NOTES AND CORRESPONDENCE

Application of Fast Fourier Transforms to the Direct Solution of a Class of Two-Dimensional Separable Elliptic Equations on the Sphere

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ABSTRACT

An efficient, direct, second-order solver for the discrete solution of a class of two-dimensional separable elliptic equations on the sphere (which generally arise in implicit and semi-implicit atmospheric models) is presented. The method involves a Fourier transformation in longitude and a direct solution of the resulting coupled second-order finite-difference equations in latitude. The solver is made efficient by vectorizing over longitudinal wavenumber and by using a vectorized fast Fourier transform routine. It is evaluated using a prescribed solution method and compared with a multigrid solver and the standard direct solver from FISHPAK.

1. Introduction

Numerical techniques used in global atmospheric models have evolved over the past several decades. Models with explicit time differencing generally require very small time steps in order to avoid linear computational instability associated with fast-moving gravity waves, particularly because of the convergence of meridians near the pole. The introduction of semi-implicit time differencing (Robert 1969) relaxed the requirement for linear computational stability and allowed larger time steps relative to explicit schemes. Further advances occurred through the introduction of semi-Lagrangian semi-implicit time-differencing schemes [see Staniforth and Cote (1991) and Bates et al. (1993) for a comprehensive review of the evolution of the semi-Lagrangian approach].

The implicit (or semi-implicit) time differencing, in general, leads to an elliptic equation (two- or three-dimensional and separable or nonseparable depending on the formulation) on the sphere (Temperton and Staniforth 1987; McDonald and Bates 1989; Tanguay et al. 1989; Bates et al. 1990; Barros et al. 1989). Thus, for the implicit schemes to be more economical compared to their explicit counterparts, efficient solvers are of paramount importance.

Algorithms for the direct solution of separable elliptic equations have been around for awhile. Swarztrauber (1974a) developed a method of direct solution of separable elliptic equations by extending the stabilized cyclic-reduction algorithm. When the coefficients of the elliptic equations are independent of one of the dimensions (which is the class of equations we are considering here), the problem can be solved with the application of Fourier analysis. Hockney (1965) used this approach to obtain the direct solution of Poisson's equation. Lindzen and Kuo (1969) suggested the use of a Fourier transform in one direction combined with the direct solution of the resulting second-order ordinary differential equations in the other, for more general elliptic equations. Le Bail (1972) applied fast Fourier transforms (FFTs) to solve a class of partial differential equations.

Until recently, fast direct solvers on the sphere have been available for the discrete Poisson- or Helmholtz-type elliptic equations (Swarztrauber and Sweet 1973, 1975; Sweet 1973, 1974; Swarztrauber 1974b; Adams et al. 1980). Thus, many implicit time-differencing schemes have been geared toward obtaining such elliptic equations (e.g., McDonald and Bates 1989). Bates et al. (1990; hereafter BSHB) were the first to obtain a more general elliptic equation for their vector semi-Lagrangian semi-implicit time-differencing scheme in the global shallow-water framework. Because no efficient direct solver was available at that time, they used the multigrid solver developed by Barros (1991); multigrid methods can be applied to more general elliptic equations and more complex domains (Phillips 1984; Fulton et al. 1986; Barros et al. 1989; Bates et al. 1990; Barros 1991). When, however, the scheme of BSHB was extended to a global multilevel primitive equation model (Bates et al. 1993), it was soon realized that the original multigrid solver was not efficient enough. This led to the development of the FFT-based direct solver presented here.

It should be emphasized that the idea of using FFTs is not new; they are routinely used to solve Poisson and Helmholtz equations on the sphere. Recently, Cote and Staniforth (1990) have described and applied this approach to a more general elliptic problem.

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In this note, we describe an FFT-based solver for a class of two-dimensional separable elliptic equations on the sphere. In section 2, we present a brief description of the solver. In section 3, we validate the solver and compare it with the multigrid method used by BSHB and the direct solver from FISHPAK (Adams et al. 1980). A summary is given in section 4.

2. The solver

a. *Method of solution*

Here, we consider the following class of elliptic equations on the sphere:

$$\frac{c_1(\theta)}{a^2\cos^2\theta}\frac{\partial^2\phi}{\partial\lambda^2}+\frac{c_2(\theta)}{a^2\cos\theta}\frac{\partial^2\phi}{\partial\lambda\partial\theta}+\frac{1}{a^2\cos\theta}\frac{\partial}{\partial\theta}\left[c_3(\theta)\cos\theta\frac{\partial\phi}{\partial\theta}\right]+\frac{c_4(\theta)}{a\cos\theta}\frac{\partial\phi}{\partial\lambda}+\frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}[c_5(\theta)\cos\theta\phi]+c_6(\theta)\phi=F, \quad (1)$$

where a is the radius of the sphere, λ and θ are the longitude and latitude, respectively, ϕ is the solution, and F is the forcing (which is known). Here, the coefficients $c_1(\theta)$, $c_2(\theta)$, $c_3(\theta)$, $c_4(\theta)$, $c_5(\theta)$, and $c_6(\theta)$ are, at most, functions of latitude. If any of these coefficients is also a function of longitude, then the problem becomes nonseparable, and the multigrid method becomes preferable (Phillips 1984).

We solve (1) on a uniform longitude and latitude grid on the sphere. Supposing that we have I equally spaced grid points along a latitude circle, any function ϕ can be expanded in a finite Fourier series in the longitudinal direction λ as

$$\phi(\lambda_n)=\sum_{-I/2}^{I/2}\hat{\phi}(k)e^{ik\lambda_n}, \quad n=1,2,\cdots I. \quad (2)$$

Here, $\hat{\phi}(k)$ is the complex amplitude for wavenumber k, and $\hat{\phi}(-k)$ is the complex conjugate of $\hat{\phi}(k)$, where $i=\sqrt{-1}$. The complex amplitude can be obtained by

$$\hat{\phi}(k)=\frac{1}{I}\sum_{1}^{I}\phi(\lambda_n)e^{-ik\lambda_n}. \quad (3)$$

Since we are considering real data, for $k=0$ (i.e., the longitudinal mean part) and $k=I/2$, only the real part of the amplitude functions exist.

For any wavenumber k, (1) reduces to a second-order ordinary differential equation in θ for the complex amplitude. Without loss of generality, let us consider a solution to (1) of the form

$$\phi=\hat{\phi}e^{ik\lambda}, \quad \text{and} \quad F=\hat{F}e^{ik\lambda}, \quad (4)$$

where $\hat{\phi}=\phi^r+i\phi^i$, $\hat{F}=F^r+iF^i$. After substituting (4) into (1), we obtain

$$\frac{-k^2c_1(\theta)}{a^2\cos^2\theta}\hat{\phi}+\frac{ikc_2(\theta)}{a^2\cos\theta}\frac{\partial\hat{\phi}}{\partial\theta}+\frac{1}{a^2\cos\theta}\frac{\partial}{\partial\theta}\left[c_3(\theta)\cos\theta\frac{\partial\hat{\phi}}{\partial\theta}\right]+\frac{ikc_4(\theta)}{a\cos\theta}\hat{\phi}+\frac{1}{a\cos\theta}\frac{\partial}{\partial\theta}[c_5(\theta)\cos\theta\hat{\phi}]+c_6(\theta)\hat{\phi}=\hat{F}. \quad (5)$$

We represent the distance from the south pole to the north pole of the sphere by a set of J equally spaced grid points. Each grid point is referred to by an integer index j with $j=1$ and $j=J$ representing the south and the north poles, respectively. For interior points of the domain, we approximate the latitudinal derivatives in (5) by second-order finite differences. Thus, we assume

$$\frac{1}{a^2\cos\theta_j}\frac{\partial}{\partial\theta}\left(c_3\cos\theta\frac{\partial}{\partial\theta}\right)\phi_j=\frac{(c_3\cos\theta)_{j+1/2}(\phi_{j+1}-\phi_j)-(c_3\cos\theta)_{j-1/2}(\phi_j-\phi_{j-1})}{a^2\cos\theta_j(\Delta\theta)^2}$$

and

$$\frac{1}{a\cos\theta_j}\frac{\partial}{\partial\theta}(c_5\cos\theta\phi)_j=\frac{(c_5\cos\theta)_{j+1/2}(\phi_{j+1}+\phi_j)-(c_5\cos\theta)_{j-1/2}(\phi_j+\phi_{j-1})}{a\cos\theta_j 2\Delta\theta}.$$

Then from (5) we obtain

$$\left[\frac{(c_3\cos\theta)_{j+1/2}}{a^2\cos\theta_j(\Delta\theta)^2}+\frac{(c_5\cos\theta)_{j+1/2}}{a\cos\theta_j 2\Delta\theta}\right]\hat{\phi}_{j+1}-\left\{\frac{k^2(c_1)_j}{a^2\cos^2\theta_j}+\frac{1}{a^2\cos\theta_j(\Delta\theta)^2}[(c_3\cos\theta)_{j+1/2}+(c_3\cos\theta)_{j-1/2}]-\frac{1}{a\cos\theta_j 2\Delta\theta}[(c_5\cos\theta)_{j+1/2}-(c_5\cos\theta)_{j-1/2}]-(c_6)_j\right\}\hat{\phi}_j+\left[\frac{(c_3\cos\theta)_{j-1/2}}{a^2\cos\theta_j(\Delta\theta)^2}-\frac{(c_5\cos\theta)_{j-1/2}}{a\cos\theta_j 2\Delta\theta}\right]\hat{\phi}_{j-1}+\frac{ik(c_2)_j}{a^2\cos\theta_j 2\Delta\theta}\hat{\phi}_{j+1}+\frac{ik(c_4)_j}{a\cos\theta_j}\hat{\phi}_j-\frac{ik(c_2)_j}{a^2\cos\theta_j 2\Delta\theta}\hat{\phi}_{j-1}=\hat{F}_j, \quad (6)$$

where $1<j<J$.

At the north pole, following the integral approach of Barros (1991) and BSHB, we obtain

$$\bar{\phi}_J\left[(c_6)_J - \frac{4}{a^2(\Delta\theta)^2}(c_3)_{J-1/2} - \frac{2}{a\Delta\theta}(c_5)_{J-1/2}\right] + \bar{\phi}_{J-1}\left[\frac{4}{a^2(\Delta\theta)^2}(c_3)_{J-1/2} - \frac{2}{a\Delta\theta}(c_5)_{J-1/2}\right] = \bar{F}_J, \quad (7)$$

where the overbar represents a longitudinal average. Similarly, for the south pole we obtain

$$\bar{\phi}_1\left[(c_6)_1 - \frac{4}{a^2(\Delta\theta)^2}(c_3)_{1+1/2} + \frac{2}{a\Delta\theta}(c_5)_{1+1/2}\right] + \bar{\phi}_2\left[\frac{4}{a^2(\Delta\theta)^2}(c_3)_{1+1/2} + \frac{2}{a\Delta\theta}(c_5)_{1+1/2}\right] = \bar{F}_1. \quad (8)$$

The polar values for the longitudinally asymmetric part of ϕ are zero.

The problem is now reduced to a system of two simultaneous second-order finite-difference equations whose solution is obtained following Lindzen and Kuo (1969) and Chao (1979). For further details and a listing of the Fortran code, see Moorthi and Higgins (1992).

b. Coding strategy

The first step of the solution procedure involves the use of a forward FFT to obtain the complex amplitudes of the forcing function F that depend only on latitude. Then, for each wavenumber, the complex amplitude of the solution is obtained by solving the coupled second-order difference equations. Finally, we use a backward FFT to obtain the solution.

We made this procedure very efficient by using the CRAY subroutine RFFTMLT to transform (both forward and backward) all latitudes simultaneously. In addition, we wrote the code to solve two coupled second-order difference equations by vectorizing over longitudinal wavenumbers. This vectorization coupled with the vectorized FFT package makes our direct solver very efficient.

It should be pointed out that if the longitudinal derivatives are approximated by finite differences, then the definitions of k and k^2 should be appropriately changed in all of the equations above. Also, the FFT has a restriction on I, namely, that $I = 2^p \times 3^q \times 5^r$, where p, q, and r are integers. There is, however, no restriction on the choice of J.

3. Evaluation of the solver

In this section, we present results from some tests to evaluate the solver. Since it is difficult to find analytical solutions to (1) against which we can compare the numerical solutions, we adopt the following "prescribed solution" procedure. Under this procedure, we assume a solution a priori and apply the differential operator on the left-hand side of (1) to obtain the forcing F. Then we obtain the numerical solution for this forcing and compare the results with the original assumed solution. Here, we consider the following simple function for ϕ:

$$\phi_{i,j} = 5.0 \times 10^4 + 1.0 \times 10^3 \cos\theta_j \cos(2\pi i/I), \quad (9)$$

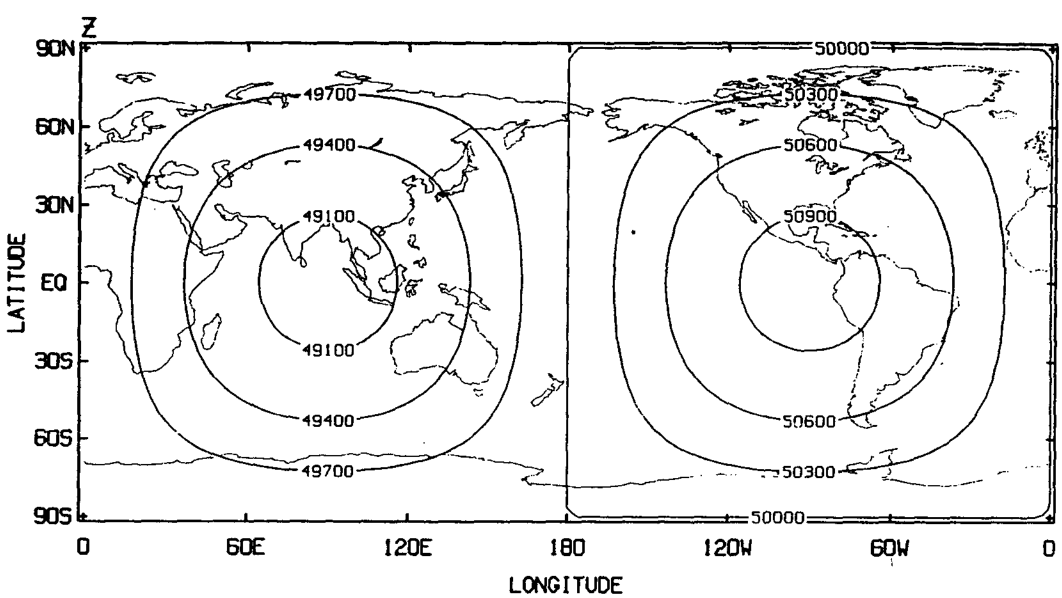

FIG. 1. Contour map of the prescribed solution [Eq. (9)] to the elliptic equation with $(I, J) = (96, 49)$. The contour interval is 300.

where i and j are longitudinal and latitudinal indices, respectively. A contour map of this function is shown in Fig. 1. The forcing $F_{i,j}$ is computed using a second-order-accurate finite-difference operator corresponding to the left-hand side of (1). In the following, we consider two cases. In case 1, we apply the solver to the elliptic equation of BSHB and compare the results with those obtained from the multigrid solver. In case 2, we apply it to the Poisson equation and compare the results with those obtained from FISHPAK.

a. Case 1

The elliptic equation of BSHB is obtained by setting

$$c_1(\theta) = G, \quad c_2(\theta) = 0.0, \quad c_3(\theta) = G$$

$$c_4(\theta) = -\frac{\partial(GF)}{a\partial\theta}, \quad c_5(\theta) = 0.0,$$

$$c_6(\theta) = -[(\Delta t/2)^2\bar{\phi}]^{-1},$$

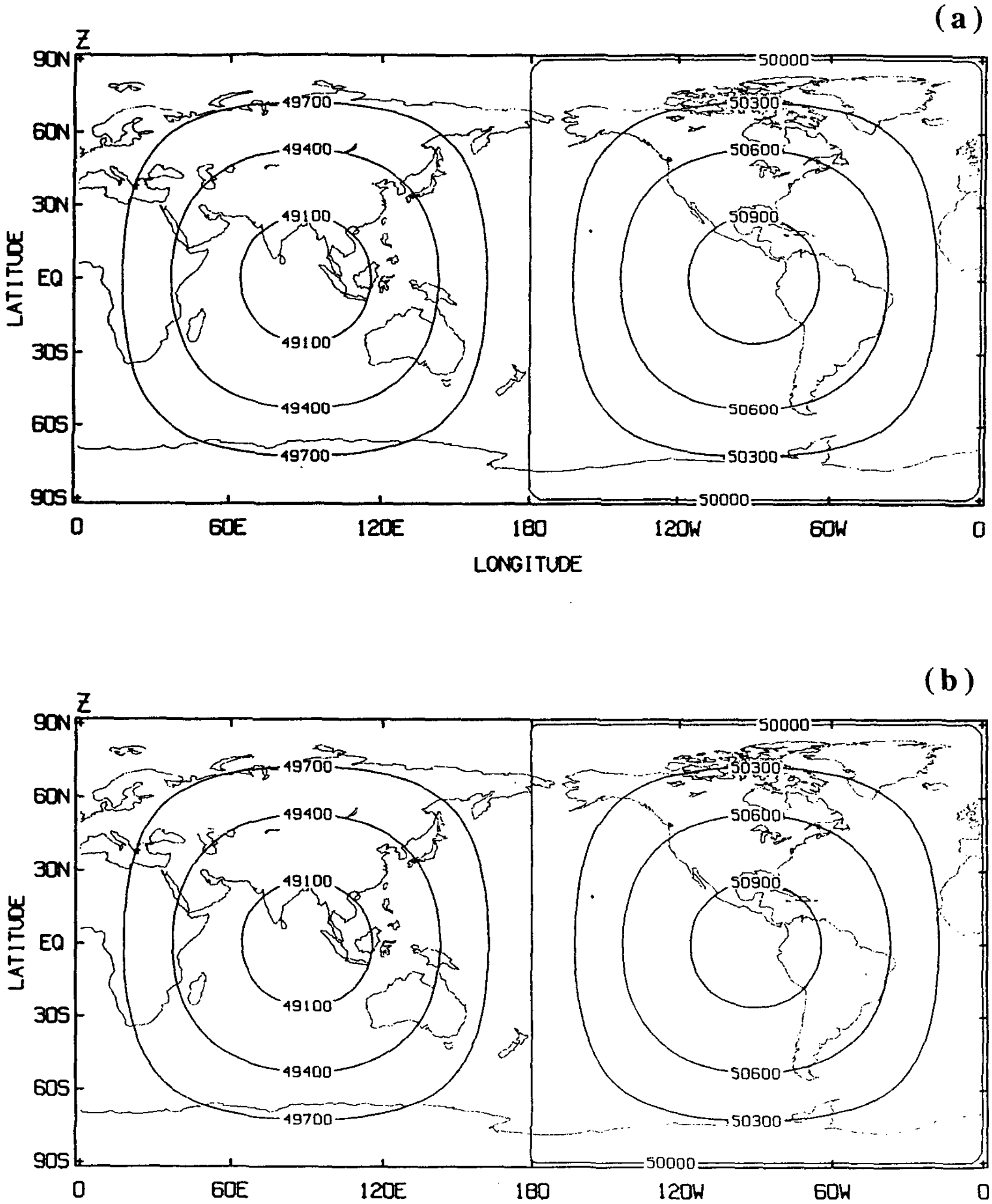

FIG. 2. Numerical solution in case 1 for (a) the direct solver and (b) the multigrid solver with $(I, J) = (96, 49)$. The contour interval is 300.

where

$$G = (1 + F^2)^{-1}, \quad \text{and} \quad F = \Delta t \Omega \sin\theta.$$

Here, $\bar{\phi} = 50\,000$ is a mean value, $\Delta t = 3600$ s, and Ω is the earth's rotation rate. The 2D multigrid solver of BSHB is used in the full-multigrid mode. We use a single V cycle with one relaxation sweep both before and after the coarse-grid correction and eight relaxation steps on the coarsest grid. These are identical to those used in BSHB and more details are available in that paper.

We solved this problem for various resolutions ranging from $(I, J) = (48, 25)$ to $(I, J) = (768, 385)$. Figures 2a and 2b show the numerical solution for the direct method and the multigrid method, respectively, with $(I, J) = (96, 49)$. The solutions appear to be almost identical to those shown in Fig. 1. The accuracy of the solution, however, is revealed in Figs. 3a and 3b, which show the difference between the exact and

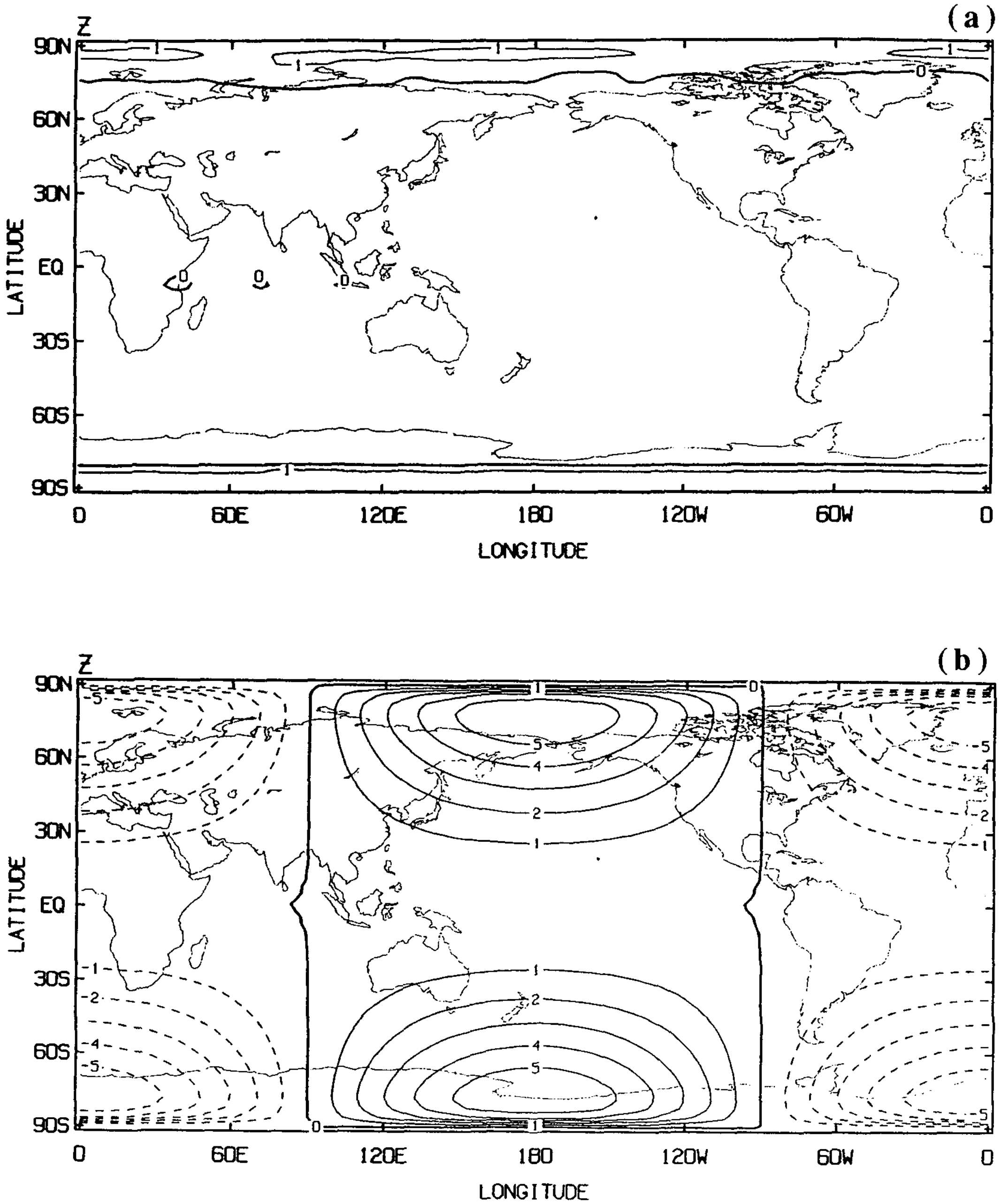

FIG. 3. As in Fig. 2 except for the difference between the numerical and prescribed solutions. The contour interval is 1. In (a) the difference is multiplied by a factor of 10^8, and in (b) the difference is multiplied by a factor of 10.

the numerical solutions for both solvers. Notice that in Fig. 3a the error is multiplied by a factor of 10^8, while in Fig. 3b, it is multiplied by a factor of 10. Thus, in this case the direct solver is several orders of magnitude more accurate. Similar results were also found at other resolutions (not shown). Furthermore, the accuracy of the multigrid solver did not improve significantly when both the number of V cycles and the number of relaxation sweeps were increased. We do, however, recognize that the level of accuracy of the spatial discretization should be the level of accuracy desired for any problem. Nevertheless, an efficient direct solver is always preferable since its solution is close to machine accuracy.

We next examine the efficiency of the solvers. For this purpose, we present in Fig. 4 the CPU time (t) taken for 500 calls to the solver on a single processor of the CRAY YMP as a function of the total number (N) of grid points (i.e., $I \times J$). In Fig. 4a, the axes are linear, while in Fig. 4b they are logarithmic. The timings are also presented in Table 1. Clearly the direct solver is faster at all resolutions examined. Also, note that for both solvers, the time t is almost a linear function of the total number of grid points N.

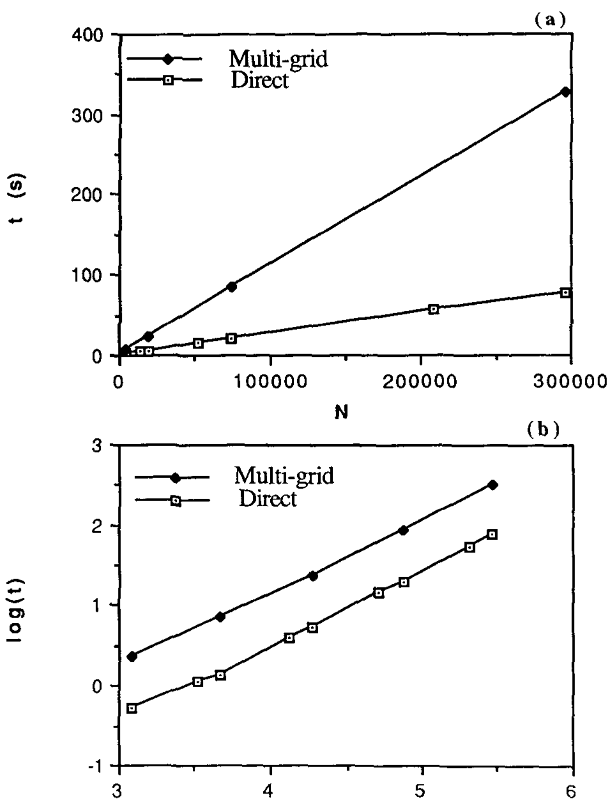

FIG. 4. The CPU time t taken for 500 calls to the direct and to the multigrid solvers on a single processor of the CRAY YMP as a function of the total number of grid points N. In (a) the axes are linear, while in (b) the axes are logarithmic.

TABLE 1. The CPU time t (s) for 500 calls to the direct solver, the multigrid solver, and the solver from FISHPAK on a single processor of the CRAY YMP as a function of resolution.

(I, J)	Direct solver	Multigrid solver	FISHPAK
(48, 25)	0.513	2.323	3.14
(72, 46)	1.110	—	—
(96, 49)	1.401	7.008	12.615
(144, 91)	4.062	—	—
(192, 97)	5.253	23.301	53.914
(288, 181)	14.464	—	—
(384, 193)	19.563	85.588	241.579
(576, 361)	55.889	—	—
(768, 385)	77.998	326.955	—

b. Case 2

Now we consider the Poisson equation, which is obtained by setting $c_1(\theta) = c_3(\theta) = 1$ and $c_2(\theta) = c_4(\theta) = c_5(\theta) = c_6(\theta) = 0$. It should be pointed out that when $c_6(\theta) = 0$ we cannot determine the constant part, and thus, there is no unique solution. In addition, if the forcing has a global mean component, then there is no solution to the problem. Therefore, in our solver we remove the global mean from the forcing F whenever $c_6(\theta) = 0$. This is equivalent to the perturbation method of Swarztrauber (1974b).

We used the forcing as in case 1 to compare the numerical solution with the assumed one. The errors in the numerical solution (not shown) are of the order 10^{-6} or less. Similar results are found for other resolutions.

For this case, we compare the efficiency of our direct solver with the direct solver from the FISHPAK package (Adams et al. 1980). As before, we show the CPU time t taken for 500 calls to both solvers as a function of the total number of grid points N (Fig. 5). Again, our direct solver is significantly faster at all resolutions, and there is a noticeable divergence between the two curves as the resolution increases.

We also confirmed that our solver works equally well for a general case in which all of the coefficients in (1) are nonzero functions of θ. Finally, we repeated each case above by assuming a prescribed solution made up of random numbers. We found for all cases that the errors were comparable to those discussed earlier.

4. Summary

An efficient, direct solver for the discrete solution of a class of two-dimensional separable elliptic equations on the sphere has been discussed. It is based on a Fourier decomposition in longitude and a direct solution of the resulting coupled second-order finite-difference equations in latitude. These equations are solved following the approach of Lindzen and Kuo (1969) and Chao (1979).

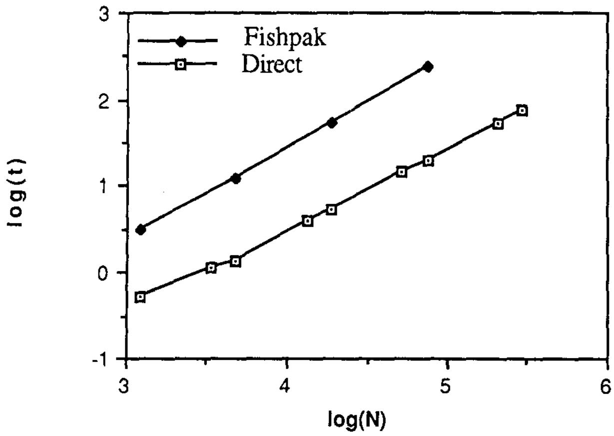

FIG. 5. As in Fig. 4b except for the present and the FISHPAK direct (subroutine HWSSSP) solvers.

For the elliptic equation of BSHB, we found that the aforementioned solver is both more efficient and accurate than the multigrid solver used by BSHB at all resolutions tested. For the special case of Poisson's equation we also found, at all resolutions, that our direct solver is more efficient than that available in FISHPAK (Adams et al. 1980).

Thus, our solver is both accurate and efficient and general enough that it could be used for any separable elliptic equation on the sphere with coefficients independent of longitude. It can also be applied to limited areas on the sphere if cyclic boundary conditions are invoked in longitude and if appropriate boundary conditions are used in latitude. It is currently being used in the global multilevel primitive equation semi-Lagrangian semi-implicit model of Bates et al. (1992) at the Goddard Laboratory for Atmospheres and in the adjoint-model development (Li et al. 1991) at the Florida State University. While the solver is accurate and flexible, it takes only a few percent of the CPU time taken by the multilevel model dynamics.

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