The Equivalency of the Tangent and Secant Lambert Conformal Map Projections

HARRY R. GLAHN

Techniques Development Laboratory, Office of Systems Development, National Weather Service, NOAA, Silver Spring, Maryland

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ABSTRACT

There is a popular misconception that the secant form of the Lambert conformal map projection is "better" than the tangent form. It is shown here that the two forms are equivalent; they are different only in the sense that the scale of the map quoted is usually true at the two secant latitudes for the secant projections and at the single tangent latitude for the tangent projection.

1. Introduction

The Lambert (conformal) map projection is well known to meteorologists. It may also be well known that both tangent and secant forms are possible. However, there seems to be a popular misconception that the secant form is "better" than the tangent because the former produces less distortion of features on the map as a function of latitude. Comments to this effect have been heard on many occasions and various references are found that discuss both secant and tangent projections and imply, if not outright state, that they are different and/or that the secant projection is better (e.g., Saucier 1955; Richardus and Adler 1972; Bowditch 1962; Maloney 1978; Pearson 1990). Actually, the two are equivalent—they are different only as to where the scale is quoted. For simplicity in the discussion below, only North Polar projections are considered, and all latitudes, unless otherwise noted, are in the Northern Hemisphere.

2. Tangent conic projections

The basic equations used here are given by Saucier (1955). Only enough review of Saucier's treatment is given for the results to be understandable. He states, "the process of map-making may be considered . . . as consisting of two steps. First, the surface of the earth is projected upon some fictitious geometric surface, the *image* surface, which is then developed by flattening into a plane surface." Consider the earth represented by the circle in Fig. 1, and a right circular cone sitting atop it with apex above (or at) the North Pole. The surface of the earth (down to the equator, say) can be projected (from some point or points) onto this image surface, then the cone cut along some meridian and flattened into a plane. The result will be a portion of a circle—a pie (about the size of the earth) with a slice cut out. Finally, this image surface must be reduced to usable size by applying a reduction factor that will give the desired map scale (e.g., 1 inch per 500 km). The projection onto the cone could be geometric, i.e., from a single point, but the resulting map would not be conformal. A certain stretching along meridians would be necessary to make it conformal; this can be thought of as a projection from a series of points along the earth's axis, these points being determined in a precise mathematical way to agree with Eq. (1).[1]

In Fig. 1 and the following sections,

ϕ = latitude,
ψ = colatitude ($90° - \phi$),
ϕ_0 = latitude of tangency,
ψ_0 = colatitude of tangency,
σ = image scale = distance on image/distance on earth ($\sigma = 1$ at ψ_0), and
n = geometric property of tangent cone = $\cos\psi_0$.

The image scale for the tangent conic projection is

$$\sigma = \frac{\sin\psi_0}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(\psi_0/2)}\right]^n. \quad (1)$$

This image scale σ is unity at ψ_0 and is larger for all other latitudes.

3. Secant conic projections

The difference between tangent and secant conic projections is that the image cone is tangent to the earth in the former and cuts the earth at two latitudes, say ψ_1 and ψ_2, in the latter (see Fig. 2). The relevant equations for the secant projection are

$$n = \frac{\log \sin\psi_1 - \log \sin\psi_2}{\log \tan(\psi_1/2) - \log \tan(\psi_2/2)}$$

and

$$\sigma = \frac{\sin\psi_1}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(\psi_1/2)}\right]^n = \frac{\sin\psi_2}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(\psi_2/2)}\right]^n.$$

Corresponding author address: Dr. Harry R. Glahn, Techniques Development Laboratory, NOAA National Weather Service, Silver Spring, MD 20910.

[1] Mathematicians view mapmaking as defining a function relating points on a flat surface to points on the earth. Such a function might be geometric in nature, or might consist of algebraic relationships between coordinate systems on the flat surface and on the earth. The concept of visual "projection" onto a flat surface may be important pedagogically but is of less importance in actual mapmaking.

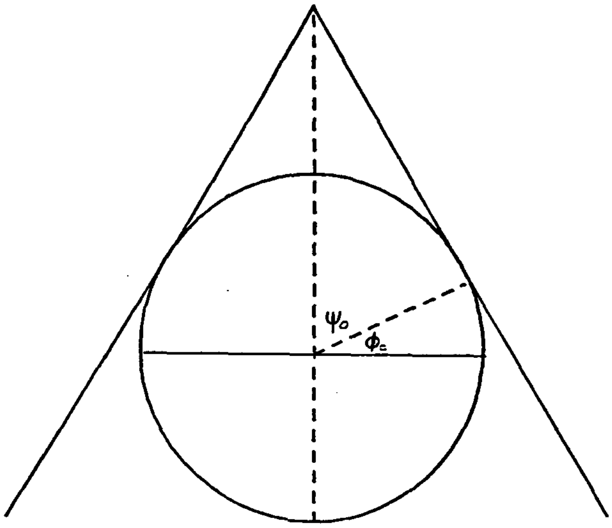

FIG. 1. Tangent cone as a basis for a conic projection.

The image scale σ is unity at ψ_1 and ψ_2, is smaller between ψ_1 and ψ_2, and is larger at other latitudes.

4. Comparison of tangent and secant projections

It is not the actual value of σ that is important, but rather how it changes with latitude. If the ratio of the image scale for a secant projection (call it σ_S) to the image scale for a tangent projection (call it σ_T) does not depend on latitude, the two projections are equivalent; that is, an actual map produced by reducing the image surface associated with the tangent projection and another map produced by reducing the image surface associated with the secant projection would be exactly the same, providing the appropriate reduction factors were used.

Consider a projection secant at ψ_1 and ψ_2. The geometric property, n, of the cone associated with that projection is the same as the n associated with a projection tangent at ψ_0 when

$$n = \cos\psi_0 = \frac{\log \sin\psi_1 - \log \sin\psi_2}{\log \tan(\psi_1/2) - \log \tan(\psi_2/2)}. \quad (2)$$

This equation can be solved for ψ_0. The ratio of σ_S to σ_T is

$$\frac{\sigma_S}{\sigma_T} = \frac{\sin\psi_1}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(\psi_1/2)}\right]^n \cdot \left\{\frac{\sin\psi_0}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(\psi_0/2)}\right]^n\right\}^{-1} = \frac{\sin\psi_1}{\sin\psi_0}\left[\frac{\tan(\psi_0/2)}{\tan(\psi_1/2)}\right]^n$$

and does not depend on ψ, so the projections are equivalent. As an example, for $\psi_1 = 60°$, $\psi_2 = 30°$, and $\psi_0 = 44.31°$; $n = 0.7156$ and $\sigma_S/\sigma_T = 0.9656$. It is interesting to note that for $\psi_1 = 60°$ and $\psi_2 = 30°$, the "equivalent" tangent projection is not $\psi_0 = 45°$, but rather 44.31°, as evaluation of Eq. (2) shows; that is, for this secant projection, σ_S is a minimum at 44.31°.

For computations on a grid, many times a "map factor" is used that can be defined as

$$m_L = \sigma/\sigma_L$$

for a grid with the scale defined at latitude L. For $L = 60°$, the map factor for a Lambert tangent at 25° is

$$m_{60} = \frac{\sigma}{\sigma_{60}} = \frac{\sin 65°}{\sin\psi}\left[\frac{\tan(\psi/2)}{\tan(65°/2)}\right]^{\cos 65°} \times \left\{\frac{\sin 65°}{\sin 30°}\left[\frac{\tan(30°/2)}{\tan(65°/2)}\right]^{\cos 65°}\right\}^{-1} = \frac{0.872\,[\tan(\psi/2)]^{0.423}}{\sin\psi}.$$

This map factor is exactly the same for any "equivalent" secant projection, of which there are many.

5. Limiting forms of the conic projection

Two limiting cases of the cone in Fig. 1 are 1) when the apex approaches infinity, the cone approaches a right circular cylinder tangent at the equator, and 2) when the apex is at the North Pole, the cone is a plane tangent at the pole. The general conic projection is the basis for the Lambert conformal map, and the limiting cases are, respectively, the bases for 1) the Mercator and 2) the polar stereographic projections.

At $\phi_0 = 0°$ ($\psi_0 = 90°$), Eq. (1) represents the tangent Mercator projection and becomes

$$\sigma_T = 1/\sin\psi = 1/\cos\phi. \quad (3)$$

The limiting form of Eq. (1) as ψ_0 approaches 0° is

$$\sigma_T = 2/(1 + \cos\psi) = 2/(1 + \sin\phi) \quad (4)$$

for the tangent polar stereographic projection.

It is noted that, in general, Eq. 1 does not represent a "geometric" projection. The projection of the earth's

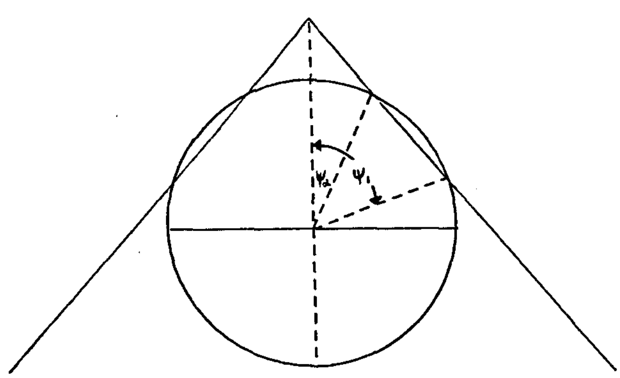

FIG. 2. Secant cone as a basis for a conic projection.

surface to the image plane is not from a single point. However, the polar stereographic is a geometric projection, the point being on the earth opposite to the point of tangency. [Bowditch (1962) uses the term "perspective" rather than "geometric" in this context.]

For the Mercator projection secant at north and south latitudes of ϕ_1,

$$\sigma_S = \cos\phi_1/\cos\phi. \qquad (5)$$

For the polar stereographic secant at north latitude ϕ_1,

$$\sigma_S = (1 + \sin\phi_1)/(1 + \sin\phi). \qquad (6)$$

Since the numerators of Eqs. 3 and 5 and of Eqs. 4 and 6 are constants, σ_S/σ_T is a constant for the Mercator and for the polar stereographic projections. Thus the tangent and secant projections for each of the limiting cases are equivalent.

6. Conclusions

Therefore, it is seen that it doesn't really matter whether a tangent or secant conic projection is used, except, possibly, for one consideration. One may want to quote the scale of the map in such a way that the magnitude of the variation from that value is the least possible over some range of latitudes. To do this, the map scale should be quoted at two latitudes, so that it varies in one direction between those latitudes and in the other direction otherwise. But again, there is no difference between the projections; it's just a matter of where the scale is quoted—that is, one could use tangent projection equations and quote the scale at other than the tangent latitude. The only practical problem in doing this is how to get the desired scale at the desired latitudes. However, this is easily handled by choosing the scale and latitudes, then computing the tangent projection to which this is equivalent.

Additional details on map projections used for meteorological purposes are given in Gerrity (1973), Hoke et al. (1981), and Glahn (1988).

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