, if the lower resolution integrations converge uniformly on the high-resolution integration, they also converge on the solution of the original differential equations. That this assumption is not always justified is implicit in the results of Hoskins (1973) and is made more explicit in the recent work of Yee and Shapiro (1981). They point out that numerical integrations initiated from a *physically* unstable state may depart drastically from the true solution.

It would, in any event, be more convincing to compare numerical integrations directly with exact analytic solutions of the original differential equations. This is precisely what Yee and Shapiro did in testing finite-difference methods of integrating the vorticity equation for nondivergent barotropic flow: their standards of comparison were the well-known solutions given by Haurwitz (1940). There is, unfortunately, only a limited class of known solutions of any self-consistent system of hydrodynamical equations. Generally, however, it is possible to specialize even very complicated physical models in such a way that they reduce to a system for which exact solutions are known. In this way, one can at least establish the adequacy (or inadequacy) of numerical methods in dealing with the most difficult aspect of the problem—i.e., the highly nonlinear scale-interactions in the simplest of flows.

Even the known nonsingular solutions (e.g., those of Haurwitz) are not always fair standards of accuracy, particularly in testing spectral methods. This arises from the fact that the Haurwitz solutions consist of two parts that do not interact nonlinearly, but whose zonal dependence is purely sinusoidal and whose meridional dependence is that of a single associated Legendre polynomial. Now, spectral methods (as applied to flow around a sphere) represent the streamfunction field as a truncated expansion in functions that have exactly the same spatial dependence as the Haurwitz solutions. Thus, if the truncated spectral representation includes the same spatial dependence as the comparison solution, there is no error of representation. In this case, the only error of numerical integration would be that of extrapolating forward in time. In short, for purposes of comparing spectral methods with, for example, finite-difference or Lagrangian methods of integration, it would be desirable to have available a less special class of exact solutions than those given by Haurwitz.

The purpose of this note is to show how to construct a more general class of exact solutions of the vorticity equation for nondivergent barotropic flow, and which are complicated enough to challenge any method of numerical integration. Briefly, they consist of a zonally-dependent velocity field that rotates around the earth's axis at a constant rate, superimposed on a zonal current with constant angular velocity around the earth's axis. The superposed zonally-dependent velocity field has the same structure as that in the Haurwitz solutions, but relative to a spherical coordinate system whose axis is tilted at a fixed arbitrary angle from the earth's axis of rotation. For almost all angles of "tilt," this guarantees that

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the usual spectral representation must be complete, whence the spectral integrations have no inherent advantage when compared with other approximations to the exact solutions.

2. The vorticity equation

The vorticity equation for nondivergent barotropic flow is the familiar equation for conservation of absolute vorticity, which may be written in the form:

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{1}{a^2\sin\theta}\left(\frac{\partial\psi}{\partial\lambda}\frac{\partial}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\lambda}\right) \times (\nabla^2\psi + 2\Omega\cos\theta) = 0, \quad (1)$$

where the streamfunction ψ is the sole dependent variable. The independent variables are time t, colatitude θ and longitude λ (increasing westward). The constants a and Ω are the earth's mean radius and angular rate of rotation, respectively. The operator ∇^2 is the Laplacian on the surface of a sphere and is independent of the coordinate system.

It is convenient to regard ψ as the sum of two distinct functions, one of which depends only on θ and not on λ or t, and the other of which depends on θ, λ and t, i.e.,

$$\psi(\theta, \lambda, t) = \Psi(\theta) + \psi'(\theta, \lambda, t).$$

Substituting the expression above into (1), we have

$$\frac{\partial}{\partial t}\nabla^2\psi' + \frac{1}{a^2\sin\theta}\left[\frac{\partial\psi'}{\partial\lambda}\frac{\partial}{\partial\theta}(\nabla^2\Psi + \nabla^2\psi' + 2\Omega\cos\theta) - \left(\frac{\partial\Psi}{\partial\theta} + \frac{\partial\psi'}{\partial\theta}\right)\frac{\partial}{\partial\lambda}\nabla^2\psi'\right] = 0. \quad (2)$$

It also simplifies matters considerably if we choose Ψ in such a way that

$$\frac{\partial\Psi}{\partial\theta} = aU\sin\theta.$$

Ψ is then the streamfunction corresponding to a steady zonal current with constant angular velocity around the earth's axis of rotation and constant speed U at the equator. Then

$$\frac{\partial}{\partial\theta}\nabla^2\Psi = -\frac{2U}{a}\sin\theta.$$

With these substitutions, (2) reduces to

$$\frac{\partial}{\partial t}\nabla^2\psi' - \left(\frac{2U}{a^3} + \frac{2\Omega}{a^2}\right)\frac{\partial\psi'}{\partial\lambda} - \frac{U}{a}\frac{\partial}{\partial\lambda}\nabla^2\psi' + \frac{1}{a^2\sin\theta}\left(\frac{\partial\psi'}{\partial\lambda}\frac{\partial}{\partial\theta}\nabla^2\psi' - \frac{\partial\psi'}{\partial\theta}\frac{\partial}{\partial\lambda}\nabla^2\psi'\right) = 0. \quad (3)$$

It should be noted that, up to this point, there is no loss of generality, since no restriction has been placed on ψ'. It is simply a solution of (3).

Let us now require that ψ' *at all times* satisfies the equation

$$\nabla^2\psi' = -k^2\psi', \quad (4)$$

where k is an eigenvalue of (4). Then the last term on the left-hand side of (3) vanishes at all times. Moreover,

$$\frac{\partial}{\partial t}\nabla^2\psi' = -k^2\frac{\partial\psi'}{\partial t} \quad \text{and} \quad \frac{\partial}{\partial\lambda}\nabla^2\psi' = -k^2\frac{\partial\psi'}{\partial\lambda},$$

with the result that (3) takes the form

$$\frac{\partial\psi'}{\partial t} + \frac{1}{k^2}\left(\frac{2U}{a^3} + \frac{2\Omega}{a^2} - \frac{k^2U}{a}\right)\frac{\partial\psi'}{\partial\lambda} = 0.$$

This equation implies that the ψ'-field rotates around the earth's axis *without change in shape* and with a constant angular frequency ν given by

$$\nu = \frac{1}{k^2}\left(\frac{2U}{a^3} + \frac{2\Omega}{a^2} - \frac{k^2U}{a}\right). \quad (5)$$

This result holds for *all* solutions of (4), singular or nonsingular.

3. The eigenfunctions

With the spherical Laplacian written in terms of derivatives with respect to the spherical coordinates θ and λ, (4) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi'}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi'}{\partial\lambda^2} = -a^2k^2\psi'.$$

Thus, if we seek solutions of the form $f(\theta)e^{im\lambda}$, $f(\theta)$ is determined by the ordinary differential equation

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df}{d\theta}\right) - \frac{m^2}{\sin^2\theta}f + k^2a^2f = 0,$$

or, with the transformation $x = \cos\theta$,

$$\frac{d}{dx}\left[(1 - x^2)\frac{df}{dx}\right] + \left(k^2a^2 - \frac{m^2}{1 - x^2}\right)f = 0. \quad (6)$$

The theory of solutions of (6) is well known (cf. MacRobert, 1967). It has nonsingular, single-valued solutions if the eigenvalues are given by $k^2a^2 = n(n + 1)$ for integer n, and provided that the integer m is less than n. Under these conditions, the solutions of (6) are the associated Legendre polynomials with argument $\cos\theta$, and ψ' is given by

$$\psi'(\lambda, \theta) = e^{im\lambda}P_n^m(\cos\theta). \quad (7)$$

These solutions, of course, correspond to those given earlier by Haurwitz.

Now, the point to be emphasized is that (4) is independent of the coordinate system and, in particular, independent of the orientation of the axis of a spherical coordinate system. It is clear, for example, that

$$\psi' = e^{im\lambda'}P_n^m(\cos\theta') \quad (8)$$

is a solution of (4), but θ' and λ' are now spherical coordinates in a system whose axis is tilted from the earth's axis of rotation. If we take the axis of the (θ', λ') system to lie in the plane of the Greenwich meridian and to be inclined at an arbitrary angle ϕ to the earth's axis,

$$\left.\begin{aligned} \lambda' &= \lambda'(\lambda, \theta, \phi) \\ \theta' &= \theta'(\lambda, \theta, \phi) \end{aligned}\right\}. \tag{9}$$

Through these trigonometric relationships we can find the explicit dependence of the generalized solutions ψ' [as given in (8)] on θ and λ. The ψ'-fields so generated still rotate around the earth's axis with the constant angular speed shown in (5), and without change in shape.

The remaining problem in spherical trigonometry is illustrated in Fig. 1. It is simply to find the surface angle λ' (longitude) and central angle θ' (colatitude), relative to a coordinate system whose axis is tilted at an angle ϕ to the earth's axis of rotation, in terms of ϕ, λ (longitude on the earth) and θ (colatitude on the earth). This requires solving the spherical triangle PQP'. For convenience, the spherical triangle PQP' is also shown in Fig. 2 as a conjunction of two right spherical triangles CQP and CP'P. The surface and central angles of the two right spherical triangles are labeled a, b, c, α, β, γ and a', b', c', α', β', γ' to emphasize the symmetry of the trigonometric relationships between them. In the present case, of course, $a = a'$ and $\gamma = \gamma'$. Referring to Fig. 1, we see that $\lambda = \pi - \beta - \beta'$, $\theta = c'$, $\lambda' = \alpha$, $\theta' = b + b'$ and $\phi = c$.

The central angle a may be eliminated between twelve of the twenty fundamental trigonometric identities to yield six relationships between b, c, α, β, b', c', α' and β'. They are:

$$\sin\alpha \sin c = \sin\alpha' \sin c', \tag{10a}$$

$$\tan b \cot\beta = \tan b' \cot\beta', \tag{10b}$$

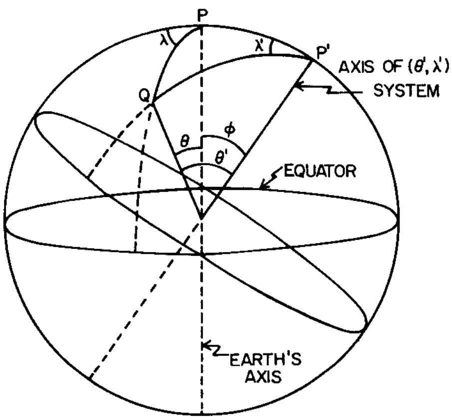

FIG. 1. The spherical triangle PQP′ relative to the spherical coordinates (θ, λ) and (θ', λ').

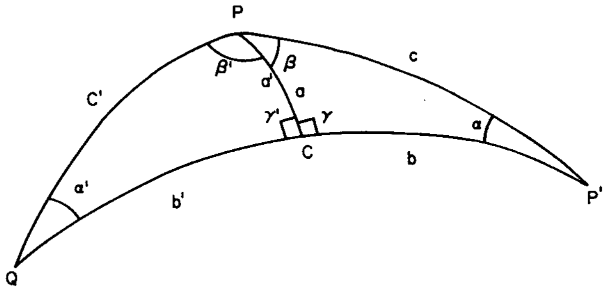

FIG. 2. The spherical triangle PQP′ as a conjunction of the right spherical triangles CPQ and CPP′.

$$\frac{\cos\alpha}{\sin\beta} = \frac{\cos\alpha'}{\sin\beta'}, \tag{10c}$$

$$\frac{\sin b}{\cot\alpha} = \frac{\sin b'}{\cot\alpha'}, \tag{10d}$$

$$\frac{\cos\beta}{\cot c} = \frac{\cos\beta'}{\cot c'}, \tag{10e}$$

$$\frac{\cos c}{\cos b} = \frac{\cos c'}{\cos b'}. \tag{10f}$$

Before calculating α, b and b', it is simplest to calculate β and β'. Letting $\mu = \pi - \lambda$ and noting that $\beta' = \mu - \beta$, we may write (10e) as

$$\begin{aligned} \cot c' \cos\beta &= \cot c \cos(\mu - \beta) \\ &= \cot c(\cos\mu \cos\beta + \sin\mu \sin\beta), \end{aligned}$$

whence

$$\beta = \arctan\left(\frac{\cot c' - \cos\mu \cot c}{\sin\mu \cot c}\right). \tag{11}$$

Since μ, c and c' are given, β can be determined immediately and β' calculated as $(\mu - \beta)$. We now regard β and β' as known.

To calculate α, we eliminate α' between (10a) and (10c) by adding the expressions for $\sin^2\alpha'$ and $\cos^2\alpha'$ and setting the sum equal to one, with the result that

$$\cos 2\alpha = \frac{2 - (\sin c/\sin c')^2 - (\sin\beta'/\sin\beta)^2}{(\sin\beta'/\sin\beta)^2 - (\sin c/\sin c')^2}. \tag{12}$$

It is readily verified that, in the special case when $\lambda = 0$, $\beta' = \pi - \beta$, $\sin^2\beta' = \sin^2\beta$ and $\cos 2\alpha = \cos 2\lambda' = 1$. In exactly the same way we may eliminate either b or b' between (10b) and (10f) to obtain[2]

$$\cos 2b = \frac{2 - (\cos c'/\cos c)^2[1 + (\cot\beta/\cot\beta')^2]}{(\cos c'/\cos c)^2[1 - (\cot\beta/\cot\beta')^2]}, \tag{13}$$

$$\cos 2b' = \frac{2 - (\cos c/\cos c')^2[1 + (\cot\beta'/\cot\beta)^2]}{(\cos c/\cos c')^2[1 - (\cot\beta'/\cot\beta)^2]}. \tag{14}$$

[2] It will be noted that (13) and (14) fail when $\cot\beta = \pm\cot\beta'$, e.g., when $\beta' = \pi - \beta$. In that event, however, $\lambda = 0$, $\lambda' = 0$ and $\theta' = \theta + \phi$.

Since c, c' and μ are given and β and β' are determined by (11), α, b and b' can be calculated from (12), (13) and (14). As noted earlier, $\lambda' = \alpha$ and $\theta' = b + b'$. This solves the problem of relating Q's coordinates in the two different systems, with poles at P and P'.

Finally, it should be noted that, for a given eigenvalue k, the meridional index n is fixed, but the zonal index m may take on any integer value less than n. Thus, any linear combination of solutions of the linear equation (4), for different values of m and for different orientations of the axis of the spherical coordinate, is also a solution of (4). No such linear combination is the general solution, but is general enough to have very complicated structure.

4. Summary

In the two foregoing sections, we have constructed a large class of exact time-dependent solutions of the nondivergent barotropic vorticity equation. Those solutions consist of a steady zonal current with constant angular speed, superposed on a zonally- and meridionally-dependent velocity field that rotates bodily around the earth's axis with constant angular speed and without change in shape. The superposed propagating field has a spatial structure that is the same as in Haurwitz' earlier solutions, but relative to a spherical coordinate system whose axis is inclined to the earth's axis of rotation at an arbitrary fixed angle.

The main virtue of these new solutions is that, for almost all angles of inclination, the usual spectral representation must be complete. Thus, they are ideal comparison solutions for testing the accuracy of spectral methods of numerical integration.

REFERENCES

Haurwitz, B., 1940: The motion of atmospheric disturbances on the spherical earth. *J. Mar. Res.*, **3**, 254–267.

Hoskins, B. J., 1973: Stability of the Rossby-Haurwitz waves. *Quart. J. Roy. Meteor. Soc.*, **99**, 723–745.

MacRobert, T. M., 1967: *Spherical Harmonics*. Pergamon Press, 345 pp.

Yee, S. Y. K., and R. Shapiro, 1981: The interaction of model dynamics and numerical errors in a nondivergent global barotropic model. *Mon. Wea. Rev.*, **109**, 687–700.

CORRIGENDUM

Winston C. Chao and Marvin A. Geller have noticed an error in their note "Utilization of Normal Mode Initial Conditions for Detecting Errors in the Dynamics Part of Primitive Equation Global Models" (*Mon. Wea. Rev.*, **110**, 304–306). In the last paragraph of Section 2, the statement "Moreover, if there is a basic flow of solid rotation with angular speed, then $\nu = 2\tilde{\sigma}(\Omega + \omega) + \omega$" is not correct. This statement is correct only for a Rossby-Haurwitz wave in a nondivergent flow on a sphere.

With a solid rotation basic state, both the eigenfrequency $\tilde{\sigma}$ and the meridional structure of the normal modes must be recalculated (see Kasahara, *J. Atmos. Sci.*, **37**, 917–929; corrigendum, **38**, 2284–2285). A term $\omega a \cos\phi$, where a is the radius of the earth and ϕ the latitude, is added to the right-hand side of the u equation; and another term $\frac{1}{2}\gamma(\cos^2\phi - \frac{2}{3})$, where $\gamma = a^2\rho_0(2\Omega + \omega)\omega$, is included as part of p_0. These changes apply to both the isentropic and the isothermal normal mode initial conditions.