# On Computing Viscous Forces in Map Coordinates with a Variable Scale

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ABSTRACT

Equations are first derived in shape-preserving coordinates for the spatial derivatives of the unit vectors, the gradient and Laplacian of a scalar, the divergence and vorticity of a vector, the advective acceleration in the equations of motion, the strain-rate tensor, and the viscous forces per unit mass. A shape-preserving projection is defined here as one in which the map scale, though spatially variable, is independent of the orientation of an infinitesimal line segment. Shape-preserving projections are also conformal. Examples are stereographic and Mercator projections. The results are then extended to the case where the map scale factors in the two horizontal coordinate directions are different.

## 1. Introduction

Since the earth's surface is approximately spherical, map projections inevitably have a spatially varying scale. This variation is often ignored in applications to a local area, if velocities are small, or if integrations are carried out for short time periods. If integrations are performed for long periods, however, systematic errors due to neglect or improper application of the variable map scale may have significant effects.

Any equation involving spatial derivations of vectors must be handled carefully. Philips (1959) gives expressions for the equations of motion and continuity in stereographic and Mercator projections. Applications where viscous forces are important, such as the studies of the dynamics of sea ice, are more complex, and are the main focus of this paper.

In section 2, equations are derived for the spatial derivatives of unit vectors in shape-preserving coordinates. Expressions are developed for the gradient and Laplacian of a scalar, and the divergence and vorticity of a vector. The divergence agrees with Phillips' (1959) expression. Phillips' result for the equation of motion is also derived as another check on the methodology.

In section 3, the strain-rate tensor is calculated and checked with equations given by Kamenkovich (1977). A general expression for the viscous forces per unit mass is derived which holds for any method of computing the stresses. It is not even necessary to assume that the stresses are proportional to the components of the strain-rate tensor, although this is customary. Smagorinsky et al.'s (1965) results are obtained as a special case.

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In section 4, the results are extended to the case where the map scale factors in the two horizontal coordinate directions differ from each other (i.e., map scale is nonisotropic).

The main contributions of this paper are the general expressions for the viscous forces (46) and (59), which can be applied with a variety of assumptions on the functional dependence of the stresses. The results of Phillips (1959), Kamenkovich (1977) and Smagorinsky et al. (1965), which they give without proof, are derived here.

## 2. Some basic derivations

Shape-preserving coordinates are defined here as coordinates in which the ratio $m$ of the length $dS$ of an infinitesimal horizontal line segment on the projection to the true length $ds$ on the earth's surface, i.e.,

$$m = \frac{dS}{ds},$$

is independent of the orientation of $ds$. Of concern here are coordinate systems in which $m$ varies spatially. For a stereographic projection with a cutting plane at 60°N,

$$m = \frac{1 + \sin 60°}{1 + \sin\theta}, \quad (1)$$

where $\theta$ is the latitude. In shape-preserving coordinates

$$dx = \frac{dX}{m}, \quad dy = \frac{dY}{m},$$

where $(dx, dy)$ are true infinitesimal distances on the earth's surface and $(dX, dY)$ are distances on the projections. It is easy to show (e.g., by taking the dot product of two line segments) that a shape-preserving projection is also conformal.

The velocity components are

$$u = \dot{x} = \frac{\dot{X}}{m}, \quad v = \dot{y} = \frac{\dot{Y}}{m},$$

where the dot denotes the substantial time derivative. The horizontal velocity is

$$\mathbf{V} = u\mathbf{i} + v\mathbf{j}, \tag{2}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors.

The horizontal gradient operator is

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} = m\left(\mathbf{i}\frac{\partial}{\partial X} + \mathbf{j}\frac{\partial}{\partial Y}\right). \tag{3}$$

When $\nabla$ operates on a scalar, $\alpha$, the results are straightforward. Thus the gradient of $\alpha$ is

$$\nabla\alpha = m\left(\mathbf{i}\frac{\partial\alpha}{\partial X} + \mathbf{j}\frac{\partial\alpha}{\partial Y}\right). \tag{4}$$

When $\nabla$ operates on a vector, however, the spatial variation of the unit vectors must be taken into account. Expressions for these quantities will now be derived.

Consider the elemental area

$$dxdy = \left(\frac{dX}{m}\right)\left(\frac{dY}{m}\right),$$

shown in Fig. 1, where $dX$ and $dY$ are constant. Then

$$\mathbf{i}_3 - \mathbf{i}_1 = -\frac{[(dx)_4 - (dx)_2]}{dy}\,\mathbf{j} = -\left[\frac{\partial}{\partial y}(dx)dy\right]\frac{\mathbf{j}}{dy}.$$

Substitute $\partial/\partial y = m\partial/\partial Y$ and $dx = dX/m$ to obtain

$$\mathbf{i}_3 - \mathbf{i}_1 = -m\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)dX\mathbf{j}.$$

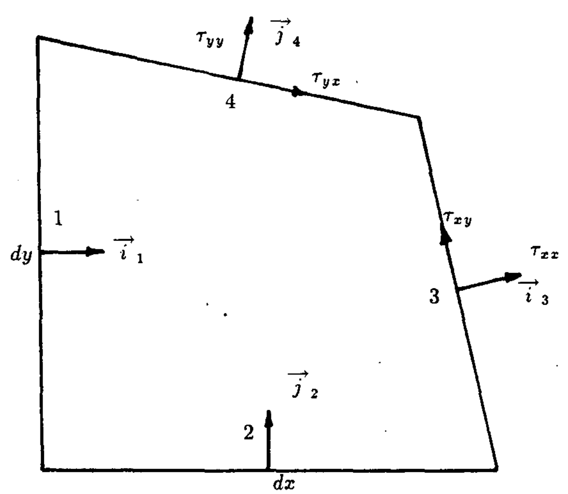


FIG. 1. Elemental area showing spatial variations of unit vectors $\mathbf{i}$ and $\mathbf{j}$, and stress components $\tau_{xx}$, $\tau_{xy}$, $\tau_{yx}$ and $\tau_{yy}$.

Therefore,

$$\frac{\partial\mathbf{i}}{\partial X} = -m\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\mathbf{j}. \tag{5}$$

Similarly,

$$\mathbf{i}_4 - \mathbf{i}_2 = \frac{[(dy)_3 - (dy)_1]}{dx}\,\mathbf{j} = \left[\frac{\partial}{\partial x}(dy)dx\right]\frac{\mathbf{j}}{dx}$$

$$= m\frac{\partial}{\partial X}\left(\frac{1}{m}\right)dY\mathbf{j}, \quad \frac{\partial\mathbf{i}}{\partial Y} = m\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\mathbf{j}, \tag{6}$$

$$\mathbf{j}_3 - \mathbf{j}_1 = \frac{[(dx)_4 - (dx)_2]}{dy}\,\mathbf{i} = \left[\frac{\partial}{\partial y}(dx)dy\right]\frac{\mathbf{i}}{dy}$$

$$= m\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)dX\mathbf{i}, \quad \frac{\partial\mathbf{j}}{\partial X} = m\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\mathbf{i}, \tag{7}$$

$$\mathbf{j}_4 - \mathbf{j}_2 = -\frac{[(dy)_3 - (dy)_1]}{dx}\,\mathbf{i} = -\left[\frac{\partial}{\partial x}(dy)dx\right]\frac{\mathbf{i}}{dx}$$

$$= -m\frac{\partial}{\partial X}\left(\frac{1}{m}\right)dY\mathbf{i},$$

and

$$\frac{\partial\mathbf{j}}{\partial Y} = -m\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\mathbf{i}. \tag{8}$$

The above equations will now be used to derive an expression for the divergence. From (2), (3) and (5)–(8), one obtains

$$\nabla\cdot\mathbf{V} = m^2\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right]. \tag{9}$$

Equation (9) is presented by Phillips (1959) without proof.

Similarly, the vorticity is

$$\zeta = \mathbf{k}\cdot\nabla\times\mathbf{V} = m^2\left[\frac{\partial}{\partial X}\left(\frac{v}{m}\right) - \frac{\partial}{\partial Y}\left(\frac{u}{m}\right)\right]. \tag{10}$$

The Laplacian of a scalar $\alpha$, i.e.,

$$\nabla^2\alpha = \nabla\cdot\nabla\alpha,$$

is the divergence of a vector whose components are $(m\partial\alpha/\partial X,\ m\partial\alpha/\partial Y)$ [see (4)]. Substituting $\nabla\alpha$ for $\mathbf{V}$ in (9) gives

$$\nabla^2\alpha = m^2\left(\frac{\partial^2\alpha}{\partial X^2} + \frac{\partial^2\alpha}{\partial Y^2}\right). \tag{11}$$

In the equation of motion, the advective acceleration takes the form

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \left[m\left(u\frac{\partial u}{\partial X} + v\frac{\partial u}{\partial Y}\right) - v\left(u\frac{\partial m}{\partial Y} - v\frac{\partial m}{\partial X}\right)\right]\mathbf{i} + \left[m\left(u\frac{\partial v}{\partial X} + v\frac{\partial v}{\partial Y}\right) + u\left(u\frac{\partial m}{\partial Y} - v\frac{\partial m}{\partial X}\right)\right]\mathbf{j}. \tag{12}$$

Expressions for the extra terms in (12) due to spatial variations in $m$ may be derived for stereographic coordinates for which

$$X = ma\cos\theta\cos\lambda \tag{13}$$

$$Y = ma\cos\theta\sin\lambda, \tag{14}$$

where $a$ is the earth's radius and $\lambda$ is the longitude. By taking differentials at constant $X$ of (1), (13) and (14), and eliminating $d\lambda$ and $d\theta$, one obtains

$$\frac{\partial m}{\partial Y} = \frac{Y}{(1+\sin 60°)a^2}. \tag{15}$$

Similarly,

$$\frac{\partial m}{\partial X} = -\frac{X}{(1+\sin 60°)a^2}. \tag{16}$$

Substituting (15) and (16) in (12) gives Eqs. (14) and (15) of Phillips (1959), which he presents without proof.

The terms in (12) due to variations in $m$ are normally small for a stereographic projection. They are of the order of 10% of the Coriolis force for $f \approx 10^{-4}$ $s^{-1}$, $(X, Y) \approx a$ and $(u, v) \approx 50$ m $s^{-1}$. Consequently they are often omitted for applications where velocities are small, although their deletion may cause systematic errors in long time integrations.

## 3. Viscous forces

The velocity $\mathbf{V}$ at a distance $(dx, dy)$ from a point where the velocity is $\mathbf{V}_0$ may be written to the first order as

$$\mathbf{V} = \mathbf{V}_0 + \frac{\partial \mathbf{V}}{\partial x}dx + \frac{\partial \mathbf{V}}{\partial y}dy$$

$$= \mathbf{V}_0 + m\frac{\partial \mathbf{V}}{\partial X}dx + m\frac{\partial \mathbf{V}}{\partial Y}dy. \tag{17}$$

From (2) and (5)–(8),

$$\frac{\partial \mathbf{V}}{\partial X} = \left[\frac{\partial u}{\partial X} + vm\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right]\mathbf{i} + \left[\frac{\partial v}{\partial X} - um\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right]\mathbf{j} \tag{18}$$

$$\frac{\partial \mathbf{V}}{\partial Y} = \left[\frac{\partial u}{\partial Y} - vm\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\right]\mathbf{i} + \left[\frac{\partial v}{\partial Y} + um\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\right]\mathbf{j}. \tag{19}$$

Using (18) and (19), we may write (17) in matrix form as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + L\begin{pmatrix} dx \\ dy \end{pmatrix},$$

where

$$L = \begin{pmatrix} m\left[\frac{\partial u}{\partial X} + vm\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right] & m\left[\frac{\partial u}{\partial Y} - vm\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\right] \\ m\left[\frac{\partial v}{\partial X} - um\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right] & m\left[\frac{\partial v}{\partial Y} + um\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\right] \end{pmatrix}$$

is the local velocity field.

The strain rate tensor $S$ is the symmetric part of $L$, and is calculated from

$$S = \frac{L + L^T}{2},$$

where $L^T$ is the transpose of $L$. One obtains

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \tag{20}$$

where

$$s_{11} = m\left[\frac{\partial u}{\partial X} + vm\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right] \tag{21}$$

$$s_{12} = \frac{m}{2}\left[\frac{\partial u}{\partial Y} - vm\frac{\partial}{\partial X}\left(\frac{1}{m}\right) + \frac{\partial v}{\partial X} - um\frac{\partial}{\partial Y}\left(\frac{1}{m}\right)\right] = s_{21} \tag{22}$$

$$s_{22} = m\left[\frac{\partial v}{\partial Y} + um\frac{\partial}{\partial X}\left(\frac{1}{m}\right)\right]. \tag{23}$$

It may be shown that (21)–(23) are a special case of Eqs. (A.9.16) of Kamenkovich (1977), which he gives without proof.

Write (20) as

$$S = S_1 + S_2 + S_3 \tag{24}$$

where

$$S_1 = \begin{pmatrix} \frac{1}{2}(s_{11}+s_{22}) & 0 \\ 0 & \frac{1}{2}(s_{11}+s_{22}) \end{pmatrix}, \tag{25}$$

$$S_2 = \begin{pmatrix} \frac{1}{2}(s_{11}-s_{22}) & 0 \\ 0 & -\frac{1}{2}(s_{11}-s_{22}) \end{pmatrix}, \tag{26}$$

and

$$S_3 = \begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix}. \tag{27}$$

Now from (21) and (23),

$$s_{11} + s_{22} = m^2\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right]. \tag{28}$$

This is equal to the horizontal divergence [see (9)]. Thus (25) may be written as

$$S_1 = \begin{pmatrix} \frac{m^2}{2}\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right] & 0 \\ 0 & \frac{m^2}{2}\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right] \end{pmatrix}, \quad (29)$$

where $S_1$ is the dilation or divergence part of the local velocity field. Also, from (21) and (23),

$$s_{11} - s_{22} = \frac{\partial}{\partial X}(mu) - \frac{\partial}{\partial Y}(mv), \quad (30)$$

so (26) becomes

$$S_2 = \begin{pmatrix} \frac{1}{2}\left[\frac{\partial}{\partial X}(mu) - \frac{\partial}{\partial Y}(mv)\right] & 0 \\ 0 & -\frac{1}{2}\left[\frac{\partial}{\partial X}(mu) - \frac{\partial}{\partial Y}(mv)\right] \end{pmatrix}. \quad (31)$$

From (22),

$$s_{12} = \frac{1}{2}\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right] = s_{21}, \quad (32)$$

so (27) may be written as

$$S_3 = \begin{pmatrix} 0 & \frac{1}{2}\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right] \\ \frac{1}{2}\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right] & 0 \end{pmatrix}. \quad (33)$$

Here $S_2$ and $S_3$ are the deformation parts of the local velocity field.

The viscous stress tensor $\tau$ is

$$\tau = \begin{pmatrix} \tau_{xx} & \tau_{yx} \\ \tau_{xy} & \tau_{yy} \end{pmatrix}, \quad (34)$$

where the first subscript denotes the direction of the unit normal and the second subscript the direction of the stress component (see Fig. 1). The diagonal components, $\tau_{xx}$ and $\tau_{yy}$, are normal stresses, whereas the off-diagonal components, $\tau_{xy}$ and $\tau_{yx}$ are tangential stresses.

Assume that

$$\tau = \tau_1 + \tau_2 + \tau_3, \quad (35)$$

where

$$\tau_1 = 2K_1S_1, \quad (36)$$

$$\tau_2 = 2K_2S_2, \quad (37)$$

$$\tau_3 = 2K_3S_3, \quad (38)$$

and $K_1$, $K_2$ and $K_3$ are viscosity coefficients. Then from (29), (31) and (33),

$$\tau_{xx} = K_1m^2\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right] + K_2\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right], \quad (39)$$

$$\tau_{xy} = K_3\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right] = \tau_{yx}, \quad (40)$$

and

$$\tau_{yy} = K_1m^2\left[\frac{\partial}{\partial X}\left(\frac{u}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m}\right)\right] - K_2\left[\frac{\partial}{\partial Y}(mu) + \frac{\partial}{\partial X}(mv)\right]. \quad (41)$$

In an application to an atmospheric model, Smagorinsky et al. (1965) omitted $\tau_1$ in (35) and assumed $K_2$ and $K_3$ equal to each other and proportional to the total deformation

$$D = [(s_{11} - s_{22})^2/4 + s_{12}^2]^{1/2}. \quad (42)$$

In a study of internal stress in sea ice, Hibler (1979) employed a plastic constitutive relation [his Eq. (4)] in which the viscosity coefficients are functions of the divergence, total deformation and ice strength. To obtain the equivalent of Hibler's Eq. (4), without the pressure term, assume $K_2 = K_3$ in (35)–(38) so that (35) becomes

$$\tau = 2K_1S_1 + 2K_2(S_2 + S_3).$$

Substitute $S_2 + S_3 = S - S_1$ from (24) to obtain

$$\tau = 2(K_1 - K_2)S_1 + 2K_2S. \quad (43)$$

This is the equivalent, in shape-preserving coordinates, of Hibler's Eq. (4), omitting the pressure term, with his bulk viscosity $\zeta$ replaced by $K_1$ and his shear viscosity $\eta$ by $K_2$. As given by Hibler's Eqs. (7)–(9), $\zeta$ and $\eta$ are assumed to depend on the ice strength, the ratio $e$ of the principal axes of the elliptical yield curve, and a quantity $\Delta$. For $e = 2$, it may be shown that

$$\Delta = \{[\nabla\cdot\mathbf{V}]^2 + D^2\}^{1/2},$$

where $\nabla\cdot\mathbf{V}$ and $D$ are given by (28) and (42), respectively.

The viscous force per unit mass, $\mathbf{F}$, will now be calculated. In Fig. 1, the element of mass is

$$dM = \rho dxdydz = \rho \frac{dXdY}{m^2} dz, \qquad (44)$$

where $\rho$ is the density and $(dX, dY)$ are constant. The viscous force on the element is

$$\mathbf{F}dM = \Big[\left(\tau_{xx}\mathbf{i}\frac{dY}{m}\right)_3 - \left(\tau_{xx}\mathbf{i}\frac{dY}{m}\right)_1 + \left(\tau_{xy}\mathbf{j}\frac{dY}{m}\right)_3 - \left(\tau_{xy}\mathbf{j}\frac{dY}{m}\right)_1 + \left(\tau_{yy}\mathbf{j}\frac{dX}{m}\right)_4 - \left(\tau_{yy}\mathbf{j}\frac{dX}{m}\right)_2 + \left(\tau_{yx}\mathbf{i}\frac{dX}{m}\right)_4 - \left(\tau_{yx}\mathbf{i}\frac{dX}{m}\right)_2\Big]dz.$$

Using (44), this becomes

$$\mathbf{F} = \frac{m^2}{\rho}\left[\frac{\partial}{\partial X}\left(\frac{\tau_{xx}\mathbf{i}}{m} + \frac{\tau_{xy}\mathbf{j}}{m}\right) + \frac{\partial}{\partial Y}\left(\frac{\tau_{yy}\mathbf{j}}{m} + \frac{\tau_{yx}\mathbf{i}}{m}\right)\right]. \qquad (45)$$

Expanding the right-hand side, employing (5)–(8), and setting $\tau_{yx} = \tau_{xy}$, one may express (45) as

$$\mathbf{F} = \frac{m^2}{\rho}\left\{\left[\frac{\partial}{\partial X}\left(\frac{\tau_{xx}}{m}\right) - \tau_{yy}\frac{\partial}{\partial X}\left(\frac{1}{m}\right) + m\frac{\partial}{\partial Y}\left(\frac{\tau_{xy}}{m^2}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial Y}\left(\frac{\tau_{yy}}{m}\right) - \tau_{xx}\frac{\partial}{\partial Y}\left(\frac{1}{m}\right) + m\frac{\partial}{\partial X}\left(\frac{\tau_{xy}}{m^2}\right)\right]\mathbf{j}\right\}. \qquad (46)$$

This is a general form for $\mathbf{F}$. It holds for any method of calculating the stress components [not just (39)–(41)].

We will now make some simplifications assuming the stress components can be expressed as (39)–(41). If either $K_1$ or $\nabla\cdot\mathbf{V}$ is zero, $\tau_{yy} = -\tau_{xx}$ from (39) and (41), and (46) simplifies to

$$\mathbf{F} = \frac{m^3}{\rho}\left\{\left[\frac{\partial}{\partial X}\left(\frac{\tau_{xx}}{m^2}\right) + \frac{\partial}{\partial Y}\left(\frac{\tau_{xy}}{m^2}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial X}\left(\frac{\tau_{xy}}{m^2}\right) - \frac{\partial}{\partial Y}\left(\frac{\tau_{xx}}{m^2}\right)\right]\mathbf{j}\right\}. \qquad (47)$$

If we further assume that $K_2 = K_3$, then (47) simplifies to Eqs. (2B3)–(2B7) of Smagorinsky et al. (1965), which they give without proof. If we assume that $K_1 = 0$, $K_2 = K_3 = K$, a constant, and $m$ is constant, then (46) reduces to the Fickian relation [cf. (11)]

$$\mathbf{F} = \frac{Km^2}{\rho}\left[\left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\right)\mathbf{i} + \left(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2}\right)\mathbf{j}\right].$$

## 4. Generalization to nonisotropic map scales

Suppose that the map scale factors in the $x$ and $y$ directions $(m_x, m_y)$ are different, but the $x$ and $y$ directions are still orthogonal. The equations following (1) are replaced by

$$dx = \frac{dX}{m_x}, \quad dy = \frac{dY}{m_y}.$$

Then instead of (5)–(8) one obtains

$$\frac{\partial\mathbf{i}}{\partial X} = -m_y\frac{\partial}{\partial Y}\left(\frac{1}{m_x}\right)\mathbf{j}, \qquad (48)$$

$$\frac{\partial\mathbf{i}}{\partial Y} = m_x\frac{\partial}{\partial X}\left(\frac{1}{m_y}\right)\mathbf{j}, \qquad (49)$$

$$\frac{\partial\mathbf{j}}{\partial X} = m_y\frac{\partial}{\partial Y}\left(\frac{1}{m_x}\right)\mathbf{i}, \qquad (50)$$

and

$$\frac{\partial\mathbf{j}}{\partial Y} = -m_x\frac{\partial}{\partial X}\left(\frac{1}{m_y}\right)\mathbf{i}. \qquad (51)$$

The gradient operator (3) now becomes

$$\nabla = m_x\mathbf{i}\frac{\partial}{\partial X} + m_y\mathbf{j}\frac{\partial}{\partial Y}. \qquad (52)$$

Instead of (9), the divergence becomes

$$\nabla\cdot\mathbf{V} = m_x m_y\left[\frac{\partial}{\partial X}\left(\frac{u}{m_y}\right) + \frac{\partial}{\partial Y}\left(\frac{v}{m_x}\right)\right]. \qquad (53)$$

Equation (12) for the advective acceleration is transformed to

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \left[m_x u\frac{\partial u}{\partial X} + m_y v\frac{\partial u}{\partial Y} - v\left(u\frac{m_y}{m_x}\frac{\partial m_x}{\partial Y} - v\frac{m_x}{m_y}\frac{\partial m_y}{\partial X}\right)\right]\mathbf{i} + \left[m_x u\frac{\partial v}{\partial X} + m_y v\frac{\partial v}{\partial Y} + u\left(u\frac{m_y}{m_x}\frac{\partial m_x}{\partial Y} - v\frac{m_x}{m_y}\frac{\partial m_y}{\partial X}\right)\right]\mathbf{j}. \qquad (54)$$

Equations (53) and (54) agree with (A-18) and (A-23) of Haltiner and Williams (1980), which they present without proof.

The elements of the strain rate tensor, given by (21)–(23), are replaced by

$$s_{11} = m_x\left[\frac{\partial u}{\partial X} + v m_y \frac{\partial}{\partial Y}\left(\frac{1}{m_x}\right)\right], \qquad (55)$$

$$s_{12} = \frac{1}{2}\left\{m_y\left[\frac{\partial u}{\partial Y} - v m_x \frac{\partial}{\partial X}\left(\frac{1}{m_y}\right)\right] + m_x\left[\frac{\partial v}{\partial X} - u m_y \frac{\partial}{\partial Y}\left(\frac{1}{m_x}\right)\right]\right\} = s_{21}, \qquad (56)$$

and

$$s_{22} = m_y\left[\frac{\partial v}{\partial Y} + u m_x \frac{\partial}{\partial X}\left(\frac{1}{m_y}\right)\right]. \qquad (57)$$

Equations (55)–(57) agree with (A.9.16) of Kamenkovich (1977), which he gives without proof. Equations (24)–(27), (34)–(38) and (42)–(43) are unchanged, provided (55)–(57) are employed in (25)–(27) and (42). Equations (45) and (46) are modified to

$$\mathbf{F} = \frac{m_x m_y}{\rho}\left[\frac{\partial}{\partial X}\left(\frac{\tau_{xx}\mathbf{i}}{m_y} + \frac{\tau_{xy}\mathbf{j}}{m_y}\right) + \frac{\partial}{\partial Y}\left(\frac{\tau_{yy}\mathbf{j}}{m_x} + \frac{\tau_{yx}\mathbf{i}}{m_x}\right)\right] \qquad (58)$$

and

$$\mathbf{F} = \frac{m_x m_y}{\rho}\left\{\left[\frac{\partial}{\partial X}\left(\frac{\tau_{xx}}{m_y}\right) - \tau_{yy}\frac{\partial}{\partial X}\left(\frac{1}{m_y}\right) + m_x\frac{\partial}{\partial Y}\left(\frac{\tau_{xy}}{m_x^2}\right)\right]\mathbf{i} + \left[\frac{\partial}{\partial Y}\left(\frac{\tau_{yy}}{m_x}\right) - \tau_{xx}\frac{\partial}{\partial Y}\left(\frac{1}{m_x}\right) + m_y\frac{\partial}{\partial X}\left(\frac{\tau_{xy}}{m_y^2}\right)\right]\mathbf{j}\right\}. \qquad (59)$$

## 5. Concluding remarks

The derivations presented here will probably be of most interest to those concerned with the dynamics of ice, or similar problems in which viscous forces are important. The main results [(46) and (59)] are general and may be applied to various topics in continuum mechanics.

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