Using Hough Harmonics to Validate and Assess Nonlinear Shallow-Water Models

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ABSTRACT

The implementation of a technique for locating programming errors in shallow-water codes, establishing the correctness of the code, and assessing the performance of the numerical model under various flow conditions is described. The right-hand side of the differential equations is modified in such a way that the exact solution of the nonlinear initial-value problem is known, so that the truncation error of the numerical scheme can be studied in detail. The exact solution is prescribed to be any linear combination of Hough harmonics which propagate in time according to their natural frequencies.

1. Introduction

The first important test of a numerical scheme for the baroclinic primitive equations is usually an implementation and assessment of the scheme for a nonlinear shallow-water model. The relevance of such a test is due to the well-known correspondence between solutions of the shallow-water equations and various atmospheric flow phenomena (e.g., Pedlosky, 1979), as well as the fact that many of the computational difficulties which arise in fully three-dimensional models are also present in two-dimensional barotropic models. Developing a computer program for even integrating the shallow-water equations is a complex project, and a large portion of the total effort is spent locating programming errors and ascertaining the correctness of the program.

The most commonly used method for validating shallow-water models is to perform a series of experiments in which the spatial and temporal resolution is increased until the numerical solution appears to converge to a limit solution. Precise information about the truncation error of the scheme cannot be obtained in this manner, since the exact limit solution is not known. Some quantitative information about the performance of the scheme can be obtained by studying the convergence of functionals of the solution whose true values are known, such as total mass or energy. Unfortunately, the convergence rate of these functionals does not necessarily imply anything about pointwise convergence of the scheme, especially if the numerical scheme is contrived to conserve these functionals exactly (e.g., Arakawa and Lamb, 1977).

The purpose of this paper is to describe an efficient and flexible implementation of a technique for detecting and isolating programming errors in fully nonlinear, global, shallow-water codes. This technique enables one to establish the correctness of the code in a simple manner, without the need to perform a large number of costly high-resolution simulation runs. In addition, it can be used as a tool to analyze the performance of a particular numerical method under a variety of flow conditions. Our implementation is easily generalized for use with three-dimensional models; however, we will limit our discussion to shallow-water models.

The basic idea is to *prescribe* the exact solution of the nonlinear initial-value problem by adding appropriate forcing terms to the differential equations. The numerical solution can then be compared with the exact solution so as to verify the correctness of the code. By choosing the exact solution appropriately, individual terms in the equations can be isolated so that selected portions of the code are tested. By performing a sequence of forced simulations with different spatial and temporal increments, the order of accuracy of the scheme can be verified. Meaningful quantitative and qualitative information about the truncation error of the scheme is provided when the prescribed exact solution resembles a physically reasonable flow.

Variants of the prescribed-solution forcing technique have appeared previously in the meteorological literature (Merilees et al., 1977; Marchesin, 1984). Our implementation is especially appropriate for atmospheric models since we use Hough harmonics to define the

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prescribed solutions. Hough harmonics are the eigensolutions of the shallow-water equations on the sphere linearized about a state of rest, and they therefore form a natural basis in which to describe physically meaningful flows.

We developed a subroutine package which allows the user to prescribe the exact solution as any linear combination of Hough harmonics that propagate in time according to their natural frequencies. At each model time step, the package automatically computes the appropriate forcing terms, without truncation error, on an arbitrary spherical grid. The package, being entirely independent from the shallow-water code under scrutiny, can be used to test, analyze and compare any number of numerical schemes. It includes a code for computing Hough functions with some features that render it especially appropriate and efficient for the application described here.

Clearly, the implementation of a prescribed-solution forcing technique must be coded and tested with extreme care, so as to insure that errors in the numerical solutions are in fact due to the shallow-water code and not to the prescribed-solution code itself. The programming of the technique we describe in this paper is straightforward, with the exception of the calculation of the Hough functions which represent the meridional structure of the prescribed solutions. Software packages which compute Hough functions are readily available (Swarztrauber and Kasahara, 1985) and can be used to implement our technique. Alternatively, the computer code and documentation developed by the authors is available upon request (da Silva and Dee, 1985).

In section 2 we review the basic ideas behind prescribed-solution forcing and the use of Hough harmonics for this purpose. We describe our implementation of these ideas in section 3. Next, in section 4, we discuss briefly how to select prescribed solutions in order to detect coding errors and how to establish the correctness of the shallow-water code. In section 5 we show how the technique may be used to analyze the performance of the numerical scheme once the code has been proven correct. A summary appears in section 6.

2. Exact solutions and Hough harmonics

We may write the initial-value problem for the shallow-water equations as

$$\left.\begin{aligned} W_t + \mathcal{M}(W) &= 0 \\ W(t=0) &= W_0 \end{aligned}\right\}, \tag{2.1}$$

where W is the 3-vector of dependent variables and $\mathcal{M}$ is the partial differential operator corresponding to the nonlinear shallow-water equations on the sphere. Let U be a given smooth time-varying vector function on the sphere, satisfying $U(t=0) = W_0$. Then the initial-value problem for W given by

$$\left.\begin{aligned} W_t + \mathcal{M}(W) &= U_t + \mathcal{M}(U) \\ W(t=0) &= W_0 \end{aligned}\right\} \tag{2.2}$$

has the known nontrivial solution $W \equiv U$.

The vector function U represents the prescribed exact solution. Once U is selected, the right-hand side of (2.2) can be evaluated and the shallow-water code designed to solve (2.1) is used to solve (2.2) instead.

To ensure that (2.2) is still a well-posed problem and that the stability and accuracy properties of the numerical scheme for solving (2.1) can be inferred from those of the forced scheme, it is important that the prescribed solution U be close to a solution of (2.1); the forcing should be small. Marchesin (1984) gives a more detailed discussion of this issue. In fact, this forcing technique is especially relevant when the prescribed solutions resemble physically meaningful flow situations. For these reasons we allow U to be any linear combination of *Hough harmonics* that evolve in time according to their natural frequencies.

Hough harmonics, which are the normal modes of the shallow-water equations on the sphere linearized about a state of rest, have been used as basis functions for spectral prediction models (Kasahara, 1977) and for objective analysis (Flattery, 1970). They were shown by Holl (1970) to form a complete set in the space of square-integrable vector functions on the sphere. Methods for computing the Hough harmonics and their frequencies to arbitrary accuracy are given, for instance, in Kasahara (1978) and Swarztrauber and Kasahara (1985).

Thus, we prescribe an exact solution of the form

$$U = U(\lambda, \phi, t) = \mathrm{Re}\left\{\sum_{n=1}^{N} a_n H_n(\lambda, \phi) e^{i\sigma_n t}\right\}, \tag{2.3a}$$

where the a_n are complex amplitudes, $H_n(\lambda, \phi)$ are selected Hough harmonics, and σ_n their respective frequencies. Each Hough harmonic is of the form

$$H_n(\lambda, \phi) = H^s_{l,r}(\lambda, \phi) = \theta^s_{l,r}(\phi) e^{is\lambda}, \tag{2.3b}$$

where $\theta^s_{l,r}(\phi)$ is the *Hough function* identified by the three indices s, l, and r. For each value of the zonal wavenumber s, there are an infinite number of Hough harmonics classified according to their frequencies as eastward gravity modes, rotational modes, and westward gravity modes. The index r indicates which of these three groups the Hough harmonic belongs to, and the meridional index l counts the modes within each group.

Hough harmonics have been used previously for the purpose of detecting coding errors in atmospheric models by Chao and Geller (1982), who specifically exploited the fact that solutions of the nonlinear model do not deviate too much from solutions of the linearized model. However, their procedure cannot test nonlinear terms in the model implementation. Merilees et al. (1977) successfully used prescribed solution forcing to test a nonlinear shallow-water model. Their prescribed solutions were Haurwitz waves and they calculated the forcing terms manually, a procedure that can lead to additional coding errors. Marchesin (1984) developed a more general procedure in which the computer calculates the forcing terms, evaluating derivatives by using highly accurate difference formulas.

3. Implementation

It is of course necessary that the forcing terms be evaluated with sufficient precision so that the error in the approximate solution computed by the shallow-water code is entirely due to the truncation error of the difference scheme. This is achieved in our implementation by using sufficiently long series expansions for the Hough functions in terms of associated Legendre functions (Kasahara, 1978). For reasons of efficiency, the expansion coefficients and frequencies of a large orthonormal set of Hough functions are precomputed and stored on a disk file. This disk file can serve as a data base for a large number of experiments.

Given a function $U(\lambda, \phi, t)$ of the form (2.3a) together with sufficiently long series expansions for the Hough functions appearing in (2.3b), it is a simple matter to compute partial derivatives of $U(\lambda, \phi, t)$ without significant truncation error. Since the prescribed solution satisfies the linear part of the equations, only the nonlinear terms need be evaluated. Derivatives with respect to λ are obtained directly from the formulas (2.3), and derivatives with respect to ϕ can be expressed in terms of the expansion coefficients by using elementary properties of the associated Legendre functions.

In a typical experiment, the user selects a number of Hough harmonics and corresponding real amplitudes and phases, thus prescribing an exact solution of the form (2.3). The selected Hough functions and their derivatives are then evaluated on a model meridian and are stored in one-dimensional arrays for future use. This computation need be performed only once, since the exact solution as well as the forcing terms are sums of separable functions of the three independent variables λ, ϕ, and t. The prescribed solution package contains subroutines which evaluate the forcing terms on the two-dimensional model grid at each model time step, with minimal computational overhead.

4. Program verification

With an exact solution at hand, it is possible to establish the correctness of a shallow-water code by verifying that the numerical solution converges to the exact solution at a rate consistent with the presumed order of accuracy of the scheme, as the mesh spacing and time increment approach zero. Furthermore, parts of the code can be tested separately by prescribing an exact solution for which selected terms in the equations vanish.

Imagine a sequence of prescribed solution experiments in which only the spatial and temporal increments are varied, using a scheme which has order of accuracy m in space and n in time. If in successive experiments the number of gridpoints in each spatial direction is say, doubled, then the part of the truncation error due to the spatial discretization should converge to zero at an asymptotic rate of $(1/2)^m$. If the timestep in each successive experiment is reduced by a factor of $(1/2)^{m/n}$, then the truncation error due to the time differencing should converge to zero at a rate of $[(1/2)^{m/n}]^n = (1/2)^m$ as well. Thus, the rms total truncation error in each of the three dependent variables should asymptote to zero at this rate (Marchesin, 1984).

As an example, we show in Table 1 the results of a sequence of prescribed solution runs which verify the accuracy of a fully implicit scheme developed by Cohn et al. (1985). Here we use a version of the scheme which is second-order accurate both in time and space. The scheme solves the shallow-water equations in flux form; the dependent variables are h, hu, and hv, where h is the free surface height and u and v are the zonal and meridional velocity components, respectively. The table shows the relative rms errors after 1 h, obtained in a sequence of three mesh refinement experiments which were performed with a prescribed solution (2.1) using only one Hough harmonic. This solution is a rotational wave with zonal wavenumber $s = 1$ and meridional index $l = 2$ which rotates at a rate of 41 deg per day. The wave varies smoothly in both spatial directions as well as in time, so that all terms in the equa-

TABLE 1. Relative rms errors at 1 h for a series of three mesh-refinement experiments. The first two columns indicate the number of grid points in the zonal direction (I) and the meridional direction (J); the third column contains the time step in minutes. The column labeled h contains the rms height errors divided by the rms of the prescribed height deviation from the mean. The last two columns contain the rms errors in the variables hu and hv, normalized by the rms of the prescribed values of these variables. Numbers in parentheses indicate reduction factors from one experiment to the next.

No. grid points			Relative rms errors (reduction factors)		
I	J	Δt	h	hu	hv
12	8	8	0.127	0.653×10^{-1}	0.261
24	16	4	0.434×10^{-1} (2.9)	0.163×10^{-1} (4.0)	0.635×10^{-1} (4.1)
48	32	2	0.117×10^{-1} (3.7)	0.395×10^{-2} (4.1)	0.151×10^{-1} (4.2)

tions play a role in these experiments. The factors by which the rms errors are reduced in successive experiments are shown in parentheses in Table 1; they are seen to rapidly converge to $2^m = 4$.

Implementation of this implicit scheme was somewhat complicated, and the results previously discussed were obtained only after several coding errors had been isolated and corrected. Most of these errors were found by prescribing exact solutions of an especially simple nature which render selected terms in the equations irrelevant to the numerical solution. For example, all zonal rotational Hough harmonics are stationary, so that these can be used to test the accuracy of spatial differencing in the meridional direction separately from the rest of the code. Error convergence rates which are inconsistent with the theoretical accuracy of the scheme are very easy to interpret in this case and lead to a quick identification of the source of error. In order to allow further simplification of the prescribed solutions, we implemented an option to set individual components of the selected Hough harmonics equal to zero. Thus, it is possible to define an exact solution which has, for example, a meridional velocity component which depends on latitude only and which has vanishing height and zonal velocity components.

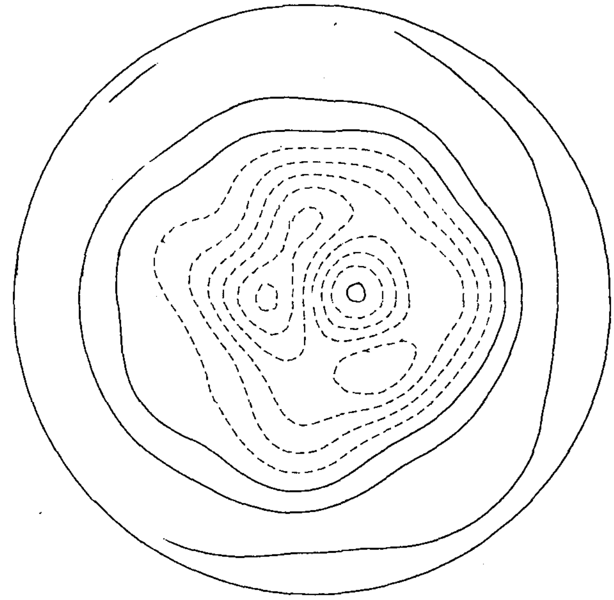

FIG. 1. Height contours (deviation from mean) in a stereographic projection on the Northern Hemisphere, showing the prescribed solution at initial time. Contours levels range from −550 to 250 m and contour interval is 100 m. Negative levels are dashed.

5. Evaluation of schemes

By prescribing exact solutions which resemble physically meaningful flows and studying the error fields produced by a particular scheme, one can gain valuable insights regarding the performance of the scheme under varying circumstances. We now present some examples of such experiments.

In Fig. 1 we show the height field over the Northern Hemisphere produced by a linear combination of nine rotational Hough harmonics. In all of the experiments described below, the prescribed solution consists of these nine waves, each of which propagates in accordance with its natural frequency. The fastest wave has a phase speed of 64 deg day^{-1}, and the wave with the finest spatial structure has zonal wavenumber 7. The largest deviation from the mean height of 8500 m occurs at the low of 7932 m, just to the left of the pole, and the maximum velocity of 36 m s^{-1} is attained in the belt of strong zonal flow near 30° north. The spatial grid is held fixed in each of the experiments, and consists of 64 equally spaced points in the zonal direction by 32 equally spaced points from pole to pole.

The first two experiments test the performance of a leapfrog scheme that uses fourth-order accurate implicit spatial difference formulas. We use a time step of 6 min, and since the scheme is explicit, stability can be maintained only by filtering techniques. The scheme was combined with a Fourier filter (technique c in Takacs and Balgovind, 1983). This filter acts only on the three latitudes nearest to the poles and effectively eliminates linear instability due to grid convergence. Nonlinear instability is controlled by periodically applying a Shapiro filter in addition to the Fourier filter.

Clearly these filters modify the computed solution and hence affect the accuracy of the scheme. The first two experiments are intended to illustrate how our prescribed solution technique might be used to study the effect of various types of filtering.

In Fig. 2 we show the height error field after 12 h, produced by the explicit scheme combined with a Fourier filter as described before, as well as a 16th order Shapiro filter which was applied at every time step (Cohn et al., 1985). All three dependent variables were filtered in both spatial directions. It is seen that the error is strongly concentrated near the pole, where deviations from the prescribed height field range between ±2.2 meters. By contrast, typical height errors in midlatitudes range from ±0.5 meters.

The effect of the Shapiro filter on the solution becomes quite evident upon comparing Figs. 2 and 3. The latter shows the error field after 12 h which results when the Shapiro filter is applied only every 2 h in both spatial directions for the height field but only in the zonal direction for the wind field. It is evident that the errors are spread much more uniformly when the Shapiro filter is applied less frequently. In fact, typical height errors in midlatitudes are quite similar in both cases, which suggests that the Shapiro filter has a much more deleterious effect near the pole than elsewhere. Errors in the height field near the pole are reduced by more than a factor of 3 when the Shapiro filter is weakened.

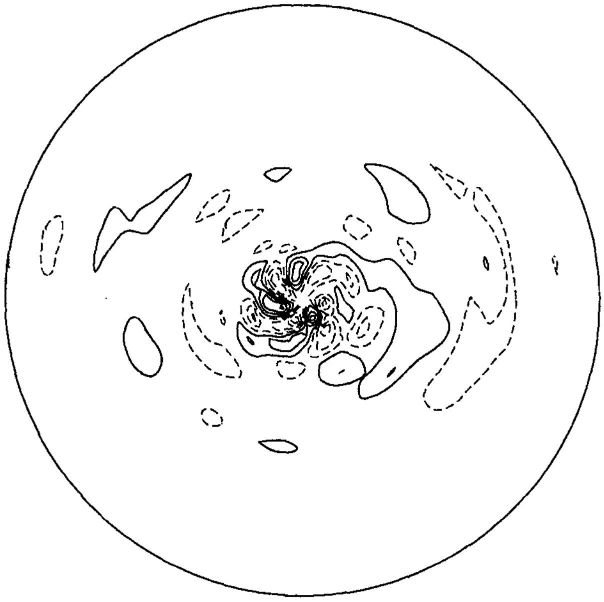

FIG. 2. Contours of height error field at 12 h, produced by the explicit scheme with a time step of 6 min, high-latitude Fourier filtering and strong Shapiro filtering. Contour levels range from −1.75 to +1.75 meters and contour interval is 0.5 m. Negative levels are dashed.

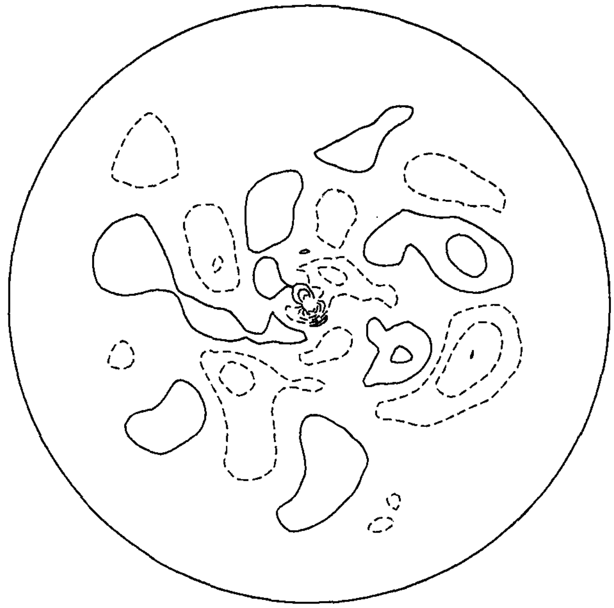

FIG. 4. Contours of height error field at 12 h, produced by the implicit scheme with a time step of 30 min and no filtering. Contour levels range from −2.25 to +1.25 m, and contour interval is 0.5 m.

Finally, we show the results of two experiments which illustrate some aspects of the performance of the fully implicit scheme (Cohn et al., 1985) referred to in section 4, but are now calculated using fourth-order accurate implicit spatial difference formulas. The scheme is unconditionally linearly stable, and in the two experiments described here no filtering whatsoever was applied to the solution. Figure 4 shows the error field after 12 h, using a time step of 30 min. In this case the height errors range from −2.6 to +1.5 m, which is comparable to the case of the explicit scheme with strong Shapiro filtering studied above. Note, however, that the time step has been increased by a factor of 5. The errors are typically about twice as large near the pole as they are in midlatitudes.

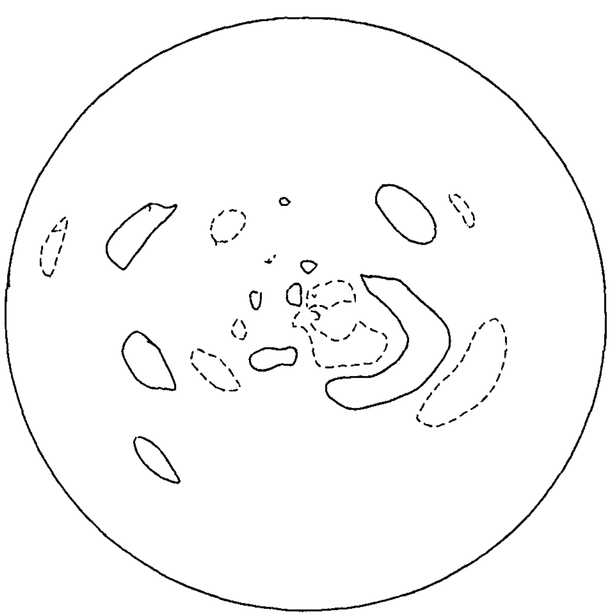

FIG. 3. As in Fig. 2, but with Shapiro filtering reduced. Contour levels range from −0.25 to +0.25 m and contour interval is 0.5 m.

In Fig. 5 we show the error field generated by the implicit scheme, now using a time step of 60 min. The errors near the pole have not increased substantially; however, height errors in midlatitudes are now about 50% larger. There is a clear wave structure in the error field which also appears in Fig. 4 but is much more pronounced here. This structure derives from the high-wavenumber part of the flow, which contains a Hough harmonic with zonal wavenumber 7 as well as one with zonal wavenumber 5. While the pointwise errors are rather small, this example does show clearly how our technique is useful in highlighting the structural details of the error field.

6. Summary

We have presented a technique which serves three purposes in the development of numerical methods for integrating the nonlinear shallow-water equations on the sphere. First, it is useful in helping to identify coding errors which inevitably arise in the implementation of the numerical method. The second purpose

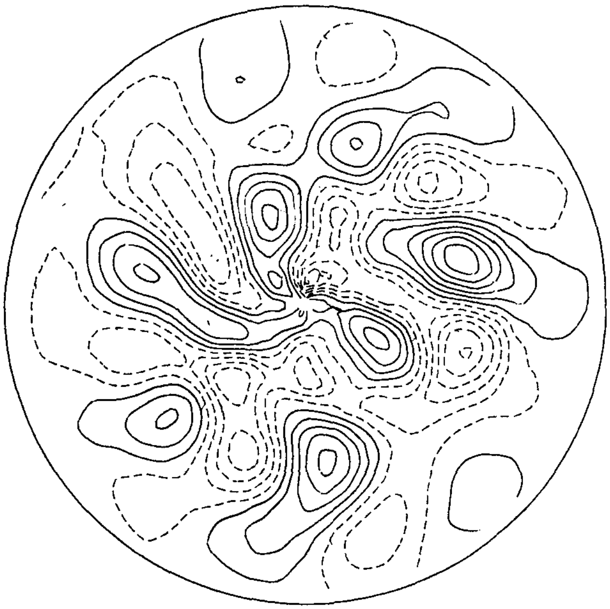

FIG. 5. As in Fig. 4, but using a time step of 60 min. Contour levels range from −2.75 to 2.75 m and contour interval is 0.5 m.

of the technique is to enable one to demonstrate conclusively the correctness of the implementation by verifying the order of accuracy of the scheme. Third, it serves as a versatile tool in analyzing the performance of numerical schemes under a variety of flow conditions. The technique is easily implemented with the help of available software packages for computing Hough functions. We expect extension to three-dimensional primitive equation models to be straightforward.

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