Evaluation of a Hydrostatic, Height-Coordinate Formulation of the Primitive Equations for Atmospheric Modeling

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ABSTRACT

The hydrostatic form of the primitive equations described by Ooyama is evaluated by comparing nonhydrostatic and hydrostatic integrations of a dry axisymmetric model with a specified entropy (heat) source. In this formulation, pressure is a diagnostic variable, so that the hydrostatic approximation can be included simply by replacing the vertical momentum equation with a diagnostic vertical velocity equation. This diagnostic equation is a one-dimensional (height) second-order elliptic equation that can be solved using a direct method. Results show that hydrostatic solutions are very sensitive to the accuracy of the method used to solve the diagnostic vertical velocity equation. However, this sensitivity can be eliminated by adding an extra term to the diagnostic equation that ensures that the solution does not drift away from hydrostatic balance due to numerical approximation. When the extra term is added, this formulation of the primitive equations allows for the design of a numerical model in height coordinates that can be used in hydrostatic and nonhydrostatic regimes.

1. Introduction

Most global and regional forecast models that are run at operational centers use the hydrostatic primitive equations with normalized pressure (σ) as a vertical coordinate. In contrast, nonhydrostatic models typically use height as a vertical coordinate and pressure is a prognostic variable (e.g., Klemp and Wilhelmson 1978). As the resolution of regional atmospheric forecast models continues to increase, operational nonhydrostatic prediction models may soon be commonplace. Regional models usually obtain lateral boundary information from global model forecasts. A difficulty that must be overcome is the mismatch in the vertical coordinate and prognostic variables used in the global hydrostatic and regional nonhydrostatic models.

One possible solution to the mismatch between models is to formulate a nonhydrostatic model in σ coordinates. Laprise (1992) described a nonhydrostatic model where the hydrostatic part of the pressure is used as the basis for the transformation to σ coordinate. Ooyama (1992) and Dudhia (1993) presented models where a reference state (either time dependent or time independent) is used to transform from height to σ coordinates.

The primary motivation for the use of σ coordinates in nonhydrostatic models is to extend existing hydrostatic models into the nonhydrostatic regime. An alternative is to extend a nonhydrostatic height-coordinate model into the hydrostatic regime. This extension may be useful in simulations of weather systems that involve a wide range of horizontal scales. For example, to model a tropical cyclone, very fine resolution is required to resolve the eyewall circulation. However, tropical cyclones interact with synoptic-scale weather systems, so that a very large computational domain is required. Nested-grid techniques are often employed to simulate this wide range of horizontal scales (e.g., Kurita et al. 1990; DeMaria et al. 1992). The time step in nonhydrostatic models is limited by the horizontally and vertically propagating sound waves. Time-splitting techniques, where the terms related to sound wave propagation are integrated with a time step smaller than for the remaining terms, are commonly used to help overcome this limitation (e.g., Klemp and Wilhelmson 1978). In a nested-grid model, the horizontal resolution is usually much lower than the vertical resolution on the outer domains. On these outer domains, the small time step in the time-splitting technique could be increased if the hydrostatic approximation were employed to filter the vertically propagating sound waves. Although the small time step could also be increased if a vertically implicit difference scheme were employed, some physical insight might be gained by comparing hydrostatic and nonhydrostatic simulations with the same vertical coordinate.

The development of hydrostatic models in height coordinates has been rather limited. The original model proposed by Richardson (1922) used this formulation, and Kasahara and Washington (1967) demonstrated that these equations can be used in a numerical predic-
tion model. However, as described by Ooyama (1990), Richardson included pressure as a prognostic variable, which makes the diagnostic equation for vertical velocity quite complicated. Ooyama presented an alternative formulation of the primitive equations where pressure is diagnostic. In this formulation, the diagnostic equation for vertical velocity is less complex than in Richardson’s case.

In this paper, the hydrostatic form of the primitive equations described by Ooyama (1990) is evaluated by comparing nonhydrostatic and hydrostatic integrations of a dry axisymmetric model with a specified entropy (heat) source. The governing equations are described in section 2 and the axisymmetric model is presented in section 3. The hydrostatic and nonhydrostatic solutions are compared in section 4.

2. Governing equations

From Ooyama (1990), the prognostic equations for the case of a dry atmosphere can be written as

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} + w \frac{\partial \mathbf{V}}{\partial z} + f \mathbf{k} \times \mathbf{V} + \frac{1}{\rho} \nabla P = \mathbf{F}_h, \tag{2.1}
\]

\[
\frac{\partial w}{\partial t} + \mathbf{V} \cdot \nabla w + w \frac{\partial w}{\partial z} + g + \frac{1}{\rho} \frac{\partial P}{\partial z} = F_z, \tag{2.2}
\]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) + \frac{\partial (w \rho)}{\partial z} = Q_p, \tag{2.3}
\]

\[
\frac{\partial \sigma}{\partial t} + \nabla \cdot (\mathbf{V} \sigma) + \frac{\partial (w \sigma)}{\partial z} = Q_s, \tag{2.4}
\]

where \( \mathbf{V} \) is the horizontal velocity vector, \( w \) is the vertical component of velocity, \( \nabla \) is the horizontal Laplacian operator, \( \rho \) is density, \( \sigma \) is entropy density (entropy per unit volume), \( P \) is pressure, \( \mathbf{F}_h \) is the horizontal frictional force, \( F_z \) is the vertical component of the frictional force, \( Q_p \) and \( Q_s \) are sources and sinks of density and entropy, respectively, \( f \) is the Coriolis parameter, \( k \) is the vertical unit vector and \( g \) is the acceleration of gravity. Note that in (2.4), \( \sigma \) represents entropy density and not the normalized pressure vertical coordinate. This notation will be used in the remainder of the paper and was chosen for consistency with Ooyama (1990).

From a thermodynamic point of view, \( P \) is a function of \( \rho \) and \( \sigma \) so that the pressure gradients in (2.1) and (2.2) can be calculated from gradients of \( \rho \) and \( \sigma \) using the chain rule as follows:

\[
\nabla P = \left( \frac{\partial P}{\partial \rho} \right) \nabla \rho + \left( \frac{\partial P}{\partial \sigma} \right) \nabla \sigma \tag{2.5}
\]

\[
\frac{\partial P}{\partial z} = \left( \frac{\partial P}{\partial \rho} \right) \frac{\partial \rho}{\partial z} + \left( \frac{\partial P}{\partial \sigma} \right) \frac{\partial \sigma}{\partial z}. \tag{2.6}
\]

The pressure coefficients \( \partial P/\partial \rho \) and \( \partial P/\partial \sigma \) in (2.5) and (2.6) are determined by differentiating the definition of the entropy density

\[
\sigma = \rho C_v \ln \left( \frac{T}{T_0} \right) - \rho R \ln \left( \frac{\rho}{\rho_0} \right), \tag{2.7}
\]

where \( T_0 \) and \( \rho_0 \) are the temperature and density of the reference state, and using the ideal gas law \( P = \rho RT \) to give

\[
\frac{\partial P}{\partial \rho} = \left( C_p - \sigma \right) \frac{RT}{C_v}, \tag{2.8}
\]

\[
\frac{\partial P}{\partial \sigma} = \frac{RT}{C_v}, \tag{2.9}
\]

where \( C_v \) and \( C_p \) are the specific heats of dry air at constant volume and constant pressure, respectively, and \( C_p - C_v = R \). In the case of a moist atmosphere described by Ooyama (1990), the temperature \( T \) is determined from the definition of entropy density, given \( \rho \), \( \sigma \), and moisture variables. In the moist case, an iterative procedure must be used to find \( T \), but in the dry case, (2.7) can be solved in closed form to give

\[
T = T_0 \left[ \left( \frac{\rho}{\rho_0} \right)^{R/C_v} \right] \exp \left( \frac{\sigma}{\rho C_v} \right). \tag{2.10}
\]

In summary, the nonhydrostatic governing equations are given by (2.1)–(2.4) where the pressure gradients are evaluated using (2.5) and (2.6), the pressure coefficients are determined from (2.8) and (2.9), and the temperature is given by (2.10).

In the case of a hydrostatic atmosphere, (2.2) is replaced by a diagnostic equation for \( w \). This diagnostic equation is derived by taking \( \partial \partial t \) of the hydrostatic equation to give

\[
\frac{\partial}{\partial z} \left( \frac{\partial P}{\partial t} \right) = -g \frac{\partial \rho}{\partial t}. \tag{2.11}
\]

Eliminating \( \partial P/\partial t \) in (2.11) using the chain rule given by

\[
\frac{\partial P}{\partial t} = \frac{\partial P}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial P}{\partial \sigma} \frac{\partial \sigma}{\partial t} \tag{2.12}
\]

and eliminating time derivatives with (2.3) and (2.4) gives

\[
\frac{\partial}{\partial z} \left( \rho c^2 \frac{\partial w}{\partial z} \right) = -\frac{\partial}{\partial z} (\rho c^2 \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla P) - g \nabla \cdot (\rho \mathbf{V}), \tag{2.13}
\]

\[
\mathcal{Q}(\mathcal{W}) + \frac{\partial}{\partial z} \left( \frac{\partial P}{\partial \rho} Q_\rho + \frac{\partial P}{\partial \sigma} Q_\sigma \right) + g Q_\sigma. \tag{2.13}
\]
where

$$\rho c^2 = \rho \frac{\partial P}{\partial \rho} + \sigma \frac{\partial P}{\partial \sigma}$$  \hspace{1cm} (2.14)$$

and \( c \) is the speed of sound. A convenient aspect of this system of equations is that the replacement of (2.2) with (2.13) is the only change necessary to convert the nonhydrostatic equations to their hydrostatic form.

3. Axisymmetric numerical model

To test the diagnostic vertical velocity equation (2.13), the system of equations described in the previous section is applied to an axisymmetric model. The hydrostatic version of the model can then be evaluated by comparison with the nonhydrostatic version.

The governing equations in cylindrical coordinates \((r, z)\) are solved numerically using a spectral method in the horizontal and finite differences in the vertical. Simple lateral boundary conditions given by

$$v, \frac{\partial v}{\partial r}, \frac{\partial w}{\partial r}, \frac{\partial \sigma}{\partial r} = 0 \quad \text{at} \quad r = 0, a \hspace{1cm} (3.1)$$

are applied so that the radial dependence of each variable can be expanded in Bessel series. In (3.1), \( u \) and \( v \) are the radial and tangential winds and \( a \) is the horizontal domain size. Each Bessel series includes 50 terms on a 1000-km radial domain, so that the horizontal resolution is about 20 km. The horizontal spectral method used here is similar to that described by Schubert and DeMaria (1985), but with a slightly less general boundary condition at \( r = a \).

To increase the accuracy of the vertical derivatives, all thermodynamic variables are divided into basic state (function of \( z \) only) and perturbation parts. The basic-state thermodynamic variables are defined by the U.S. standard atmosphere and their vertical derivatives are determined analytically. Vertical derivatives of all other variables are calculated using either second- or fourth-order centered finite differences on an unstaggered, evenly spaced vertical grid. At the top and bottom model levels, one-sided differences are applied. For cases with fourth-order differences, second-order differences are applied at the level just below (above) the model top (bottom). The model top \( z_t \) is at 12 km, and either 7, 13, 25, or 49 vertical levels are included (\( \Delta z = 2.0, 1.0, 0.5 \) or 0.25 km, respectively). At the lower and upper boundaries, \( w = 0 \).

In the hydrostatic case, the elliptic equation (2.13) is solved using a direct method. The term on the left side of (2.13) is expanded to give \( \rho \sigma^2 (\partial^2 w/\partial z^2) + \partial (\rho \sigma^2) / \partial z (\partial w/\partial z) \) and then written in terms of either second- or fourth-order vertical derivatives. This discretization results in a linear system of the form \( \mathbf{A} \mathbf{W} = \mathbf{F} \), where \( \mathbf{A} \) is an \( N - 2 \times N - 2 \) banded matrix (\( N \) is the number of vertical levels), \( \mathbf{W} \) is a vector of length \( N - 2 \) containing the vertical velocity \( w \) at each level except the model top and bottom, and \( \mathbf{F} \) is a forcing vector of length \( N - 2 \). The matrix \( \mathbf{A} \) is tridiagonal (pentadiagonal) for the case with second-order (fourth-order) vertical differences. Here \( \mathbf{W} \) is determined by performing an L-U decomposition (e.g., Atkinson 1978) of the matrix \( \mathbf{A} \) at each horizontal grid location.

A simulation that crudely represents a developing tropical cyclone will be used to test the axisymmetric model. The model is initialized with no radial wind and a tangential wind given by

$$v(r, z, 0) = v_m F(r) G(z),$$  \hspace{1cm} (3.2)

where

$$F(r) = \frac{r}{r_m} \exp \left\{ \frac{1}{b} \left[ 1 - \left( \frac{r}{r_m} \right)^b \right] \right\} \hspace{1cm} (3.3)$$

$$G(z) = \frac{z}{z_t}.$$  \hspace{1cm} (3.4)

The function \( F(r) \) is zero at \( r = 0 \), reaches a maximum value of 1 at \( r = r_m \) and then decays exponentially for larger \( r \). The parameter \( b \) controls the rate of decay at large radii. The function \( G(z) \) is a maximum of 1 at \( z = 0 \) and decreases linearly to zero at \( z = z_t \). The maximum tangential wind \( v_m \) in (3.2) is set to 10 m s\(^{-1}\), the radius of maximum tangential wind \( r_m \) is 100 km, and the parameter \( b \) is set to 0.6. The initial values of the prognostic thermodynamic variables \( \rho \) and \( \sigma \) are determined by assuming gradient and hydrostatic balance at \( t = 0 \).

The entropy source \( Q_s \), is specified as

$$Q_s(r, z, t) = BF(r) H(z) S(t), \hspace{1cm} (3.5)$$

where

$$H(z) = \sin \left( \frac{\pi z}{z_t} \right) \exp \left( \frac{-\gamma z}{z_t} \right) \left[ \sin \left( \frac{\pi z_m}{z_t} \right) \exp \left( \frac{-\gamma z_m}{z_t} \right) \right]^{-1} \hspace{1cm} (3.6)$$

$$S(t) = \alpha \left[ \exp(1 - \alpha t) \right] \hspace{1cm} (3.7)$$

and

$$\gamma = \frac{\pi}{\tan(\pi z_m/z_t)}. \hspace{1cm} (3.8)$$

If \( \gamma = 0 \) in (3.6) then the function \( H(z) \) is sinusoidal and the maximum entropy source is at \( z = z_t/2 \). If \( \gamma > 0 \) then the maximum entropy source occurs below \( z_t/2 \). When \( \gamma \) is determined from (3.8), the maximum entropy source occurs at \( z = z_m \). The function \( H(z) \) is normalized so that the maximum value is 1 at \( z = z_m \). In all of the model simulations presented later, \( z_m \) was set to 5 km which gives \( \gamma = 0.8418 \).

The horizontal structure of the entropy source \( F(r) \) in (3.5) has the same functional form as for the initial tangential wind defined by (3.3). The radius of maxi-
Fig. 1. Radial–height cross section of tangential wind (m s$^{-1}$) for the control simulation (N-25L-2nd) at t = 0 h (top), 36 h (middle), and 72 h (bottom).

The maximum entropy source $r_m$ is set to 60 km, and the parameter $b$ is set to 1.0. Thus, the maximum entropy source occurs inside the initial radius of maximum tangential wind and the entropy source decays more rapidly with radius than the initial tangential wind.

The temporal structure of the entropy source $S(t)$ is zero at $t = 0$, increases to a maximum value of 1 at $t = \alpha^{-1}$, and decreases exponentially after this time. The value of $\alpha$ is chosen so that the entropy source reaches its maximum at 36 h.
The constant $B$ in (3.5) is the maximum value of the entropy source. If the entropy density is increased by an amount $\Delta \sigma$ at constant pressure, then the corresponding temperature change $\Delta T$ can be determined from

$$\Delta \sigma = \left( \frac{\partial \sigma}{\partial T} \right)_p \Delta T, \tag{3.9}$$

where

$$\left( \frac{\partial \sigma}{\partial T} \right)_p = \frac{\rho}{T} \left[ C_p + R \ln \left( \frac{\rho}{\rho_0} \right) - C_v \ln \left( \frac{T}{T_0} \right) \right]. \tag{3.10}$$

Equation (3.10) was derived from (2.7) and the ideal gas law. The constant $B$ is chosen so that the equivalent heating rate at the level of the maximum entropy source is $10$ K day$^{-1}$, where (3.9) and (3.10) evaluated for the basic-state atmosphere were used to estimate the temperature change due to an entropy change. This maximum entropy source occurs at $r = 60$ km, $z = 5$ km, and $t = 36$ h.

The density source term $Q_\rho$ in (2.3) is set to zero. The friction terms in (2.1) and (2.2) are also set to zero, except for linear fourth-order horizontal diffusion terms that are included to prevent energy from accumulating in the highest wavenumbers. The fourth-order diffusion coefficient is set to $3.0 \times 10^{12}$ m$^4$ s$^{-1}$.

A leapfrog time-differencing scheme (with a forward scheme for the diffusion terms) is used with a time step of $10$ s for the hydrostatic model and $1$ s for the 25-level nonhydrostatic model. A time filter of the type described by Asselin (1972) with a filter parameter of 0.2 is also included to damp the computational mode of the leapfrog scheme. The time filter is required only for certain hydrostatic simulations as will be described later but is included in all of the simulations for consistency.
Fig. 3. The time evolution of the difference between the maximum vertical velocity (cm s$^{-1}$) in the control simulation and that in several hydrostatic simulations. A negative difference indicates that the maximum vertical velocity in the hydrostatic run is less than that in the control run. Differences with magnitudes that exceed 6 cm s$^{-1}$ are not plotted.

4. Numerical results

In this section, the hydrostatic version of the axisymmetric model is evaluated by comparison with the nonhydrostatic version. Several different vertical resolutions with either second- or fourth-order vertical differences will be used. For convenience, the model runs will be referred to by using N or H for nonhydrostatic or hydrostatic, nL where $n$ is the number of vertical levels, and second or fourth for second- or fourth-order differences. For example, H-25L-2nd refers to a hydrostatic run with 25 vertical levels and second-order vertical differences.

As a control simulation, the nonhydrostatic version of the axisymmetric model with 25 vertical levels and second-order vertical differences (N-25L-2nd) with the specified entropy source was integrated to 72 h with a 1-s time step. Figure 1 shows the time evolution of the tangential wind for the control run. The maximum tangential winds increase from 10 to approximately 27 m s$^{-1}$ and the radius of maximum tangential wind decreases from 100 to 75 km at 72 h. At upper levels the tangential wind becomes anticyclonic.

Figure 2 shows the time evolution of the vertical velocity for the control simulation. There is upward motion for radii less than 200 km and weak subsidence at larger radii. The maximum vertical velocity occurs at about 36 h, when the entropy source is a maximum.

Figures 1 and 2 show that the horizontal scale of the response is about 50 km and the vertical scale is 5 km. Thus, nonhydrostatic effects are probably not important and the hydrostatic model run should be similar to the control run. The control run was repeated, but with increased vertical resolution (49 levels), and the results were nearly identical to those in Figs. 1 and 2. This similarity indicates that 25 levels are sufficient to accurately represent the vertical structure of the solution in this idealized simulation.

To test the hydrostatic version of the model, the control run was repeated but with the prognostic vertical velocity equation (2.2) replaced by the diagnostic equation (2.13). Because the hydrostatic approximation filters the vertically propagating sound waves, the time step was increased to 10 s. The model solution in the hydrostatic run remained fairly close to that in the control run until about 4 h. After that time, the solution in the hydrostatic run diverged rapidly from that in the control run, and the model was on the verge of becoming unstable by 72 h. Examples of this behavior can be seen in Fig. 3 which shows the difference between the maximum vertical velocity in the control simulation and that in several hydrostatic simulations. Some of the vertical velocity differences for the latter parts of the simulations are not plotted because the vertical velocities in the hydrostatic runs became too large (up to 10 m s$^{-1}$ compared with a maximum of approximately 6 cm s$^{-1}$ in the control run).

The behavior of the hydrostatic model does not appear to be related to the time differencing. Reductions in the time step and variations in the Asselin filter parameter did not have much effect on the H-25L-2nd simulation. In all cases, the solution still diverged from the control run after 4 h.

The behavior of the hydrostatic model is sensitive to the accuracy of the numerical solution of the diagnostic vertical velocity equation. When the accuracy of the vertical differencing is increased by including fourth-order differences or by doubling the vertical resolution, the time at which the hydrostatic solutions diverge from the control occurs later in the simulation. Figure 3 shows that the H-25L-4th solution diverged after 22 h, the H-49L-2nd run diverged after 25 h, and the H-49L-4th run remained similar to the control simulation.

Some insight into the errors in the hydrostatic simulations can be obtained from the perturbation temperature fields as shown in Fig. 4 for the H-49L-2nd simulation at the time when the errors were increasing rapidly. At 36 h, the temperature has increased by approximately 3 K in the middle layers near the vortex center in the control simulation. As described in the previous section, the maximum entropy source corresponds to a maximum heating rate of 10 K per day at 36 h. The fact that the actual temperature changes are less than expected from the heating rate indicates that much of the heating is compensated by adiabatic cooling due to the upward vertical velocity. As can be seen
in Fig. 3, the vertical velocity tends to be underestimated in the early part of the H-49L-2nd simulation. Thus, the adiabatic cooling is underestimated in the H-49L-2nd run, which leads to an overestimate of the temperature increase, as can be seen in Fig. 4. Although the vertical velocity underestimates are small initially, eventually the accumulated temperature error becomes significant. These temperature errors are due to errors in both the entropy density and the dry air density, which appear in the coefficients of the diagnostic vertical velocity equation (2.13). When the errors in the thermodynamic variables become large enough, a feedback occurs where errors in the thermodynamic variables lead to errors in the vertical velocity, which lead to even larger errors in the thermodynamic variables.

The vertical velocity errors in Fig. 3 for the early part of the H-25L-4th simulation are less systematic than for the H-25L-2nd and H-49L-2nd runs. However, the rapid divergence of the vertical velocity for this case also begins when the perturbation temperatures deviate by about 1 K from those in the control run. Thus, the feedback between the thermodynamic fields and the diagnostic vertical velocity is also important for rapid error growth in the H-25L-4th run.

In the formulation of his system of equations, Ooyama (1989, personal communication) recognized that the vertical velocity errors due to discretization of the continuous equations might accumulate in the hydrostatic model, and recommended a computational remedy. To see this, let the deviation from hydrostatic balance $\epsilon$ be defined by

$$\epsilon = \frac{\partial P}{\partial z} + \rho g. \quad (4.1)$$

The diagnostic vertical velocity equation was derived from the following conditions:

$$\epsilon = 0 \quad \text{for} \quad t = 0, \quad (4.2a)$$

$$\frac{\partial \epsilon}{\partial t} = 0 \quad \text{for} \quad t \geq 0. \quad (4.2b)$$
Equations (4.2) indicate that for the continuous case, \( \epsilon \) will remain zero during the integration so that hydrostatic balance will be maintained. However, when the equations are discretized, small numerical errors might be introduced, so that \( \epsilon \) may gradually drift from zero and hydrostatic balance will not be maintained. To test this hypothesis, \( \epsilon \) was calculated for the control and the hydrostatic simulations and the maximum value at any point in the domain was determined. Figure 5 shows that for the control run, the deviations from hydrostatic balance remain fairly small. For this case, the maximum value of \( \epsilon \) is 0.026 kg m\(^{-2}\) s\(^{-2}\) at \( t = 72 \) h. For comparison, the magnitude of each individual term in the hydrostatic equation is about 10 kg m\(^{-2}\) s\(^{-2}\). Figure 5 also shows that the nonhydrostatic control run remains closer to hydrostatic balance than any of the hydrostatic runs. Thus, the solution of the diagnostic vertical velocity equation does not appear to be accurate enough to prevent the gradual drift away from hydrostatic balance, even with fourth-order differences and 49 vertical levels.

Ooyama's method for reducing the drift from hydrostatic balance is to replace the condition (4.2) with

\[
\frac{\partial \epsilon}{\partial t} = -\kappa \epsilon, \tag{4.3}
\]

where \( \kappa \) is a nudging coefficient. The diagnostic vertical velocity equation then becomes

\[
\frac{\partial}{\partial z} \left( \rho c^2 \frac{\partial w}{\partial z} \right) + \frac{\partial}{\partial z} (\epsilon w) = -\frac{\partial}{\partial z} \left( \rho c^2 \nabla \cdot \mathbf{V} + \mathbf{V} \cdot \nabla P \right) - g \nabla \cdot (\rho \mathbf{V}) + \frac{\partial}{\partial z} \left( \frac{\partial P}{\partial \rho} Q_r + \frac{\partial P}{\partial \sigma} Q_s \right) + gQ_v + \kappa \epsilon. \tag{4.4}
\]

Comparing (4.4) with (2.13) it can be seen that there are two extra terms in (4.4). The extra term on the right side is the nudging term in (4.3). The extra term on the left side appears because the condition \( \epsilon = 0 \) was assumed in the derivation of (2.13). The condition (4.3) does not require that \( \epsilon \) be identically equal to zero. These additional terms do not add any numerical difficulties to the solution of (4.4). The linear system has the same banded structure as for (2.13), and the direct solution using the L-U decomposition is applied as before. The additional terms do, however, change the properties of the time integration. When the nudging term is added, the leapfrog time differencing scheme without the Asselin filter becomes unstable. This might be expected since (4.3) has the form of a linear decay equation, although the leapfrog scheme is not applied directly to (4.3). Fortunately, the inclusion

![Figure 5](image-url)

**Figure 5.** The time evolution of the maximum deviation from hydrostatic balance \((10^{-2} \times \text{kg m}^{-2} \text{s}^{-2})\) for the control simulation and for several hydrostatic simulations. Deviations that exceed \(20 \times 10^{-2}\) kg m\(^{-2}\) s\(^{-2}\) are not plotted.

![Figure 6](image-url)

**Figure 6.** The time evolution of the maximum deviation from hydrostatic balance \((10^{-2} \times \text{kg m}^{-2} \text{s}^{-2})\) for the control simulation and for the hydrostatic simulation H-25L-2nd with several values of the e-folding time.
of the Asselin time filter removes the instability. The instability can also be removed by using an alternate time-differencing scheme. Experiments showed that the hydrostatic equations with the nudging term can be integrated with the second-order Adams–Bashforth scheme. However, for consistency, the leapfrog scheme with the Asselin filter was used in all versions of the model, as described previously.

When (4.4) is used to diagnose the vertical velocity, the nudging coefficient $\kappa$ must be specified. The inverse of this coefficient represents the $e$-folding time of the hydrostatic imbalance $\epsilon$. The $e$-folding time should be greater than the model time step, but shorter than the time required for the hydrostatic model solutions to diverge from the nonhydrostatic run. Figure 6 shows the time evolution of the maximum value of the hydrostatic imbalance $\epsilon$ for the H-25L-2nd run with various values of the $e$-folding time. This figure shows that as the $e$-folding time is reduced, the solution remains closer to hydrostatic balance. However, even the case with the largest $e$-folding time (400 s) remains much closer to hydrostatic balance than the most accurate hydrostatic solution in Fig. 5 (H-49L-4th).

The simulation H-25L-2nd was also run with an $e$-folding time of 800 s, but the solution rapidly diverged from the control run after approximately 24 h, similar to the case without the nudging term. Thus, it appears that the nudging coefficient $\kappa$ must be sufficiently large to maintain hydrostatic balance. When the $e$-folding time was reduced to 50 s, it was necessary to increase the Asselin filter parameter to 0.3 to maintain stability. This result suggests that the nudging coefficient cannot be made arbitrarily large. As will be discussed later in this section, the maximum value of $\kappa$ is limited by the size of the time step. In the test problem presented here, hydrostatic balance was adequately maintained for $\kappa^{-1}$ ranging from 50 to 400 s, which corresponds to 5 to 40 time steps.

Figure 7 shows the difference between the maximum vertical velocity in the control simulation and that in the H-25L-2nd simulation with several $e$-folding times.

Figure 6 showed that the nudging term in the hydrostatic equation does not totally eliminate the hydrostatic imbalance, and the imbalance is larger than that in the control run. To gain further insight into the nature of the imbalance, Fig. 8 shows a cross section of $\epsilon$ at 36 h for the control simulation and for the H-25L-2nd simulation with the nudging term. In both cases in Fig. 8, the largest value of $\epsilon$ occurs near the bottom boundary at small radii. This result indicates that the imbalance might be due to a slight inconsistency between the model solution and the imposed boundary conditions. For the hydrostatic case, this region of imbalance extends a little further into the model domain, perhaps due to the fact that $w$ is evaluated using an elliptic equation. However, over most of the model domain in Fig. 8, $\epsilon$ is considerably smaller than the maximum values shown in Fig. 6.

The primary benefit of the hydrostatic approximation is the computational efficiency that is gained by filtering the vertically propagating sound waves. Because the vertical velocity (and the prognostic variables which were not shown) are nearly identical in the nonhydrostatic and hydrostatic simulations, the slight increase in $\epsilon$ that results from the hydrostatic approximation appears to be a small price to pay for the increased efficiency.

As a further test of the hydrostatic model with the nudging term, two additional simulations with low vertical resolution were performed (H-7L-2nd with $\kappa^{-1} = 100$ s and N-7L-2nd). In each of these simulations, the amplitude of the entropy source was doubled, so that the maximum tangential wind increased to approximately 47 m s$^{-1}$ at 72 h (compared with 27 m s$^{-1}$ in the control run). Figure 9 shows that even with low vertical resolution and increased forcing, the vertical
velocity in the hydrostatic run is close to that in the nonhydrostatic run. Thus, the addition of the nudging term removes the extreme sensitivity to the accuracy of the solution of the diagnostic equation for vertical velocity in the hydrostatic case.

The vertical velocity in the control simulation is the quasi-balanced response to the slowly varying entropy source. However, some small amplitude gravity wave activity is also generated, which is responsible for the vertical velocity differences shown in Fig. 7. To test the hydrostatic model in a simulation where the vertical velocity is primarily due to transient gravity wave activity, the entropy source was set to zero and the H-25L-2nd version of the model with an e-folding time of 100 s was initialized with the tangential wind given by (3.2), but with the perturbation entropy and density set to zero. In this case, gravity wave activity will be generated as the model adjusts toward gradient balance. The initial maximum tangential wind $v_m$ was increased from 10 to 20 m s$^{-1}$ to increase the amplitude of the gravity wave response. Figure 10 shows the vertical velocity field near the beginning of this 72-h simulation. The vertical velocity is associated with a gravity wave disturbance that propagates outward at 50 m s$^{-1}$ and reaches the outer boundary at 6 h. In the rest of the simulation, the gravity wave disturbance is reflected back and forth by the inner and outer domain boundaries, and is slowly reduced in amplitude by the fourth-order horizontal diffusion and the Asselin time filter. This simulation shows that the nudging term with an e-folding time of 100 s is sufficient to prevent the rapid error growth due to the feedback with the thermodynamic fields in a simulation where the vertical velocity is primarily due to transient gravity wave activity.

This simulation was repeated with the nonhydrostatic version of the model, and the results were qualitatively similar to the hydrostatic case. However, there were some differences in the small-scale structure of
the vertical velocity field (as shown in Fig. 11). The vertical velocity field at 4 h for the hydrostatic simulation with an $e$-folding time of 50 s is also shown in Fig. 11 and is similar to that with the 100 s $e$-folding time in Fig. 10. This similarity suggests that the $e$-folding time is small enough so that the hydrostatic solution is accurate. The differences between the nonhydrostatic and hydrostatic solutions in Fig. 11 are probably due to nonhydrostatic effects on the propagating gravity waves.

The reduction of the $e$-folding time from 100 to 50 s did have a small effect on the amplitude of the gravity wave response. For the response to the slowly varying entropy source described previously, there was almost no difference between the hydrostatic simulations with $e$-folding times of 100 and 50 s. Thus, the transient gravity wave part of the solution requires a smaller $e$-folding time than the slowly varying part. This result suggests that the $e$-folding time should be as small as possible ($\kappa$ should be as large as possible).

Further experiments were performed, which showed that the hydrostatic model could be integrated with an $e$-folding time of 25 s, although it was necessary to increase the Asselin filter parameter to 0.5. Attempts to integrate the model with an $e$-folding time of 12.5 s were unsuccessful, perhaps because this time is shorter than the value of $2\Delta t$ (20 s) used in the individual leapfrog time steps. Thus, the largest value of $\kappa$ is limited to an $e$-folding time that corresponds to 2–3 time steps.

The above results show that the hydrostatic model gives reasonable results provided that the additional term is included in the diagnostic vertical velocity equation. One of the reviewers (Dr. Rene Laprise) suggested an alternate method for reducing the drift from hydrostatic balance. Equation (2.13) was derived from the continuous forms of (2.3), (2.4), and (2.11) and then written in finite-difference form. Thus, the discrete form of (2.13) is not necessarily consistent with the discrete forms of (2.3) and (2.4). To determine if this
inconsistency contributes to the drift from hydrostatic balance, the diagnostic vertical velocity equation was rederived from the discrete prognostic and hydrostatic equations, as described in the appendix. The H-25L-2nd simulation (analogous to the control run shown in Figs. 1 and 2) was repeated, but with (2.13) replaced by (A.6) for the case with $\kappa = 0$. The vertical velocity in this simulation diverged rapidly from that in the control simulation after about 3 h, similar to the results for the H-25L-2nd simulation shown in Fig. 3. The H-25L-2nd simulation using (A.6) with $\kappa^{-1} = 100$ s was also run, and the results were very similar to those from the H-25L-2nd simulation using (4.4) with $\kappa^{-1} = 100$ s. However, the hydrostatic imbalance shown in Fig. 6 was approximately 50% less when (A.6) was used in place of (4.4). These results indicate that the drift from hydrostatic balance is not directly related to the discretization of the vertical velocity equation, and that the nudging term is still required in (4.4) or (A.6). The advantage of (A.6) relative to (4.4) is that the hydrostatic imbalance is somewhat reduced. However, the disadvantage is that the linear system is somewhat more difficult to solve (the tridiagonal system becomes a pentadiagonal system in the second-order case), and is perhaps less accurate because the second derivatives are evaluated over a grid interval of $2\Delta z$ rather than $\Delta z$.

5. Concluding remarks

The hydrostatic form of the primitive equations described by Ooyama (1990) was evaluated by comparing nonhydrostatic and hydrostatic integrations of a dry axisymmetric model with a specified entropy (heat) source. This model test problem was designed to crudely represent a developing tropical cyclone. In this formulation of the primitive equations, pressure is a diagnostic variable, so that the hydrostatic approximation can be included simply by replacing the vertical momentum equation with a diagnostic vertical velocity.
equation. This diagnostic equation is a one-dimensional (height) second-order elliptic equation that can be solved using a direct method.

Results showed that the hydrostatic solutions are very sensitive to the accuracy of the method used to solve the diagnostic vertical velocity equation. Small errors in the vertical velocity accumulate with time, and eventually a feedback with the thermodynamic variables leads to large vertical velocity errors. The source of this difficulty is related to the condition used to derive the diagnostic vertical velocity equation. When the equations are in continuous form, this condition insures that hydrostatic balance is maintained. However, when the equations are discretized, the system can drift away from hydrostatic balance.

The problem with the hydrostatic solutions can be eliminated by adding an extra term to the diagnostic equation for vertical velocity that insures that the solution does not drift away from hydrostatic balance due to numerical approximation. When this extra term is added, the hydrostatic solutions become similar to the nonhydrostatic runs, and the sensitivity to the accuracy of the numerical solution of the vertical velocity equation is removed. These results demonstrate that Ooyama's formulation of the primitive equations allows for the design of a numerical model in height coordinates that can be used in hydrostatic and nonhydrostatic regimes, provided that the modified vertical velocity equation is used in the hydrostatic case.

The goal of this study was to evaluate the diagnostic vertical velocity equation in the hydrostatic version of Ooyama's formulation of the primitive equations, in the context of a simplified numerical model. Further study is necessary to determine the effects of moisture and an irregular lower boundary (terrain) on this system of equations.

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APPENDIX

Derivation of the Diagnostic Vertical Velocity Equation from the Finite-Difference Form of the Prognostic Equations

Because the horizontal derivatives are evaluated from the spectral amplitudes in both the diagnostic vertical velocity equation and the prognostic equations, the horizontal derivatives are presented in continuous form. For simplicity, only the case with second-order vertical finite differences is discussed. As described in section 3, all of the variables are defined on an evenly spaced unstaggered vertical grid. The vertical derivative of a function $f(z)$ at level $z_i$ can be written as

$$\left(\delta_z f\right)_i = \frac{f_{i+1} - f_{i-1}}{2\Delta z},$$  \hspace{1cm} (A.1)

where $f_i = f(z_i)$. Appropriate one-sided differences are used at the top and bottom model levels. The vertically discretized versions of (2.3), (2.4), and (4.3) can be written as

$$\left(\frac{\partial \rho}{\partial t}\right)_i + \nabla \cdot (\mathbf{V}_i \rho_i) + w_i (\delta_z \rho)_i + \rho_i (\delta_z w)_i = \left(Q_{\rho}\right)_i$$ \hspace{1cm} (A.2)

$$\left(\frac{\partial \sigma}{\partial t}\right)_i + \nabla \cdot (\mathbf{V}_i \sigma_i) + w_i (\delta_z \sigma)_i + \sigma_i (\delta_z w)_i = \left(Q_{\sigma}\right)_i,$$ \hspace{1cm} (A.3)

$$\delta_z \left(\frac{\partial P}{\partial t}\right)_i + g \frac{\partial \rho}{\partial t}_i = -\kappa \epsilon_i,$$ \hspace{1cm} (A.4)

where

$$\epsilon_i = (\delta_z P)_i + \rho_i g.$$ \hspace{1cm} (A.5)

Now, using the chain rule (2.12) to eliminate $\partial P/\partial t$ in (A.4) and then eliminating time derivatives using (A.2) and (A.3) gives the desired form of the vertical velocity equation that can be written as

$$\delta_z [\rho_i c_i ^2 (\delta_z w)_i] + \delta_z [(\delta_z P)_i w_i] + g \rho_i (\delta_z w)_i + gw_i (\delta_z \rho)_i = F_i,$$  \hspace{1cm} (A.6)

where

$$F_i = -\delta_z [\rho_i c_i ^2 \nabla \cdot \mathbf{V}_i + \mathbf{V}_i \cdot \nabla P_i]$$

$$- g \nabla \cdot (\rho_i \mathbf{V}_i) + \delta_z \left[ \left(\frac{\partial P}{\partial \rho}\right)_i (Q_{\rho})_i + \left(\frac{\partial P}{\partial \sigma}\right)_i (Q_{\sigma})_i \right] + g (Q_{\rho})_i + \kappa \epsilon_i.$$ \hspace{1cm} (A.7)

For the case with $\kappa = 0$ it can be seen that the term $F_i$ in (A.6) is the discrete form of the forcing on the right side of (2.13) and the first term on the left side of (A.6) is the discrete form of the left side of (2.13). However, the next three terms on the left side of (A.6) do not appear in (2.13). These terms cancel in the continuous case when $\epsilon = 0$ but do not cancel in the discretized case. Also note that when the vertical velocity equation was derived from the continuous form of the prognostic equations, the first term on the left side of (2.13) was expanded into two terms before it was discretized, as was described in section 3. This expansion was performed so that the second derivative of $w$ could be represented by a finite difference of the form $(w_{i+1} + w_{i-1} - 2w_i)/\Delta z^2$ for the case with second-order accuracy. When this form is used, the resulting linear system is tridiagonal. However, the linear system that results from (A.6) is pentadiagonal. If fourth-order vertical differences were used, the analog of (A.6) would be heptadiagonal.

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