Comments on "Truncation Errors in Finite-Difference Estimates of Geostrophic Wind and Relative Vorticity"

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ABSTRACT

Peppler and Smith (1984) discussed truncation errors associated with second-order and fourth-order finite difference approximations used to calculate the geostrophic wind and relative vorticity. They found that these errors were, in general, smaller for longer wavelengths, finer-grid resolution, and fourth-order differencing. Second-order differencing produced fields that were numerically less than their corresponding analytic values and yielded errors which decreased with reduced grid interval. Fourth-order differencing decreased the errors when the grid interval was reduced, but only *while the wavelength was ten times or more greater than the grid interval.*

Results presented here indicate that the wind speed and vorticity errors estimated by the fourth-order scheme decreased when the grid interval was decreased *independent of wavelength.* Comparison with Peppler and Smith's actual computations showed only one difference: in their finite-difference equations the coefficients were truncated to the nearest thousandth (for example, 0.083 was used in place of $^1/_{12}$). On the other hand, we used 1.0/12.0 and allowed the computer's word length to determine the value of the coefficient, thus, preserving greater accuracy.

1. Introduction

Peppler and Smith (1984) analyzed truncation errors in geostrophic wind and relative vorticity estimates by comparing second-order and fourth-order finite-differencing schemes. Three grid resolutions were used and the horizontal wave dimensions were varied in order to study sensitivity to different scales of motion. The finite-difference estimates were compared with exact calculations derived from Sanders' (1971) analytic equations describing geostrophic wind and relative vorticity.

Peppler and Smith found that the truncation errors were, in general, smaller for longer wavelengths, finer-grid resolution, and fourth-order differencing. Second-order differencing produced fields that were numerically less than their corresponding analytic values and yielded errors which decreased with reduced grid interval. Fourth-order differencing errors decreased when the grid interval was reduced also, but only until the wavelength was ten times or more greater than the grid interval. Here the wind speeds and vorticities estimated by the fourth-order scheme were overestimated and were accompanied by increased truncation errors at finer grids.

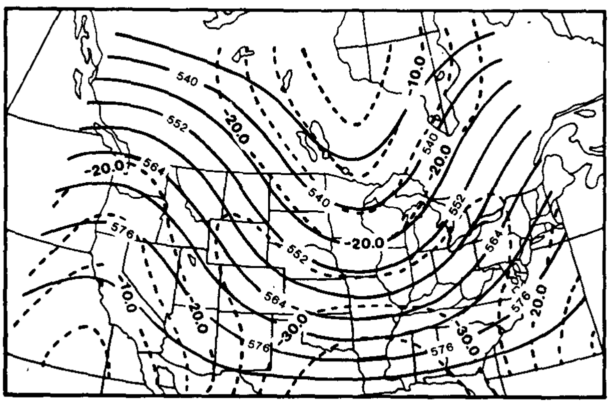
Since previous studies (Bermowitz, 1969; Kalnay-Rivas et al., 1976, 1977; Williamson, 1978; and Campana, 1979) show that higher-order, smaller-mesh calculations yield better results, we questioned whether fourth-order differencing at wavelengths ten times or greater than the grid interval increases the truncation errors at finer-grid resolutions and overestimates the wind speeds and vorticities. As a result, the present study attempts to reproduce Peppler and Smith's results using the same equations and procedures. Most of our results are identical to Peppler and Smith's results. The only exception is that we did not observe wavelengths ten times or greater than the grid interval, increased fourth-order truncation errors at finer grid resolutions and overestimated wind speeds and vorticities. Rather, we found that the fourth-order truncation errors decrease at finer grid resolutions and fourth-order differencing produces fields that are numerically less than the analytic fields. A comparison of Peppler and Smith's actual computations showed only one difference: the coefficients in their finite-differencing equations were truncated to the nearest thousandth (in other words, 0.083 was used in place of $^1/_{12}$). We, on the other hand, allow the computer to truncate the coefficients (in the results presented in this paper the Cray was used with its 64-bit word length). However, when we use their truncated values as the coefficients, we reproduce all of Peppler and Smith's results. This suggests that errors in the coefficients in the finite-difference scheme propagate to the solution and limit the precision of the solution that can be obtained. A comparison is made between the two sets of results.

2. Computational procedures

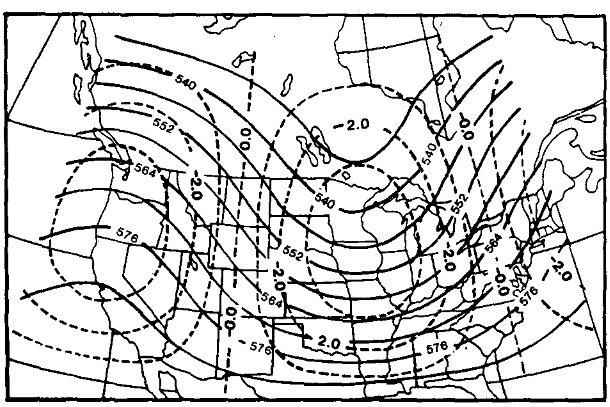
As previously stated, the same procedures are used in order to reproduce Peppler and Smith's results (see Peppler and Smith, 1984, for more detail). In short, "exact" geostrophic wind and relative vorticity values prescribed by analytic functions suggested by Sanders (1971) are compared with second- and fourth-order values using latitude–longitude grid distances of 1.25°, 2.50°, and 5.00°. Also, in order to study sensitivity to different scales of motion, the horizontal wave dimensions are varied to accommodate one, two, and four sinusoidal waves within the grid network. Analytic geostrophic wind speed and relative vorticity computations are shown for wavenumber 1 in Fig. 1.

3. Results

To be consistent with Peppler and Smith, two types of error statistics are examined: difference maps and mean absolute errors. Difference maps are computed by subtracting the analytic value from the finite-difference value at each grid point. The mean absolute errors are calculated by averaging the absolute difference values over all grid points.



(a)



(b)

FIG. 1. Analytic computations for 500 mb height (solid lines, dam) and (a) geostrophic wind speed (dashed lines, m s^{-1}) for wavenumber 1 and (b) relative vorticity (dashed lines, 10^{-5} s^{-1}) for wavenumber 1.

a. Difference maps

The difference maps of geostrophic wind speed and relative vorticity are shown in Figs. 2 and 3. Only difference maps for wavenumber 1 for all grid sizes will be shown, since wavenumbers 2 and 4 have similar characteristics.

The second-order wind speed difference maps are very similar to Peppler and Smith's second-order difference maps. In all cases, the estimated values of wind speed are less than the analytic values (negative differences) at most grid points (Fig. 2a). Therefore, in agreement with Peppler and Smith, second-order truncation errors tend to yield underestimates of the wind speed.

The fourth-order wind speed differences exhibit patterns similar to those found for the second-order calculations (compare Figs. 2a and 2b). However, unlike Peppler and Smith, who show overestimated wind speeds at wavelengths greater than the grid interval, the present estimated wind speed values are less than the analytical values in all cases for most grid points. Therefore, fourth-order truncation errors tend to yield underestimates of the wind speed; however, these fourth-order wind speed estimates are more accurate than the second-order estimates by one, two, or three orders of magnitude.

The second-order vorticity difference maps are similar to Peppler and Smith's second-order vorticity difference maps. The trough regions of the vorticity field are underestimated and the ridge regions are overestimated when using second-order finite differencing (Fig. 3a). Therefore, both positive and negative vorticity areas are damped by second-order differencing, thus leading to an overall smoothing of the vorticity field and reduction of vorticity gradients.

The patterns of the fourth-order vorticity differences are similar to the patterns of the second-order vorticity (compare Figs. 3a and 3b). The trough regions of the vorticity field are underestimated and the ridge regions are overestimated, even at wavelengths ten times or more greater than the grid interval, where Peppler and Smith observed different results. Therefore, fourth-order differencing also damps both positive and negative vorticity areas leading to an overall smoothing of the vorticity field. However, these fourth-order vorticity estimates are more accurate than the second-order estimates by one, two, or three orders of magnitude.

b. Mean absolute errors

A comparison of Peppler and Smith's mean absolute error results with the present results are presented in Table 1. In general, the present results are very similar to the results of Peppler and Smith for the second-order estimates. However, the differences appear when comparing the fourth-order estimates, where in some cases the estimates differ by one to three orders of magnitude.

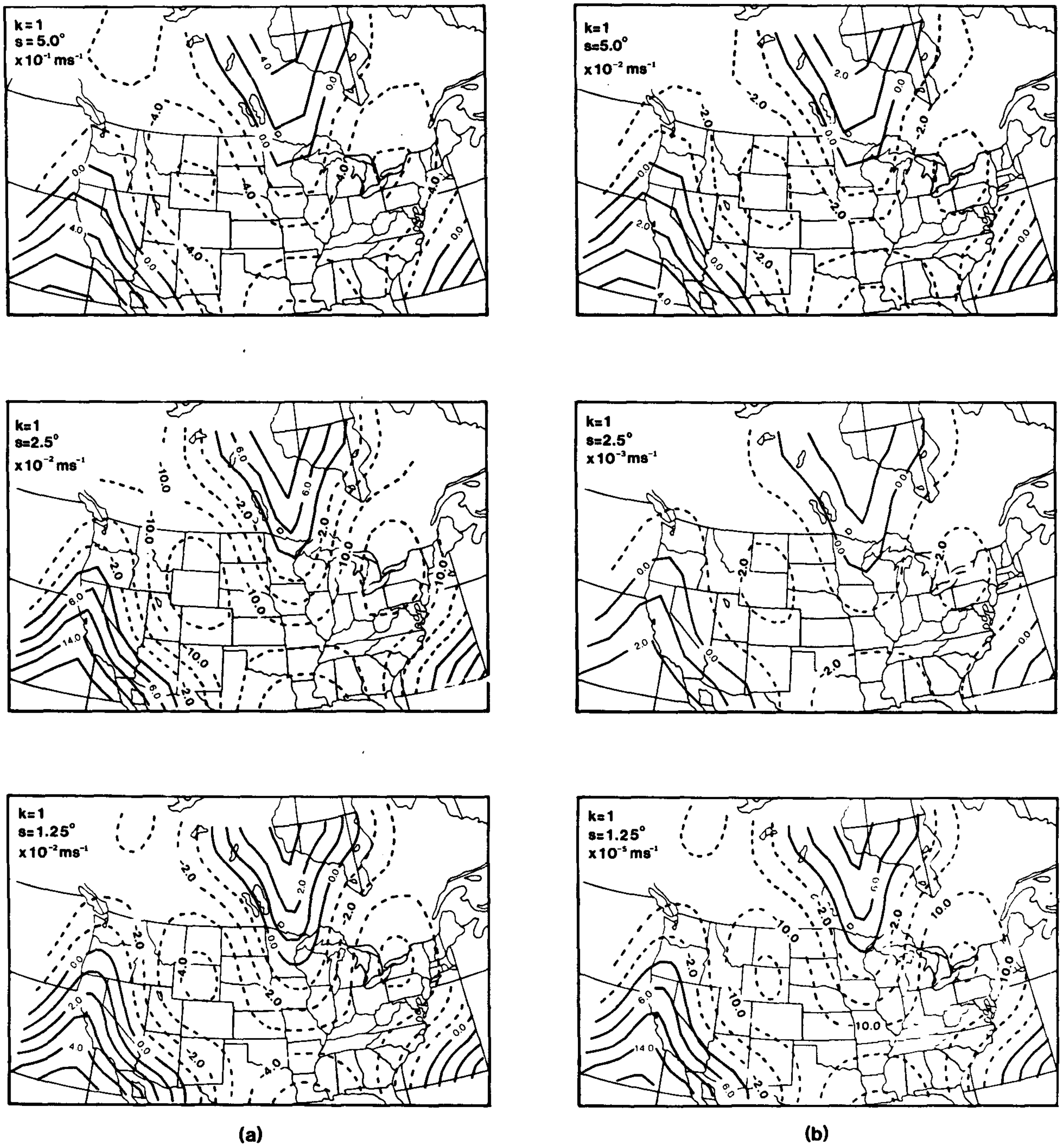


FIG. 2. Geostrophic wind difference (error) maps for wavenumber 1: k = wavenumber, s = grid resolution. Note the magnitude changes from map to map. (a) Second-order estimates minus analytic values and (b) fourth-order estimates minus analytic values. Dashed contours indicate negative differences while solid contours denote positive differences.

Similar to Peppler and Smith, the present results show that the fourth-order estimates are more accurate than the second-order estimates, generally by one, two or three orders of magnitude. Both results show that truncation error is less for longer waves and finer grid resolution in all cases of the second-order results. The present results show that this is also true for fourth-order results.

4. Summary

Peppler and Smith made comparisons between second-order and fourth-order finite differencing schemes

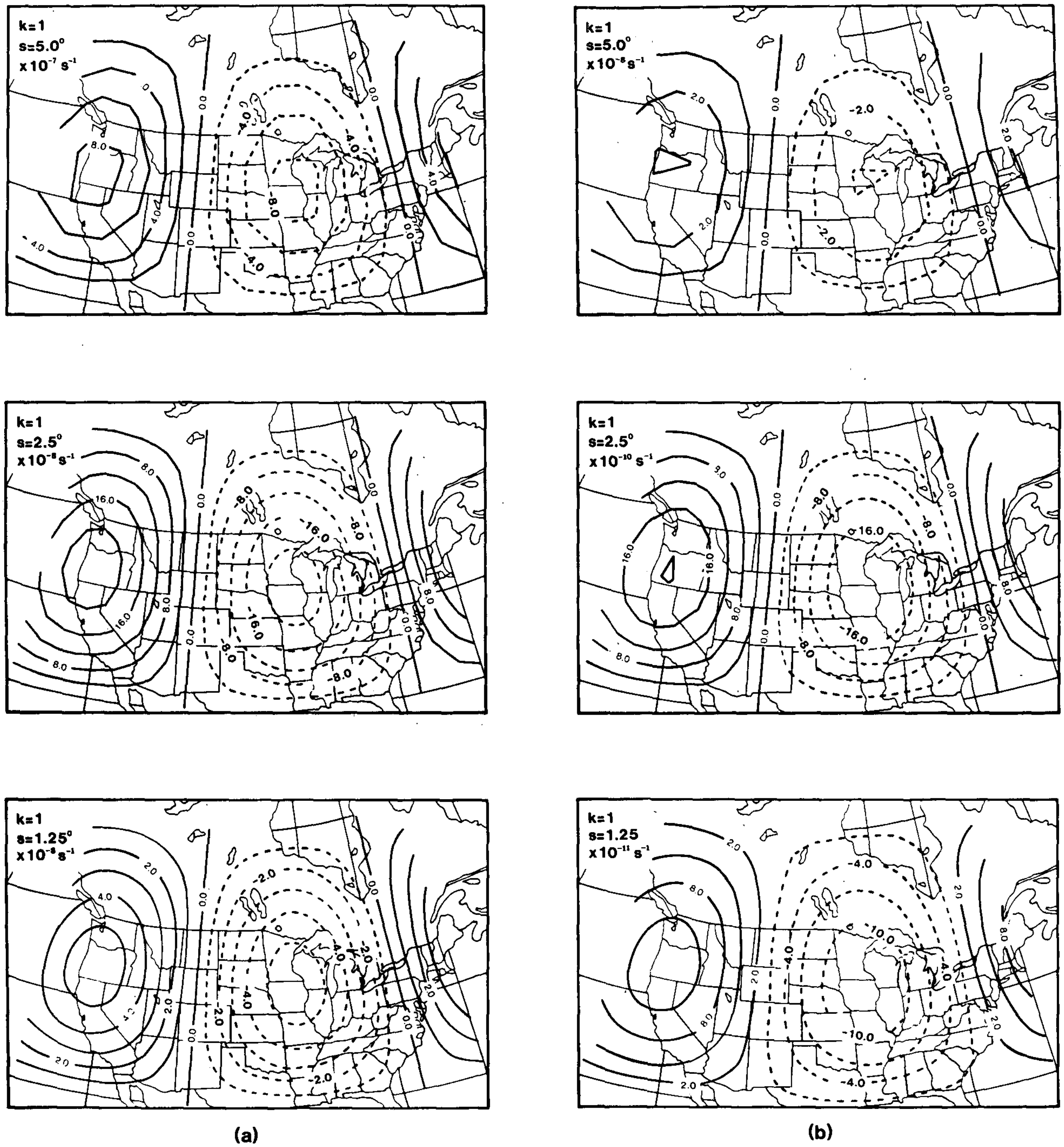


FIG. 3. As in Fig. 2 except for relative vorticity difference (error).

using three grid resolutions in order to analyze truncation errors in geostrophic wind and relative vorticity estimates. To study sensitivity to different scales of motion, the horizontal wave dimensions were varied. The present study attempted to reproduce Peppler and Smith's results by using the same equations and procedures.

In general, a comparison of the second-order truncation errors with Peppler and Smith's second-order truncation errors shows very similar results: second-order truncation errors produced wind speeds and vorticities less than their analytic values, which resulted in a smoothing of both fields. In addition, these errors were reduced as the wavelength increased and the grid interval decreased.

A comparison of the fourth-order truncation errors with Peppler and Smith's fourth-order truncation errors produces different results. Peppler and Smith showed

TABLE 1. Comparison of mean absolute errors: wind speed $|V_g|$ in m s^{-1}, vorticity ζ_g in s^{-1}.

Finite difference order	Variable	Grid increment (deg)	Peppler and Smith/Lario et al.		
			Wavenumber 1	Wavenumber 2	Wavenumber 3
2nd	$\|V_g\|$	5.00	$3.78\times10^{-1}/3.78\times10^{-1}$	1.94/1.94	$1.37\times10^{1}/1.37\times10^{1}$
		2.50	$9.38\times10^{-2}/9.38\times10^{-2}$	$5.40\times10^{-1}/5.40\times10^{-1}$	4.64/4.64
		1.25	$2.34\times10^{-2}/2.34\times10^{-2}$	$1.42\times10^{-1}/1.42\times10^{-1}$	1.26/1.26
4th	$\|V_g\|$	5.00	$2.78\times10^{-2}/2.04\times10^{-2}$	$3.58\times10^{-1}/3.82\times10^{-1}$	9.10/9.12
		2.50	$3.79\times10^{-2}/1.28\times10^{-3}$	$2.09\times10^{-2}/2.81\times10^{-2}$	$9.10\times10^{-1}/9.49\times10^{-1}$
		1.25	$3.89\times10^{-2}/8.00\times10^{-5}$	$4.71\times10^{-2}/1.87\times10^{-3}$	$2.40\times10^{-2}/6.75\times10^{-2}$
2nd	ζ_g	5.00	$3.09\times10^{-7}/3.09\times10^{-7}$	$3.28\times10^{-6}/3.28\times10^{-6}$	$4.57\times10^{-5}/4.57\times10^{-5}$
		2.50	$8.38\times10^{-8}/8.38\times10^{-8}$	$9.27\times10^{-7}/9.27\times10^{-7}$	$1.39\times10^{-5}/1.39\times10^{-5}$
		1.25	$2.19\times10^{-8}/2.19\times10^{-8}$	$2.40\times10^{-7}/2.40\times10^{-7}$	$3.75\times10^{-6}/3.75\times10^{-6}$
4th	ζ_g	5.00	$1.03\times10^{-9}/1.11\times10^{-8}$	$4.14\times10^{-7}/4.40\times10^{-7}$	$2.09\times10^{-5}/2.09\times10^{-5}$
		2.50	$1.35\times10^{-8}/7.63\times10^{-10}$	$7.55\times10^{-9}/3.26\times10^{-8}$	$1.81\times10^{-6}/1.91\times10^{-6}$
		1.25	$1.59\times10^{-8}/5.00\times10^{-11}$	$4.26\times10^{-8}/2.14\times10^{-9}$	$1.76\times10^{-8}/1.34\times10^{-7}$

that when the grid interval was decreased to the point that the wavelength was ten times or more greater than the interval, fourth-order differences produced overestimates of the wind speed and vorticity, as well as increases in the truncation errors. Our results show that fourth-order differencing generally produces underestimates of wind speed and vorticity. Furthermore, like the second-order results, truncation errors are reduced as the wavelength increases and the grid interval decreases. In addition, the fourth-order results are more accurate by one, two, or three order of magnitudes. These results agree with previous studies which show that higher-order, smaller-mesh calculations yield better results (Bermowitz, 1969: Kalnay-Rivas et al., 1976, 1977; Williamson, 1978; and Campana, 1979). The cause of this discrepancy was Peppler and Smith's use of truncated coefficients in their finite-difference approximations which became the major source of error in their fourth-order, high-resolution calculations. Thus, care must be used when coding the finite-difference approximations to the geophysical equations that require high-precision solutions.

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