Vertical Differencing of the Primitive Equations in Sigma Coordinates

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ABSTRACT

A vertical finite-difference scheme for the primitive equations in sigma coordinates is obtained by requiring that the discrete equations retain some important properties of the continuous equations. A family of schemes is derived whose members conserve total energy, maintain an integral constraint on the vertically integrated pressure gradient force, have a local differencing of the hydrostatic equation, and give exact forms of the hydrostatic equation and the pressure gradient force for particular atmospheres. The proposed scheme is a member of this family that in addition conserves the global mass integral of the potential temperature under adiabatic processes.

1. Introduction

The problem of choosing a finite-difference scheme can be posed as that of deriving consistent difference equations that maintain discrete analogues of some of the constraints satisfied by the continuous equations. Arakawa (1972) and Arakawa and Lamb (1977) used this approach to derive a vertical difference scheme for the primitive equations in σ-coordinates. They chose the following integral constraints:

(I) That the pressure gradient force generate no circulation of vertically integrated momentum along a contour of the surface topography.

(II) That the finite-difference analogues of the energy conversion term have the same form in the kinetic energy and thermodynamic equations.

(III) That the global mass integral of the potential temperature θ be conserved under adiabatic processes.

(IV) That the global mass integral of a function of θ, such as θ^2 or $\ln\theta$, be conserved under adiabatic processes.

Constraint (I) is on the form of the pressure gradient force in the momentum equation, and constraints (III) and (IV) are on the form of the thermodynamic equation. It then follows from constraint (II) that the form of the hydrostatic equation cannot be freely specified. In this way the above four constraints are nearly sufficient to determine the vertical difference scheme, leaving one free to specify only the pressure at which the potential temperature (or the temperature) is carried as a prognostic variable. [Phillips (1974) and Tokioka (1978) made use of this freedom to improve the original scheme proposed by Arakawa (1972).]

When all four of the above constraints are imposed, the thickness immediately above the surface depends on the potential temperature in all layers, and therefore the hydrostatic equation is non-local. In principle, there is nothing wrong with a non-local dependence. It appears for example, in higher-order schemes. But the way in which the non-local dependence appears in the above scheme can seriously affect the local accuracy of the hydrostatic equation, as will be shown in Section 4. Although the resulting errors in the pressure gradient force may not be the most serious occurring over steep topography, one may question whether such a loss of local accuracy is justified.

In this paper we will examine a general family of schemes for which the hydrostatic equation is local. As should be clear from the preceeding discussion, no scheme in this family can satisfy all four of the above constraints. Relaxing constraint (II) allows one to difference the hydrostatic equation directly; this, however, is not the only way of eliminating the non-local dependence. Even when (II) is satisfied, the final form of the hydrostatic equation depends on the schemes used for the pressure gradient force and the thermodynamic equation and, therefore, on any constraints they are subject to. It turns out, in fact, that it is possible to obtain local schemes that satisfy any three of the four constraints.

Of the four, we regard constraints (I) and (II), which involve the pressure gradient force, as the most important. In sigma coordinates the pressure gradient force appears as the sum of two terms that near steep topography are of comparable magnitude and opposite sign. Discretization errors there, even when they produce small errors in either term, can result in a large error in the pressure gradient force. If (I) and (II) are not satisfied by the difference equations, these errors can lead to large, spurious sources or sinks of total energy and vertically integrated vorticity. To limit somewhat the types of schemes considered, we will restrict our attention to those that satisfy (I) and (II).

After writing the continuous equations in Section 2 and reviewing in Section 3 the results of Arakawa and Lamb (1977) for schemes that satisfy all four constraints, we will obtain in Section 4 conditions for the class of schemes that satisfy (I) and (II). In Section 5 we will narrow this class by requiring that the hydrostatic equation be local, and in Section 6 we will define a family of schemes for which, in addition, the pressure gradient force vanishes for particular atmospheres when the temperature is a function of pressure only. In Section 7 we obtain schemes within this family that satisfy (III) or some form of (IV). For convenience, the equations for the scheme that satisfies (III), which we have found particularly useful, are collected in Section 8.

2. Continuous equations

a. The vertical coordinate

We will use a σ-coordinate defined by

$$p = p_I + \pi\sigma. \tag{2.1}$$

Here p is the pressure, p_I is a constant pressure at the top of the model atmosphere,[1] and π is $p_S - p_I$, p_S being the surface pressure. When $p_I = 0$ the coordinate reduces to the σ-coordinate originally proposed by Phillips (1957). From (2.1) we have

$$dp = \pi d\sigma, \tag{2.2}$$

where d denotes the differential under constant horizontal coordinates and time.

The material time derivative in the σ-coordinate is

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)_\sigma + \dot\sigma\frac{\partial}{\partial\sigma}, \tag{2.3}$$

where $\mathbf{v}$ is the horizontal velocity and $\dot\sigma \equiv D\sigma/Dt$. By operating (2.3) on (2.1), we obtain the following expression for the vertical pressure velocity:

$$\omega \equiv \frac{Dp}{Dt} = \sigma\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\pi + \pi\dot\sigma. \tag{2.4}$$

Gradients in the p-coordinate and σ-coordinate systems are related by $\nabla_p = \nabla\sigma + \nabla_p\sigma\partial/\partial\sigma$. Operating ∇_p on (2.1), we obtain $\nabla_p\sigma = (-\sigma/\pi)\nabla\pi$. Then

$$\nabla_p = \nabla_\sigma - \frac{\sigma}{\pi}\nabla\pi\frac{\partial}{\partial\sigma}. \tag{2.5}$$

[1] In Arakawa and Lamb (1977), p_I is a constant pressure level below which the σ-coordinate is used. In this paper, p_I is treated as the top of the model atmosphere for simplicity.

b. The continuity equation

Using (2.2), (2.4) and (2.5), the continuity equation, $\nabla_p\cdot\mathbf{v} + \partial\omega/\partial p = 0$, yields

$$\frac{\partial\pi}{\partial t} + \nabla_\sigma\cdot(\pi\mathbf{v}) + \frac{\partial(\pi\dot\sigma)}{\partial\sigma} = 0. \tag{2.6}$$

Assuming that the top and bottom of the model atmosphere are material surfaces, the appropriate boundary conditions are $\pi\dot\sigma = 0$ at $\sigma = 0$ and $\sigma = 1$. Integrating (2.6) then gives the auxilliary relations

$$\frac{\partial\pi}{\partial t} = -\nabla\cdot\int_0^1 \pi\mathbf{v}d\sigma, \tag{2.7}$$

$$\pi\dot\sigma = -\nabla_\sigma\cdot\int_0^\sigma \pi\mathbf{v}d\sigma - \sigma\frac{\partial\pi}{\partial t}. \tag{2.8}$$

c. The equation of state

The model atmosphere is assumed to be a perfect gas so that $\alpha = RT/p$, where α is the specific volume, T the temperature, and R the gas constant. The potential temperature is defined by $\theta = T/P$, where $P \equiv (p/p_0)^\kappa$, p_0 is a standard pressure, κ is R/c_p, and c_p is the specific heat at constant pressure. The equation of state may then be written as $\alpha = c_p\theta dP/dp$. Since P is a function of p only, the derivative dP/dp can be taken under constant σ or under constant horizontal coordinates and time; we then obtain the two alternative forms

$$\alpha = c_p\theta\frac{1}{\sigma}\left(\frac{\partial P}{\partial\pi}\right)_\sigma, \tag{2.9}$$

$$\alpha = c_p\theta\frac{1}{\pi}\frac{\partial P}{\partial\sigma}. \tag{2.10}$$

d. The hydrostatic equation

Letting ϕ be the geopotential and using (2.2) we may rewrite the hydrostatic equation, $-d\phi = \alpha dp$, as

$$-d\phi = \pi\alpha d\sigma, \tag{2.11}$$

or, using forms (2.9) and (2.10) of the equation of state,

$$-d\phi = c_p\theta\frac{\pi}{\sigma}\left(\frac{\partial P}{\partial\pi}\right)_\sigma d\sigma, \tag{2.12}$$

$$-d\phi = c_p\theta dP. \tag{2.13}$$

e. The pressure gradient force

The pressure gradient force is given by $-\nabla_p\phi$. Application of (2.5) to ϕ and use of (2.12) give

$$-\nabla_p\phi = -\nabla_\sigma\phi + \frac{\sigma}{\pi}\frac{\partial\phi}{\partial\sigma}\nabla\pi \tag{2.14}$$

$$= -\nabla_\sigma\phi - c_p\theta\left(\frac{\partial P}{\partial\pi}\right)_\sigma\nabla\pi. \tag{2.15}$$

Eq. (2.14) may also be written as

$$-\nabla_p\phi = -\nabla_\sigma\phi - \frac{1}{\pi}\left[\phi - \frac{\partial(\sigma\phi)}{\partial\sigma}\right]\nabla\pi \quad (2.16)$$

$$= -\frac{1}{\pi}\nabla_\sigma(\pi\phi) + \frac{1}{\pi}\frac{\partial(\sigma\phi)}{\partial\sigma}\nabla\pi. \quad (2.17)$$

Integration of $\pi \times$ (2.17) with respect to σ from 0 to 1 gives

$$-\int_0^1 \pi\nabla_p\phi d\sigma$$

$$= -\nabla\left[\int_0^1 \pi\phi d\sigma - \phi_S\pi\right] - \pi\nabla\phi_S, \quad (2.18)$$

where ϕ_S is ϕ at $\sigma = 1$. The first term on the right-hand side of (2.18) is a gradient vector, and a line integral of its tangential component taken along an arbitrary closed curve on the sphere vanishes. Only the second term can contribute to such a line integral, but the contribution is zero when the line integral is taken along ϕ_S = constant. Thus the pressure gradient force satisfies constraint (I) of Section 1.

The kinetic energy generation by the pressure gradient force, per unit $d\sigma/g$, is obtained through multiplication of (2.14) by $\pi\mathbf{v}$. After some manipulation, including use of the continuity equation (2.6), we obtain

$$-\pi\mathbf{v}\cdot\nabla_p\phi = -\nabla_\sigma\cdot(\pi\mathbf{v}\phi)$$

$$-\frac{\partial}{\partial\sigma}\left[\left(\pi\dot{\sigma} + \sigma\frac{\partial\pi}{\partial t}\right)\phi\right] - \pi\omega\alpha. \quad (2.19)$$

Here we have used the relation

$$\pi\omega\alpha = -\frac{\partial\phi}{\partial\sigma}\left[\sigma\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\pi + \pi\dot{\sigma}\right], \quad (2.20)$$

which can be derived from (2.4) and (2.11).

f. The first law of thermodynamics

The first law of thermodynamics under adiabatic processes can be written in terms of the potential temperature as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)_\sigma\theta + \dot{\sigma}\frac{\partial\theta}{\partial\sigma} = 0. \quad (2.21)$$

Using the continuity equation (2.6), we obtain the flux form corresponding to (2.21),

$$\frac{\partial}{\partial t}(\pi\theta) + \nabla_\sigma\cdot(\pi\mathbf{v}\theta) + \frac{\partial(\pi\dot{\sigma}\theta)}{\partial\sigma} = 0. \quad (2.22)$$

Since $(\pi\dot{\sigma}) = 0$ at $\sigma = 0$ and $\sigma = 1$, integration of (2.22) with respect to σ from 0 to 1 gives

$$\frac{\partial}{\partial t}\int_0^1 \pi\theta d\sigma = -\nabla\cdot\int_0^1 \pi\mathbf{v}\theta d\sigma. \quad (2.23)$$

Since the right-hand side of (2.23) vanishes when integrated with respect to horizontal area over the entire globe, we obtain constraint (III) of Section 1.

From (2.21) we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)_\sigma F(\theta) + \dot{\sigma}\frac{\partial F(\theta)}{\partial\sigma} = 0, \quad (2.24)$$

where $F(\theta)$ is an arbitrary function of θ. Following a procedure similar to the derivation of (2.23), we can show that constraint (IV) of Section 1 is satisfied.

From the first law of thermodynamics (2.21), the continuity equation (2.6), and the relation $\theta = T/P$, we can derive

$$\frac{\partial}{\partial t}(\pi c_pT) + \nabla_\sigma\cdot(\pi\mathbf{v}c_pT) + \frac{\partial(\pi\dot{\sigma}c_pT)}{\partial\sigma} = \pi\omega\alpha, \quad (2.25)$$

where

$$\pi\omega\alpha = c_p\theta\left[\pi\left(\frac{\partial P}{\partial\pi}\right)_\sigma\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)_\sigma\pi + \frac{\partial P}{\partial\sigma}\pi\dot{\sigma}\right], \quad (2.26)$$

which can be derived from (2.9), (2.10) and (2.4). It is easy to see that the two forms of $\pi\omega\alpha$ given by (2.20) and (2.26) are identical by using the hydrostatic equation in the forms given by (2.12) and (2.13). The appearance of $\pi\omega\alpha$ in (2.19) and (2.25) with opposite signs leads to conservation of the total energy, as far as the first law of thermodynamics and the kinetic energy generation by the pressure gradient force are concerned, and thus constraint (II) of Section 1 is satisfied.

3. General formulation of schemes that satisfy constraints (I) and (II)

We divide the region $0 \leqslant \sigma \leqslant 1$ into L layers separated by $L - 1$ levels of prescribed constant σ, as shown by Fig. 1. We will identify quantities defined for the layers by integer subscripts, $1, 2, \ldots, l - 1, l, l + 1, \ldots, L - 1, L$; and quantities defined at the top ($\sigma = 0$), interface, and bottom ($\sigma = 1$) levels by half-integer subscripts $1/2, 1\,1/2, \ldots, l - 1/2, l + 1/2, \ldots, L - 1/2, L + 1/2$. A caret will be used for variables at these half-integer levels. The following definition will be used throughout for an arbitrary variable A:

$$(\delta A)_l \equiv \hat{A}_{l+1/2} - \hat{A}_{l-1/2}. \quad (3.1)$$

We will restrict our attention to the case where all three-dimensional prognostic variables are defined for the layers, and the vertical σ-velocity ($\dot{\sigma}$) is defined at the half-integer levels. With this staggering of the variables, the simplest centered differencing will be used whenever appropriate. The finite-difference form of the continuity equation (2.6) is then written as

$$\frac{\partial\pi}{\partial t} + \nabla\cdot(\pi\mathbf{v}_l) + \left[\frac{\delta(\pi\dot{\sigma})}{\delta\sigma}\right]_l = 0. \quad (3.2)$$

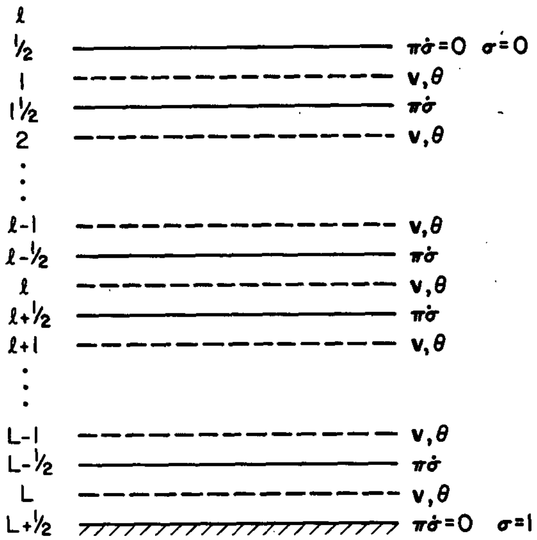

FIG. 1. The vertical grid, showing the indexing convention and the staggering of the variables.

Here and where there is no danger of confusion it is understood that the equation is applied to all layers ($l = 1, 2, \ldots, L$). From (3.2), with $\hat{\sigma}_{1/2} = 0$, $\hat{\sigma}_{L+1/2} = 1$ and $(\pi\dot{\sigma})_{1/2} = (\pi\dot{\sigma})_{L+1/2} = 0$, we obtain

$$\frac{\partial\pi}{\partial t} = -\nabla\cdot\sum_{l=1}^{L}(\pi\mathbf{v}\delta\sigma)_l \tag{3.3}$$

and

$$(\pi\dot{\sigma})_{l+1/2} = -\nabla\cdot\sum_{l'=1}^{l}(\pi\mathbf{v}\delta\sigma)_{l'} - \hat{\sigma}_{l+1/2}\frac{\partial\pi}{\partial t}. \tag{3.4}$$

These are finite-difference analogues of (2.7) and (2.8). From (3.3) and the definition $\pi \equiv p_S - p_I$, where p_I is constant, it is easy to see that the global integral of mass is conserved.

As mentioned in Section 1, we will consider only schemes that satisfy constraints (I) and (II). To satisfy (I), the pressure gradient force is differenced from the form given by (2.17) as

$$-(\nabla_p\phi)_l = -\frac{1}{\pi}\nabla(\pi\phi_l) + \frac{1}{\pi}\left[\frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l\nabla\pi. \tag{3.5}$$

Multiplying by $\pi(\delta\sigma)_l$, summing over l, and using $\hat{\phi}_{L+1/2} = \phi_S$ gives a discrete form of (2.18):

$$-\sum_{l=1}^{L}(\pi\delta\sigma\nabla_p\phi)_l = -\nabla\Big[\sum_{l=1}^{L}(\pi\phi\delta\sigma)_l - \phi_S\pi\Big] - \pi\nabla\phi_S. \tag{3.6}$$

Since the first term on the right-hand side is a gradient vector, no matter how $\hat{\phi}$ and ϕ are related to the prognostic thermodynamic variable, (3.5) is the most general discrete form of the pressure gradient force that satisfies constraint (I). Eq. (3.5) may also be written as

$$-(\nabla_p\phi)_l = -\nabla\phi_l - \frac{1}{\pi}\left[\phi - \frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l\nabla\pi, \tag{3.7}$$

which is a discrete form of (2.16).

To obtain the kinetic energy generation by the pressure gradient force we multiply (3.7) by $\pi\mathbf{v}_l$. Some manipulation parallel to the derivation of (2.19) gives

$$\begin{aligned}-(\pi\mathbf{v}\cdot\nabla_p\phi)_l = {} & -\nabla\cdot(\pi\mathbf{v}\phi)_l - \left[\frac{\delta\{(\pi\dot{\sigma} + \sigma\partial\pi/\partial t)\phi\}}{\delta\sigma}\right]_l \\ & - \left[\phi - \frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi \\ & - \frac{1}{(\delta\sigma)_l}[(\pi\dot{\sigma})_{l+1/2}(\phi_l - \hat{\phi}_{l+1/2}) \\ & + (\pi\dot{\sigma})_{l-1/2}(\hat{\phi}_{l-1/2} - \phi_l)]. \end{aligned} \tag{3.8}$$

Comparing (3.8) with (2.19), the last two terms can be seen to represent a discrete form of $\pi\omega\alpha$, as given by (2.20). To conserve total energy, then, the discrete form of the thermodynamic equation (2.25) must be

$$\begin{aligned}\frac{\partial}{\partial t}(\pi c_pT_l) + \nabla\cdot(\pi\mathbf{v}_lc_pT_l) + \left[\frac{\delta(\pi\dot{\sigma}c_pT)}{\delta\sigma}\right]_l \\ = \left[\phi - \frac{\delta(\sigma\phi)}{\delta\sigma}\right]_l\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi \\ + \frac{1}{(\delta\sigma)_l}[(\pi\dot{\sigma})_{l+1/2}(\phi_l - \hat{\phi}_{l+1/2}) \\ + (\pi\dot{\sigma})_{l-1/2}(\hat{\phi}_{l-1/2} - \phi_l)]. \end{aligned} \tag{3.9}$$

We note that at this stage no specific discrete form of the hydrostatic equation has been used. Eqs. (3.7) and (3.9) thus represent a broad family of schemes for the pressure gradient force and the thermodynamic equation that satisfy constraints (I) and (II). We will consider two ways to select a scheme from among this family:

1) By specifying a finite-difference scheme for the thermodynamic equation that has some desired properties, such as satisfying integral constraints (III) and (IV), and then determining the discrete form of the hydrostatic equation by requiring that the specified scheme become identical to (3.9).

2) By directly specifying a finite-difference scheme for the hydrostatic equation that satisfies some criterion of local accuracy, and then determining the discrete form of the thermodynamic equation through (3.9).

Arakawa (1972) and Arakawa and Lamb (1977) followed the first approach, which we will review briefly in the next section. The second approach will be discussed in later sections.

4. Potential temperature conserving schemes

Corresponding to (2.22) we adopt the following discrete form of the thermodynamic equation:

$$\frac{\partial}{\partial t}(\pi\theta_l) + \nabla\cdot(\pi \mathbf{v}_l\theta_l) + \left[\frac{\delta(\pi\dot{\sigma}\theta)}{\delta\sigma}\right]_l = 0. \quad (4.1)$$

Using the continuity equation (3.2), Eq. (4.1) can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\theta_l + \frac{1}{\pi(\delta\sigma)_l}[(\pi\dot{\sigma})_{l+1/2}(\hat{\theta}_{l+1/2} - \theta_l) + (\pi\dot{\sigma})_{l-1/2}(\theta_l - \hat{\theta}_{l-1/2})] = 0, \quad (4.2)$$

which is a finite-difference analogue of (2.21).

Multiplying (4.1) by $(\delta\sigma)_l$, summing over l, and applying the boundary conditions $(\pi\dot{\sigma})_{1/2} = (\pi\dot{\sigma})_{L+1/2} = 0$, we can see that constraint (III) is satisfied no matter how the half-integer level potential temperatures, $\hat{\theta}_{l+1/2}$ for $l = 1, \ldots, L-1$, are related to the prognostically determined integer level potential temperatures θ_l for $l = 1, \ldots, L$. Further constraints may be applied through the choice of $\hat{\theta}$. When $\hat{\theta}$ is a proper interpolation of θ in the sense that

$$\hat{\theta}_{l-1/2} = \hat{\theta}_{l+1/2} = \Theta \quad \text{when}$$
$$\theta_{l-1} = \theta_l = \theta_{l+1} = \Theta, \quad (4.3)$$

where Θ is a constant, Eq. (4.2) gives $\partial\theta_l/\partial t = 0$. So that a three-dimensionally isentropic atmosphere remains isentropic, as is the case with the continuous equation. More specifically, the choice of $\hat{\theta}$ can be used to maintain conservation of some function of the potential temperature. Arakawa and Lamb (1977, p. 222)[2] showed that the choice

$$\hat{\theta}_{l+1/2} = \frac{(F'_{l+1}\theta_{l+1} - F_{l+1}) - (F'_l\theta_l - F_l)}{F'_{l+1} - F'_l}, \quad (4.4)$$

where $F(\theta)$ is an arbitrary function of θ and $F'(\theta) \equiv dF/d\theta$, leads to conservation of the global mass integral of $F(\theta)$ [constraint (IV)]. Schemes generated using (4.4) also satisfy the condition (4.3) in the limit as $\theta_{l+1} - \theta_l \to 0$. As an example we take $F(\theta) = \theta^2$. Eq. (4.4) then gives

$$\hat{\theta}_{l+1/2} = \tfrac{1}{2}(\theta_{l+1} + \theta_l), \quad (4.5)$$

which leads to θ^2 conservation.

Once $F(\theta)$ is chosen, Eqs. (4.1) and (4.4) completely determine the scheme for the thermodynamic equation. Satisfying total energy conservation then requires that the hydrostatic equation relate potential temperatures and geopotentials in such a way that (4.1) may be written in the form (3.9). To obtain that form of the hydrostatic equation we define layer values of the temperature as

$$T_l = \theta_l P_l, \quad (4.6)$$

where P_l is an as yet unspecified form of $(p/p_0)^\kappa$ for the layer l. At this point we assume only that P_l can be expressed in terms of $\hat{\sigma}_{l-1/2}$ and $\hat{\sigma}_{l+1/2}$, which are prescribed constants, and the variable π. Then as Arakawa and Lamb (1977) showed, (4.2) has the same form as (3.9) when

$$\left[\phi - \frac{\delta(\phi\sigma)}{\delta\sigma}\right]_l = \pi c_p\left(\theta\frac{dP}{d\pi}\right)_l \quad (4.7)$$

and

$$\phi_l - \hat{\phi}_{l+1/2} = c_p(\hat{T}_{l+1/2} - P_l\hat{\theta}_{l+1/2}),$$
$$l = 1, 2, \ldots, L-1, \quad (4.8)$$
$$\hat{\phi}_{l-1/2} - \phi_l = c_p(P_l\hat{\theta}_{l-1/2} - \hat{T}_{l-1/2}),$$
$$l = 2, 3, \ldots, L. \quad (4.9)$$

Thus constraints (I)–(IV) are satisfied independently of the choice of P_l.

Substitution of (4.7) in the pressure gradient force (3.7) gives

$$-(\nabla_p\phi)_l = -\nabla\phi_l - c_p\left(\theta\frac{dP}{d\pi}\right)_l\nabla\pi. \quad (4.10)$$

Since $\hat{\phi}$ and $\hat{T}$ do not appear in (4.10), it is convenient to eliminate them between (4.8) and (4.9). Incrementing l by one in (4.9) and adding it to (4.8) yields

$$\phi_l - \phi_{l+1} = c_p(P_{l+1} - P_l)\hat{\theta}_{l+1/2},$$
$$l = 1, 2, \ldots, L-1, \quad (4.11)$$

which is a finite-difference analogue of (2.13). To determine ϕ_L, Eqs. (4.7) and (4.11) are substituted into

$$\phi_L - \phi_S = \sum_{l=1}^{L}[\phi\delta\sigma - \delta(\sigma\phi)]_l - \sum_{l=1}^{L-1}\hat{\sigma}_{l+1/2}(\phi_l - \phi_{l+1}), \quad (4.12)$$

which is an identity since $\hat{\sigma}_{1/2} = 0$, $\hat{\sigma}_{L+1/2} = 1$ and $\phi_S \equiv \hat{\phi}_{L+1/2}$. Then we obtain

$$\phi_L - \phi_S = \sum_{l=1}^{L}\pi c_p\left[\theta\frac{dP}{d\pi}\delta\sigma\right]_l - \sum_{l=1}^{L-1}c_p(P_{l+1} - P_l)(\hat{\sigma}\hat{\theta})_{l+1/2}. \quad (4.13)$$

When the expression for $\hat{\theta}_{l+1/2}$ in terms of θ_l and θ_{l+1} and that for P_l in terms of $\hat{\sigma}_{l+1/2}$, $\hat{\sigma}_{l-1/2}$ and π are specified, Eqs. (4.13) and (4.11) determine ϕ_l and the pressure gradient force (4.10) for all layers.

[2] In Arakawa and Lamb (1977), k, $k+1$ and $k+2$ were used in place of l, $l + \tfrac{1}{2}$ and $l+1$.

Assuming that $\hat{\theta}_{l+1/2}$ is taken from (4.4) to satisfy some form of (IV), there remains only to choose P_l. Phillips (1974) considered the form

$$P_l = \frac{1}{1+\kappa}\left[\frac{\delta(pP)}{\delta p}\right]_l, \qquad (4.14)$$

where $\hat{P} \equiv (\hat{p}/p_0)^\kappa$. Use of the relation $(\delta p)_l = \pi(\delta\sigma)_l$ and differentiation of (4.14) with respect to π yield

$$\pi\left(\frac{dP}{d\pi}\delta\sigma\right)_l = [\delta(P\sigma) - P\delta\sigma]_l. \qquad (4.15)$$

The thickness between level L and the surface, given by (4.13), depends on all L values of θ_l, so that the scheme is non-local. Substituting (4.5), (4.14) and (4.15) into (4.13), using $\hat{\sigma}_{1/2} = 0$, $\hat{\sigma}_{L+1/2} = 1$ and $\hat{P}_{L+1/2} = P_S$, and rearranging terms, we obtain, for the θ^2-conserving scheme

$$(\phi_L - \phi_S)g^{-1} = \mathcal{A} + \mathcal{B}, \qquad (4.16)$$

where

$$\mathcal{A} \equiv c_p\theta_L g^{-1}(P_S - P_L), \qquad (4.17)$$

$$\mathcal{B} \equiv \sum_{l=1}^{L-1} c_p(\theta_l - \theta_{l+1})g^{-1} \times [\hat{P}_{l+1/2} - \tfrac{1}{2}(P_l + P_{l+1})]\hat{\sigma}_{l+1/2}. \qquad (4.18)$$

As Tokioka (1978) showed, one of the advantages of the choice (4.14) is that the hydrostatic equation (4.13) becomes exact for isentropic atmospheres, as can easily be seen from (4.16) in the case of the θ^2-conserving scheme. When $\theta_l = \Theta$ for $l = 1, \ldots, L$, where Θ is a constant, $\mathcal{B}$ in (4.16) vanishes, and $\mathcal{A}$ gives the true value of ϕ_L for an isentropic atmosphere, $\phi_L \equiv \phi(P_L) = c_p\Theta(P_S - P_L)$.

But for non-isentropic atmospheres, $\mathcal{B}$ in (4.16), which depends on the differences of θ between all pairs of adjacent layers, can be quite large. As an example, let us consider an isothermal atmosphere, for which the true value of $(\phi_L - \phi_S)g^{-1}$ is given by

$$\mathcal{C} \equiv c_p T g^{-1}\ln(P_S/P_L), \qquad (4.19)$$

and a vertical grid which has equal $\delta\sigma$. The upper panel of Fig. 2 shows the error of $(\phi_L - \phi_S)g^{-1}$ for $p_I = 0$ when $\mathcal{B}$ is included, $\mathcal{A} + \mathcal{B} - \mathcal{C}$; and that when $\mathcal{B}$ is excluded, $\mathcal{A} - \mathcal{C}$. Here $c_pTg^{-1} = 30$ km has been used. The most striking aspect in this figure is that as the number of layers, L, increases, $\mathcal{A} - \mathcal{C}$ rapidly decreases, but $\mathcal{A} + \mathcal{B} - \mathcal{C}$ does not. As $L \to \infty$, $\mathcal{A} - \mathcal{C}$ approaches zero, but $\mathcal{A} + \mathcal{B} - \mathcal{C}$ approaches a large finite value of ~690 m. This means that the scheme (4.13) is not convergent.

The situation improves when p_I is not zero. The lower panel of Fig. 2 shows the case when $p_I = 100$ mb. Here $\mathcal{A} + \mathcal{B} - \mathcal{C}$ approaches 0 as $L \to \infty$, and therefore the scheme is convergent. But its rate of convergence is much slower than $\mathcal{A} - \mathcal{C}$. For $L = 10$, for example, $\mathcal{A} = 407.7$ m, $\mathcal{B} = 55.7$ m and $\mathcal{C} = 405.0$ m. This shows that $\mathcal{A}$ is a good approximation to $\mathcal{C}$, but $\mathcal{A} + \mathcal{B}$ is not.

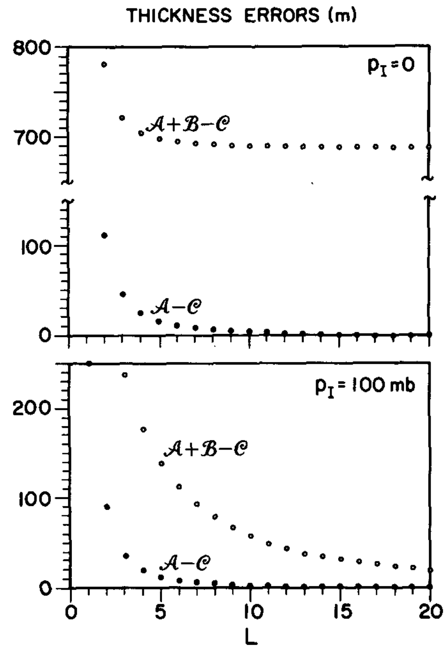

FIG. 2. Errors in the height of the lowest layer for an isothermal atmosphere. $\mathcal{C}$ is the exact value, obtained by integrating the hydrostatic equation. $\mathcal{A}$ and $\mathcal{B}$ are the two terms that appear in the finite-difference hydrostatic equation of Arakawa and Lamb (1972). $\mathcal{B}$ contains the non-local contributions to the lowest height. L is the number of layers between 1000 mb and p_I. Note that for $p_I = 0$ the lowest thickness will not tend to zero as the vertical resolution is increased.

From the above example it is clear that the non-local dependence of $\mathcal{B}$ in (4.16) can introduce a large error in $\phi_L - \phi_S$, and therefore in ϕ_L. Still, a large error due to $\mathcal{B}$ is not serious if it is horizontally uniform and does not affect the pressure gradient force. But $\mathcal{B}$ can vary significantly as a result of horizontal gradients of temperature, lapse rate, or surface pressure, and thereby produce errors in the pressure gradient force in sigma coordinates.

In the following sections we explore what global constraints may be applied without giving up the local accuracy in the hydrostatic equation. We proceed by seeking schemes for which the lowest thickness, $\phi_L - \phi_S$, depends only on θ_L, without placing any restriction on the thermodynamic equation beyond satisfying (3.9).

5. General schemes with local differencing of the hydrostatic equation

In this section we derive a family of schemes by using the approach (2) mentioned in Section 3. We begin by taking the following general forms for the hydrostatic equation:

$$\phi_l - \phi_{l+1} = c_p(A_{l+1/2}\theta_l + B_{l+1/2}\theta_{l+1}),\quad l = 1, 2, \ldots, L-1, \quad (5.1)$$

and

$$[\phi\delta\sigma - \delta(\sigma\phi)]_l = c_p C_l \theta_l, \quad (5.2)$$

where the coefficients A, B and C are assumed to depend only on the $\hat{\sigma}$'s and π. We note that (5.1) and (5.2) include as special cases the θ^2-conserving scheme considered in Section 4, for which $A_{l+1/2} = B_{l+1/2} = \frac{1}{2}(P_{l+1} - P_l)$ and $C_l = (\delta\sigma)_l\pi(dP/d\pi)_l$ [see (4.5), (4.7) and (4.11)].

For the remainder of the paper, however, we will consider only a restricted family of schemes for which $\phi_L - \phi_S$ depends only on θ_L, and not on the potential temperature in the other layers. Substituting (5.1) and (5.2) into the identity (4.12) and rearranging terms, we obtain

$$\phi_L - \phi_S = c_p\theta_L(C_L - \hat{\sigma}_{L-1/2}B_{L-1/2}) + \sum_{l=1}^{L-1} c_p\theta_l(C_l - \hat{\sigma}_{l+1/2}A_{l+1/2} - \hat{\sigma}_{l-1/2}B_{l-1/2}). \quad (5.3)$$

Here the form of $B_{1/2}$ does not matter since $\hat{\sigma}_{1/2} = 0$. The dependency of $\phi_L - \phi_S$ on θ_l for $l < L$ can be eliminated by choosing

$$C_l = \hat{\sigma}_{l+1/2}A_{l+1/2} + \hat{\sigma}_{l-1/2}B_{l-1/2},\quad l = 1, 2, \ldots, L-1. \quad (5.4)$$

Using (5.4) in (5.3) gives

$$\phi_L - \phi_S = c_p A_{L+1/2}\theta_L, \quad (5.5)$$

where $A_{L+1/2}$ has been defined by arbitrarily applying (5.4) at $l = L$. With this definition and (5.4), (5.2) may be written for all layers as

$$\hat{\sigma}_{l+1/2}(\phi_l - \hat{\phi}_{l+1/2}) + \hat{\sigma}_{l-1/2}(\hat{\phi}_{l-1/2} - \phi_l) = c_p(\hat{\sigma}_{l+1/2}A_{l+1/2} + \hat{\sigma}_{l-1/2}B_{l-1/2})\theta_l. \quad (5.6)$$

Once $A_{l+1/2}$ for $l = 1, 2, \ldots, L$ and $B_{l+1/2}$ for $l = 1, 2, \ldots, L-1$ are specified, (5.1) and (5.5) can be used to obtain ϕ_l for $l = 1, 2, \ldots, L$, and (5.6) to obtain $\hat{\phi}_{l-1/2}$ for $l = 1, 2, \ldots, L$.

Some authors (e.g. Tokioka, 1978; Simmons and Burridge, 1981) prefer to first obtain $\hat{\phi}$ from a hydrostatic equation written for the layer thickness, $\hat{\phi}_{l-1/2} - \hat{\phi}_{l+1/2}$, and then interpolate for ϕ_l. We can show that the hydrostatic equation in the above formulation can also be written in this way. If we consider (5.1), (5.5) and (5.6) a set of equations for the quantities $\phi_l - \hat{\phi}_{l+1/2}$ and $\hat{\phi}_{l-1/2} - \phi_l$ for $l = 1, 2, \ldots, L$, the solutions are

$$\phi_l - \hat{\phi}_{l+1/2} = c_p A_{l+1/2}\theta_l, \quad (5.7)$$

$$\hat{\phi}_{l-1/2} - \phi_l = c_p B_{l-1/2}\theta_l, \quad (5.8)$$

as can be easily verified by substitution. From these two relations we then obtain

$$\hat{\phi}_{l-1/2} - \hat{\phi}_{l+1/2} = c_p(A_{l+1/2} + B_{l-1/2})\theta_l, \quad (5.9)$$

$$\phi_l = \frac{A_{l+1/2}\hat{\phi}_{l-1/2} + B_{l-1/2}\hat{\phi}_{l+1/2}}{A_{l+1/2} + B_{l-1/2}}, \quad (5.10)$$

which have the desired form.

Using (5.2) and (5.4), the pressure gradient force as given by (3.7) becomes

$$-(\nabla_p\phi)_l = -\nabla\phi_l - \frac{c_p}{\pi}\frac{\theta_l}{(\delta\sigma)_l} \times (\hat{\sigma}_{l+1/2}A_{l+1/2} + \hat{\sigma}_{l-1/2}B_{l-1/2})\nabla\pi. \quad (5.11)$$

And substitution of (5.2), (5.4), (5.7) and (5.8) into the thermodynamic equation (3.9) gives

$$\frac{\partial}{\partial t}(\pi c_p T_l) + \nabla\cdot(\pi \mathbf{v}_l c_p T_l) + \left[\frac{\delta(\pi\dot{\sigma}c_p\hat{T})}{\delta\sigma}\right]_l = c_p\frac{\theta_l}{(\delta\sigma)_l}\left[A_{l+1/2}\left\{\hat{\sigma}_{l+1/2}\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi + (\pi\dot{\sigma})_{l+1/2}\right\} + B_{l-1/2}\left\{\hat{\sigma}_{l-1/2}\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi + (\pi\dot{\sigma})_{l-1/2}\right\}\right]. \quad (5.12)$$

Eqs. (5.11), (5.12) and either (5.1) and (5.5) or (5.9) and (5.10) represent a fairly general family of schemes that satisfy constraints (I) and (II) and have a completely local form of the hydrostatic equation. In these equations the coefficients A and B, P that relates θ and T through $\theta = T/P$, and $\hat{T}$ that enters through the third term of the left hand side of (5.12) may be freely chosen. In the next two sections, we discuss how these freedoms may be effectively used.

6. Other constraints on the pressure gradient force

It is well known that in σ-coordinates the pressure gradient force, which appears as the sum of two large compensating terms, can be subject to serious discretization errors. In this section we will use some of the freedom available in the choice of A, B and P to reduce such errors. Following an approach used by many authors (e.g., Corby *et al.*, 1972; Nakamura, 1978; Phillips, 1974; Sundqvist, 1976; Simmons and Burridge, 1981), we will evaluate the accuracy of the discrete pressure gradient force by examining its errors for three-dimensionally hydrostatic atmospheres, in which the temperature is a function of pressure only.

Let $\phi = \Phi(P)$ be the geopotential of such an atmosphere, where $\Phi(P)$ is a function of $P \equiv (p/p_0)^\kappa$ only, so that $\nabla_p\Phi = 0$. *If the finite-difference hydrostatic equation that relates ϕ_l's to θ_l's is exact for that atmosphere*, the following replacements can be made in the discrete pressure gradient force (5.11),

$$\phi_l \rightarrow (\Phi)_{P=P_l}, \quad (6.1)$$

$$\theta_l \longrightarrow -\frac{1}{c_p}\left(\frac{d\Phi}{dP}\right)_{P=P_l}. \tag{6.2}$$

Since

$$\nabla(\Phi)_{P=P_l} = \left(\frac{d\Phi}{dP}\right)_{P=P_l}\left(\frac{dP_l}{d\pi}\right)\nabla\pi, \tag{6.3}$$

the requirement that $(\nabla_p\phi_l)$ vanish becomes, after dropping the common factor $(d\Phi/dP)_{P=P_l}\nabla\pi$,

$$\frac{dP_l}{d\pi} = \frac{1}{\pi(\delta\sigma)_l}(\hat{\sigma}_{l+1/2}A_{l+1/2} + \hat{\sigma}_{l-1/2}B_{l-1/2}). \tag{6.4}$$

For given $A_{l+1/2}$ and $B_{l-1/2}$, Eq. (6.4) can be integrated for P_l. If, however, the given forms of $A_{l+1/2}$ and $B_{l-1/2}$ involve P with an integer subscript other than l, Eq. (6.4) for $l = 1, 2, \ldots, L$ form a set of coupled ordinary differential equations that is not easily integrable. We wish to require that this not be the case. This "integrability" requirement, however, need not be applied directly to (6.4). Let $f(P)$ be an arbitrary function of P, $f'(P) \equiv df(P)/dP$, $f_l \equiv f(P_l)$ and $f_l' \equiv f'(P_l)$. Then, from (6.4),

$$\frac{df_l}{d\pi} = \frac{f_l'}{\pi(\delta\sigma)_l}(\hat{\sigma}_{l+1/2}A_{l+1/2} + \hat{\sigma}_{l-1/2}B_{l-1/2}) \tag{6.5}$$

or

$$\frac{d(\pi f_l)}{d\pi} = \frac{1}{(\delta\sigma)_l} \times [\hat{\sigma}_{l+1/2}(f_l + f_l'A_{l+1/2}) - \hat{\sigma}_{l-1/2}(f_l - f_l'B_{l-1/2})]. \tag{6.6}$$

We now require that (6.6) be easily integrable by choosing $A_{l+1/2}$ and $B_{l-1/2}$ so that the right-hand side does not depend explicitly on the unknown f_l's. Guided by the differential relation

$$\left[\frac{\partial(\pi f)}{\partial\pi}\right]_\sigma = \left[\frac{\partial(\sigma f)}{\partial\sigma}\right]_\pi, \tag{6.7}$$

valid for any function f of P only (and, therefore, of $p = p_I + \pi\sigma$ only), we choose

$$A_{l+1/2} = \frac{\hat{f}_{l+1/2} - f_l}{f_l'}, \tag{6.8}$$

$$B_{l-1/2} = \frac{f_l - \hat{f}_{l-1/2}}{f_l'}, \tag{6.9}$$

where $\hat{f} = f(\hat{P})$ with $\hat{P} \equiv (\hat{p}/p_0)^\kappa$, and $\hat{p} = p_I + \pi\hat{\sigma}$. Then (6.6) becomes

$$\frac{d(\pi f_l)}{d\pi} = \left[\frac{\delta(\sigma f)}{\delta\sigma}\right]_l, \tag{6.10}$$

which is now easily integrable with respect to π. Performing the integration and subsequently using $\hat{\sigma}d\pi = d\hat{p}$ and $\pi\delta\sigma = \delta p$, we obtain

$$f_l = \frac{1}{\pi}\left[\frac{\delta(\sigma\int f d\pi)}{\delta\sigma}\right]_l \tag{6.11a}$$

$$= \left[\frac{\delta\int f dp}{\delta p}\right]_l. \tag{6.11b}$$

Then P_l can be determined from $f(P_l) = f_l$.

When a specific form is chosen for the function $f(P)$, the finite difference scheme for the hydrostatic equation and the pressure gradient force will be fully determined. For now, we write the equations in terms of $f(P)$. Eqs. (5.5) and (5.1) become

$$\phi_L - \phi_S = c_p(f_S - f_L)\left(\frac{\theta}{f'}\right)_L, \tag{6.12}$$

$$\phi_l - \phi_{l+1} = c_p\left[(\hat{f}_{l+1/2} - f_l)\left(\frac{\theta}{f'}\right)_l + (f_{l+1} - \hat{f}_{l+1/2})\left(\frac{\theta}{f'}\right)_{l+1}\right] \quad \text{for} \quad l = 1, 2, \ldots, L-1, \tag{6.13}$$

where $f' \equiv df/dP$, as previously defined. And (5.11) becomes

$$-(\nabla_p\phi)_l = -\nabla\phi_l - c_p\left(\frac{\theta}{f'}\right)_l\nabla f_l. \tag{6.14}$$

From the continuous hydrostatic equation (2.13) rewritten in the form $-d\phi/df = c_p\theta/f'$, we can easily see that (6.12) and (6.13) are exact for atmospheres with a constant θ/f', so that the replacements (6.1) and (6.2) are in fact possible for such atmospheres. It then follows that (6.14), with f_l given by (6.11), vanishes when θ/f' is constant.

The thermodynamic equation (5.12) may now be written as

$$\frac{\partial}{\partial t}(\pi c_pT_l) + \nabla\cdot(\pi\mathbf{v}_lc_pT_l) + \left[\frac{\delta(\pi\dot{\sigma}c_p\hat{T})}{\delta\sigma}\right]_l = \frac{c_p}{(\delta\sigma)_l}\left(\frac{\theta}{f'}\right)_l \times \left[(\hat{f}_{l+1/2} - f_l)\left\{\hat{\sigma}_{l+1/2}\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi + (\pi\dot{\sigma})_{l+1/2}\right\} + (f_l - \hat{f}_{l-1/2})\left\{\hat{\sigma}_{l-1/2}\left(\frac{\partial}{\partial t} + \mathbf{v}_l\cdot\nabla\right)\pi + (\pi\dot{\sigma})_{l-1/2}\right\}\right], \tag{6.15}$$

where $\hat{T}$ also remains to be specified.

Later we will propose a rationale for determining $f(P)$ and $\hat{T}$, but first let us see how the choice of f affects the pressure gradient force (6.14). The following equation, which can be derived through some manipulation of (6.10) and (6.11a), including integration by parts, will be useful:

$$\nabla f_l = \left[\frac{\delta\left(\sigma \int \pi \frac{df}{d\pi} d\pi\right)}{\pi\delta\sigma}\right]_l \nabla \ln\pi. \quad (6.16)$$

When $p_l = 0$, we have $\hat{p} = \pi\hat{\sigma}$, and (6.16) may then be rewritten as

$$\nabla f_l = \left[\frac{\delta \int p \frac{df}{dp} dp}{\delta p}\right]_l \nabla \ln\pi. \quad (6.17)$$

The equation for the vertical difference of the pressure gradient force for the above family of schemes can be derived from (6.14) and (6.13),

$$(\nabla_p\phi)_l - (\nabla_p\phi)_{l+1} = c_p\left[(\hat{f}_{l+1/2} - f_l)\nabla\left(\frac{\theta}{f'}\right)_l + (f_{l+1} - \hat{f}_{l+1/2})\nabla\left(\frac{\theta}{f'}\right)_{l+1} + \left\{\left(\frac{\theta}{f'}\right)_l - \left(\frac{\theta}{f'}\right)_{l+1}\right\}\nabla\hat{f}_{l+1/2}\right]. \quad (6.18)$$

It is easy to see that the right-hand side of (6.18) vanishes when $(\theta/f')_l = (\theta/f')_{l+1} =$ constant, as expected. In addition when $p_l = 0$, so that (6.17) holds, it is possible to choose $f(P)$ to make the right-hand side of (6.18) vanish for more general atmospheres. By way of example we consider the following two choices, which are of special interest.

a. $f = \ln p$

A member of the family of schemes presented in this section is the scheme proposed by Simmons and Burridge (1981), which can be obtained by choosing $f = \ln p$ $(=\kappa^{-1} \ln P +$ constant). Then, from $f' \equiv df/dP$ and $T = P\theta$, we obtain $\theta/f' = \kappa T$, and therefore, the finite-difference hydrostatic equations (6.12) and (6.13) are exact for isothermal atmospheres. Eqs. (6.12), (6.13) and (6.14) become

$$\phi_L - \phi_S = RT_l \ln\left(\frac{p_S}{p_L}\right), \quad (6.19)$$

$$\phi_l - \phi_{l+1} = R\left[T_l \ln\left(\frac{\hat{p}_{l+1/2}}{p_l}\right) + T_{l+1} \ln\left(\frac{p_{l+1}}{\hat{p}_{l+1/2}}\right)\right]$$

$$\text{for} \quad l = 1, 2, \ldots, L-1, \quad (6.20)$$

$$-(\nabla_p\phi)_l = -\nabla\phi_l - RT_l\nabla \ln p_l, \quad (6.21)$$

while (6.11b) gives

$$\ln p_l = \left[\frac{\delta(p \ln p - p)}{\delta p}\right]_l. \quad (6.22)$$

When $p_l = 0$ (i.e., $\pi = p_S$), $f = \ln p$ gives

$$\nabla\hat{f}_{l+1/2} = \nabla \ln p_S, \quad (6.23)$$

and also, from (6.17),

$$\nabla f_l = \nabla \ln p_S. \quad (6.24)$$

We then find, as Simmons and Burridge (1981) pointed out, that the right-hand side of (6.18) vanishes when $\theta/f' = \kappa(a + bf)$, and therefore, $T = a + b \ln p$, where a and b are arbitrary constants.

b. $f = P$

Another member of this family of schemes can be obtained by choosing $f = P$. Then $\theta/f' = \theta$, and therefore, the finite-difference hydrostatic equations (6.12) and (6.13) are exact for isentropic atmospheres. Eqs. (6.12), (6.13) and (6.14) become

$$\phi_L - \phi_S = c_p\theta_L(P_S - P_L), \quad (6.25)$$

$$\phi_l - \phi_{l+1} = c_p[(\hat{P}_{l+1/2} - P_l)\theta_l + (P_{l+1} - \hat{P}_{l+1/2})\theta_{l+1}]$$

$$\text{for} \quad l = 1, 2, \ldots, L-1, \quad (6.26)$$

$$-(\nabla_p\phi)_l = -\nabla\phi_l - c_p\theta_l\nabla P_l, \quad (6.27)$$

while (6.11b) gives

$$P_l = \frac{1}{1+\kappa}\left[\frac{\delta(pP)}{\delta p}\right]_l. \quad (6.28)$$

This P_l is identical to that proposed by Phillips (1974) given by (4.14). When $p_l = 0$, $f = P$ gives

$$\nabla\hat{f}_{l+1/2} = \kappa\hat{f}_{l+1/2}\nabla \ln p_S, \quad (6.29)$$

and also, from (6.17) and (6.28),

$$\nabla f_l = \kappa f_l\nabla \ln p_S. \quad (6.30)$$

We then find that the right-hand side of (6.18) vanishes when $\theta/f' = a + b/f$, and therefore, $\theta = a + b/P$, where a and b are arbitrary constants. Since $\theta = T/P$, $\theta = a + b/P$ means $T = b + aP$, so that it includes both isothermal and isentropic atmospheres.

As we pointed out earlier, $\hat{T}$ on the left-hand side of (6.15) remains free to be chosen. Simmons and Burridge more or less arbitrarily chose $\hat{T}_{l+1/2} = ½(T_l + T_{l+1})$ for their scheme. Since, at least for the two examples presented above, the different choices of $f(P)$ lead to similar properties for the pressure gradient force, and since the choice of $\hat{T}_{l+1/2}$ can affect only the scheme for the thermodynamic equation, it seems reasonable to use these last two remaining choices to impose a further constraint on the thermodynamic equation. This is the subject of the next section.

7. Integral constraints on the thermodynamic equation

We feel that the minimum additional requirement on the thermodynamic equation is that a three-dimensionally isentropic atmosphere remain isentropic. This includes vanishing of the static stability when the potential temperature is constant in height. Using (3.2), $T = P\theta$ and (6.4), we may rewrite (5.12) as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_l \cdot \nabla\right)\theta_l = -\frac{1}{P_l}\frac{1}{\pi(\delta\sigma)_l}[(\pi\dot{\sigma})_{l+1/2}\{\hat{T}_{l+1/2} - \theta_l(A_{l+1/2} + P_l)\} + (\pi\dot{\sigma})_{l-1/2}\{\theta_l(P_l - B_{l-1/2}) - \hat{T}_{l-1/2}\}], \quad (7.1)$$

where, for convenience, we use $A_{l+1/2}$ and $B_{l-1/2}$ in place of the expressions in (6.8) and (6.9). Then the above requirement becomes that

$$\hat{T}_{l+1/2} - \theta_l(A_{l+1/2} + P_l) = 0, \quad (7.2)$$

$$\theta_l(P_l - B_{l-1/2}) - \hat{T}_{l-1/2} = 0. \quad (7.3)$$

for an isentropic atmosphere. Incrementing l in (7.3), adding it to (7.2), and using $\theta_{l+1} = \theta_l$, we obtain

$$A_{l+1/2} + B_{l+1/2} = P_{l+1} - P_l. \quad (7.4)$$

Since this relation is independent of θ, it must be viewed as a general constraint on A and B, and thus involves only the choice of $f(P)$. For $f = P$, (6.8) and (6.9) give

$$A_{l+1/2} = \hat{P}_{l+1/2} - P_l, \quad (7.5)$$

$$B_{l-1/2} = P_l - \hat{P}_{l-1/2}, \quad (7.6)$$

so that scheme (b) of Section 6 satisfies (7.4), while scheme (a), which gives $A_{l+1/2} + B_{l+1/2} = \ln(p_{l+1}/p_l)$, does not.

With the choice (7.5) and (7.6), Eq. (7.1) becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_l \cdot \nabla\right)\theta_l = -\frac{1}{\pi(\delta\sigma)_l}\left[(\pi\dot{\sigma})_{l+1/2}\frac{1}{P_l}(\hat{T}_{l+1/2} - \theta_l\hat{P}_{l+1/2}) + (\pi\dot{\sigma})_{l-1/2}\frac{1}{P_l}(\theta_l\hat{P}_{l-1/2} - \hat{T}_{l-1/2})\right]. \quad (7.7)$$

$\hat{T}_{l+1/2}$ may then be chosen so that (7.7) can be put in the flux form (4.1), thus guaranteeing conservation of the global mass integral of the potential temperature. But a more general formulation is possible. Let $G(\theta)$ be an arbitrary function of θ and $G' = dG/d\theta$. Multiplying (7.7) by G', we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_l \cdot \nabla\right)G_l = -\frac{1}{\pi(\delta\sigma)_l}\left[(\pi\dot{\sigma})_{l+1/2}\left(\frac{G'}{P}\right)_l(\hat{T}_{l+1/2} - \theta_l\hat{P}_{l+1/2}) + (\pi\dot{\sigma})_{l-1/2}\left(\frac{G'}{P}\right)_l(\theta_l\hat{P}_{l-1/2} - \hat{T}_{l-1/2})\right]. \quad (7.8)$$

If we then take

$$\hat{T}_{l+1/2} - \theta_l\hat{P}_{l+1/2} = \left(\frac{P}{G'}\right)_l(\hat{G}_{l+1/2} - G_l), \quad (7.9)$$

$$\theta_l\hat{P}_{l-1/2} - \hat{T}_{l-1/2} = \left(\frac{P}{G'}\right)_l(G_l - \hat{G}_{l-1/2}), \quad (7.10)$$

which may be considered as choices for the $\hat{T}$'s and the newly defined $\hat{G}$'s, Eq. (7.8) may be rewritten as

$$\frac{\partial}{\partial t}(\pi G_l) + \nabla \cdot (\pi \mathbf{v}_l G_l) + \left[\frac{\delta(\pi\dot{\sigma}G)}{\delta\sigma}\right]_l = 0. \quad (7.11)$$

Thus (7.9) and (7.10) result in conservation of the global mass integral of $G(\theta)$. The form of $\hat{G}_{l+1/2}$, which appears in (7.11), can be obtained by combining (7.9) and (7.10):

$$\hat{G}_{l+1/2} = \frac{\left(\frac{P}{G'}\right)_{l+1}G_{l+1} - \left(\frac{P}{G'}\right)_l G_l - \hat{P}_{l+1/2}(\theta_{l+1} - \theta_l)}{\left(\frac{P}{G'}\right)_{l+1} - \left(\frac{P}{G'}\right)_l}. \quad (7.12)$$

When, in particular, $G(\theta) \equiv \theta$, Eq. (7.11) becomes identical to (4.1) and constraint (III) is satisfied. From (7.12), the corresponding form of $\hat{\theta}_{l+1/2}$ is

$$\hat{\theta}_{l+1/2} = \frac{(P_{l+1} - \hat{P}_{l+1/2})\theta_{l+1} + (\hat{P}_{l+1/2} - P_l)\theta_l}{P_{l+1} - P_l}. \quad (7.13)$$

In summary, the choice $f(P) = P$ results in schemes for which an isentropic atmosphere remains isentropic and, in addition, allows the choice of $\hat{T}_{l+1/2}$ to be used to conserve any one function of θ. We have found no comparably effective use of the choice of $\hat{T}$ for other choices of $f(P)$.

8. Summary

Following the work of Arakawa (1972), we derived a vertical finite-difference scheme for the primitive equations in sigma coordinates by requiring that the discrete equations satisfy analogues of some important properties of the continuous equations. We considered a vertical grid with the distribution of variables shown in Fig. 1 and used straightforward centered differencing wherever it was naturally suggested by the staggering. Within this framework we attempted to maintain as much generality as possible.

We began by imposing requirement (I) of Section 1—that the pressure gradient force produce no vertically integrated circulation along a contour of the topography. A special case of (I) is that the scheme conserve vertically integrated angular momentum in the absence of mountains.

If after applying this constraint the scheme for the thermodynamic equation is specified from other considerations, the hydrostatic equation will be determined from requirement (II) of Section 1—that the resulting energy conversion term in the thermodynamic equation have the same form as that in the kinetic energy equation. When Arakawa (1972) did this by requiring that the global mass integrals of the potential temperature θ and its square be conserved, he obtained a non-local form of the hydrostatic equation for the thickness between the lowest layer and the surface. In Section 4 we showed that this non-local dependence can seriously affect the local accuracy of the hydrostatic equation. Therefore, we proceeded in Section 5 by imposing locality on the form of the hydrostatic equation.

We next considered in Section 6 the local accuracy of the pressure gradient force. The pressure gradient force in sigma coordinates is the sum of nearly compensating terms, one involving the gradient of the geopotential along a sigma surface and the other the gradient of the surface pressure, and can thus be subject to large discretization errors. Although such errors cannot be eliminated in general, they can be required to vanish for particular atmospheres. We followed this approach and defined a family of schemes, characterized primarily by a single function $f(P)$, where $P = (p/p_0)^\kappa$, for which the pressure gradient force vanishes when $\theta = adf/dP$, for any constant a. The scheme of Simmons and Burridge (1981) belongs to this family, corresponding to the choice $f = \ln p$. For this scheme and also for the one defined by $f = P$, the pressure gradient force also vanishes for more general atmospheres when the top of the model is at $p = 0$. For the Simmons and Burridge scheme it vanishes when $T = a + b \ln p$, and for the scheme with $f = P$ it vanishes when $T = a + bP$, where a and b are arbitrary constants.

As there appeared to be little guidance in the pressure gradient force for the choice of $f(P)$, we turned our attention in Section 7 to the thermodynamic equation. We found that among the choices we considered $f(P) = P$ is the only one that satisfies what we feel is the minimum constraint on the thermodynamic equation—that an isentropic atmosphere remain isentropic. What's more, $f(P) = P$ allowed us to use the other remaining freedom, the choice of the interpolated temperature that defines the vertical temperature flux, to conserve the global mass integral of any one function of θ, say $G(\theta)$.

Both $G(\theta) = \ln\theta$ and $G(\theta) = \theta$ seem reasonable choices. In the current UCLA general circulation model we are using $G(\theta) = \theta$, applying the scheme to the modified sigma coordinate described by Suarez and Arakawa (1979).[3]

For convenience, we repeat here the equations for the θ-conserving scheme. The thermodynamic equation is simply (4.1):

[3] In the modified sigma coordinate, p_S is replaced by the pressure at the top of the planetary boundary layer, p_B.

$$\frac{\partial}{\partial t}(\pi\theta_l) + \nabla\cdot(\pi\mathbf{v}_l\theta_l) + \left[\frac{\delta(\pi\dot{\sigma}\theta)}{\delta\sigma}\right]_l = 0. \qquad (8.1)$$

From (6.27), the pressure gradient force becomes

$$-(\nabla_p\phi)_l = -\nabla\phi_l - c_p\theta_l\frac{dP_l}{d\pi}\nabla\pi. \qquad (8.2)$$

And from (6.25) and (6.26) the hydrostatic equation is

$$\phi_L - \phi_S = c_p\theta_L(P_S - P_L) \qquad (8.3)$$

and

$$\phi_l - \phi_{l+1} = c_p\hat{\theta}_{l+1/2}(P_{l+1} - P_l). \qquad (8.4)$$

The form of $\hat{\theta}_{l+1/2}$ used in (8.1) and (8.4) is

$$\hat{\theta}_{l+1/2} = \frac{(\hat{P}_{l+1/2} - P_l)\theta_l + (P_{l+1} - \hat{P}_{l+1/2})\theta_{l+1}}{P_{l+1} - P_l}, \qquad (8.5)$$

where $\hat{P}_{l+1/2} = (\hat{p}_{l+1/2}/p_0)^\kappa$. And from (6.28) the required form of P_l is

$$P_l = \frac{1}{1+\kappa}\left[\frac{\hat{P}_{l+1/2}\hat{p}_{l+1/2} - \hat{P}_{l-1/2}\hat{p}_{l-1/2}}{\hat{p}_{l+1/2} - \hat{p}_{l-1/2}}\right]. \qquad (8.6)$$

Finally, from (6.4) or by differentiation of (8.6),

$$\frac{dP_l}{d\pi} = \frac{\hat{\sigma}_{l+1/2}(\hat{P}_{l+1/2} - P_l) + \hat{\sigma}_{l-1/2}(P_l - \hat{P}_{l-1/2})}{\hat{p}_{l+1/2} - \hat{p}_{l-1/2}}. \qquad (8.7)$$

To summarize, the scheme defined by (8.1)–(8.7) has the following properties:

- Constraints (I), (II) and (III) of Section 1 are satisfied.
- The hydrostatic equation for the lowest thickness has a local form.
- The hydrostatic equation is exact for vertically isentropic atmospheres.
- The pressure gradient force is exact for three-dimensionally isentropic atmospheres and, when the model top is at $p = 0$, for atmospheres for which $T = a + b(p/p_0)^\kappa$.

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