An Inverse Balance Equation in Sigma Coordinates for Model Initialization

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ABSTRACT

In many numerical simulation studies there is a need to obtain the balanced initial fields in numerical models with $\sigma$ as the vertical coordinate. By employing an equation that establishes the balance between the mass and wind fields directly on $\sigma$ surfaces, we can obtain mass fields from wind fields, or vice versa. Since the balance equation is generally nonlinear, numerical methods must be used to obtain approximate solutions. However, the term corresponding to the divergence of the pressure gradient in the $\sigma$ system is more complicated compared to that in pressure or height coordinates. As a result, previous studies have made various approximations for this term. In this paper, a relatively accurate and consistent numerical scheme is proposed to solve the inverse balance equation in $\sigma$ coordinates. Several numerical calculations show that the proposed scheme is more accurate and more consistent than that used by Kurihara and Bender and Kurihara et al.

1. Introduction

The balance equation has been widely used to determine the initial fields of wind and mass in primitive equation models (Sundqvist 1975). Since many numerical models currently in use have $\sigma$ as the vertical coordinate, the best choice to establish a properly balanced initial state is to perform the initialization procedure on the $\sigma$ surfaces, as indicated by Sundqvist (1975). However, when one uses the inverse balance equation to obtain mass fields from wind fields, there is a coexistence of geopotential and temperature in the inverse equation, and some mathematical approximations are necessary to find the numerical solution.

In an early paper, Kurihara and Bender (1980, hereafter referred to as KB) proposed a numerical scheme to solve the inverse balance equation in the $\sigma$ coordinate derived by Sundqvist (1975) to derive the mass fields from a specified wind field of a symmetric vortex. In a recent paper (Kurihara et al. 1993, hereafter referred to as KBR), the simplified divergence equation was used in which the local divergence tendency and the frictional force were both included. However, the numerical schemes to solve the inverse balance problem are similar. Some simplifications are employed in determining the temperature fields. Although the approximations are acceptable, there are some problems when the wind is very strong.

In this paper, the hydrostatic relation is incorporated into the inverse divergence equation so that a more accurate and more consistent numerical scheme can be used to solve the inverse problem in a multilevel primitive equation model using $\sigma$ as the vertical coordinate. In the next section, the numerical scheme used by KB and KBR is briefly reviewed. A proposed new scheme is described in section 3. Some numerical results from both schemes are compared in section 4. Finally, a summary and concluding remarks are presented in section 5.

2. A review

The divergence equation in the $\sigma$-coordinate system takes the following form (KB and KBR):

$$ G = \nabla^2 \Phi + \nabla \cdot (RT \nabla \ln p_\sigma), \quad (1) $$

where $R$ is the gas constant, $\Phi$ is the geopotential of a constant $\sigma$ surface, $T$ is temperature, $p_\sigma$ is surface pressure, and the quantity $G$ is obtained from the momentum distribution. The right-hand side is derived from the momentum equation in which the pressure gradient force is split into two terms to facilitate the diagnosis of both $p_\sigma$ and $T$. The mass dependent right-hand side of (1) balances $G$, which takes the form

$$ G = -\frac{\partial D}{\partial t} + \nabla \cdot \left[ - (\nabla \cdot \nabla) \nabla \Phi - \sigma \frac{\partial \nabla}{\partial \sigma} - \left( f + \frac{\tan \phi}{a} \right) (k \times \mathbf{V}) + F \right], \quad (2) $$

where $F$ represents the divergence of frictional force and $\sigma$ is defined as $p/p_\sigma$. The other symbols have the same meanings as those used in KBR.

Equation (1) can be referred to as the nonlinear balance equation if $G$ takes the following form (Sundqvist 1975; KB)

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\[ G(u, v) = 2J_{x, y}(u, v) + f\zeta - u\beta \]
\[ -\frac{\partial}{\partial\phi} [(u^2 + v^2) \sin\phi], \quad (3) \]
where \( J(a, b) \) is the Jacobian operator, \( \zeta \) is the relative vorticity in the vertical, \( \beta \) is the meridional gradient of Coriolis parameter, \( \alpha \) is equal to \( a \cos\phi \), and \( v \) is the meridional component of the wind.

From (1) we can see that regardless of whether \( G \) takes the form (2) or (3), (1) does not change and therefore \( G \) can be calculated from any given wind fields. Therefore, in the present study, we will only be concerned with the numerical solution of (1). Diagnostic formulas for the surface pressure and geopotential or equivalently the temperature, are derivable from (1). KB replaced \( T \) in (1) by \( T_0(\sigma) \), and then derived \( p_\ast \) and \( \Phi \) by the simplified Poisson equations. Namely, the equations for \( \sigma = 1 \) and \( \sigma < 1 \) were written:
\[ \nabla^2 p_\ast = \left( \frac{p_\ast}{RT_{0\ast}} \right) (G_\ast - g\nabla^2 z_\ast), \quad \sigma = 1 \quad (4) \]
\[ \nabla^2 \Phi = G - \left( \frac{T_0}{T_{0\ast}} \right) (G_\ast - g\nabla^2 z_\ast), \quad \sigma < 1, \quad (5) \]
where the asterisk denotes the surface value, \( g \) is the acceleration of gravity, and \( p_\ast \) is the mean surface pressure that is taken to be constant. The right-hand side of the above equations can be calculated from a given wind field, and then \( p_\ast \) and \( \Phi \) are obtained by a relaxation method using appropriate boundary conditions. Once the initial field of geopotentials on constant \( \sigma \) surfaces are obtained, the initial temperature field can be derived easily through the hydrostatic relation
\[ \frac{\partial \Phi}{\partial \ln \sigma} = -RT. \quad (6) \]

Then, vertical interpolation is used to obtain the temperature fields on each \( \sigma \) surface.

It is clear that the temperature in the second term of the right-hand side of (1) is considered to be approximately constant on a constant \( \sigma \) surface. This approximation is acceptable if the anomaly of temperature on the \( \sigma \) surface is very small. However, when the temperature anomaly is very large, such as in tropical cyclones, the error resulting from this approximation may become significantly large. Although several other improvements have been conducted in recent work by KBR, the approximation used to solve (1) is still not fully consistent. The equations corresponding to (4) and (5) were written as
\[ \nabla \cdot (RT_{\sigma}^0 \nabla \ln p_\ast) = G_{\sigma} - \nabla^2 \Phi_{\sigma}^0 \quad (7) \]
\[ \nabla^2 \Phi = G - \nabla \cdot (RT_0^0 \nabla \ln p_\ast), \quad (8) \]
where the superscript 0 denotes the values obtained from a combination of global model assimilation and the hurricane model integration, and the subscript \( d \) denotes the values at the \( \sigma_d \) level. It can be seen from (7) and (8) that the temperature on the right-hand side of (1) is no longer constant on a \( \sigma \) surface, but it is not the same as that obtained from the hydrostatic relation (6). We suggest that improvements in the evaluation of temperature in (1) may be obtained by retaining temperature anomalies as the unknown in the second term of the right-hand side of (1).

3. Proposed new scheme

To include the temperature anomaly as the unknown in the second term of the right-hand side in (1), we differentiate (1) with respect to \( \sigma \) and use the hydrostatic relation (6), to obtain
\[ -\nabla^2 T + \nabla \cdot \left( \frac{\partial T}{\partial \ln \sigma} \nabla \ln p_\ast \right) = \frac{\partial G}{R \partial \ln \sigma}. \quad (9) \]

Further, assuming
\[ T = T_0(\sigma) + T' \quad (10) \]
and substituting (10) into (9), we can obtain
\[ \nabla^2 T' - \nabla \cdot \left( \frac{\partial T'}{\partial \ln \sigma} \nabla \ln p_\ast \right) = \frac{\partial T_0}{\partial \ln \sigma} \nabla^2 \ln p_\ast - \frac{\partial G}{R \partial \ln \sigma}. \quad (11) \]

We divide the model atmosphere into \( M \) layers and carry \( u, v, T' \) in the middle of each layer (see Fig. 1), then (11) on the \( k \)th level is given by

\[ \sigma = 0.0 \]
\[ \sigma = 1.0 \]
\[ \sigma = 0.1 \]
\[ \sigma = 0.2 \]
\[ \sigma = 0.3 \]
\[ \sigma = 0.4 \]
\[ \sigma = 0.5 \]
\[ \sigma = 0.6 \]
\[ \sigma = 0.7 \]
\[ \sigma = 0.8 \]
\[ \sigma = 0.9 \]
\[ \sigma = 1.0 \]

**Fig. 1.** The vertical structure of the model atmosphere.
\[
\n\nabla^2 T'_k - \nabla \cdot \left( \frac{\partial T'}{\partial \ln \sigma} \right)_k \nabla \ln p_\ast \\
= \left( \frac{\partial T_0}{\partial \ln \sigma} \right)_k \nabla^2 \ln p_\ast - \left( \frac{\partial G}{R \partial \ln \sigma} \right)_k.
\]

(12)

Using centered differencing for the right-hand side and one-sided differencing for the left-hand side of (12) for derivatives in the vertical, for \(k = M\), we have

\[
\nabla^2 T'_k + \lambda a_1 \nabla \cdot (T'_k \nabla \ln p_\ast) = A_k,
\]

(13)

where

\[
A_k = a_2 \left( T_{0k+1/2} - T_{0k-1/2} \right) \nabla^2 \ln p_\ast \\
+ \lambda a_1 \nabla \cdot (T'_{k+1/2} \nabla \ln p_\ast) \\
- \frac{G_{k+1/2} - G_{k-1/2}}{R},
\]

(14)

\[
a_1 = \frac{1}{\ln(\sigma_{k+1/2}/\sigma_k)}, \quad a_2 = \frac{1}{\ln(\sigma_{k+1/2}/\sigma_{k-1/2})},
\]

(15)

and \(\lambda = 0\) or \(1\). If \(\lambda = 0\), (13) is equivalent to (5), while \(\lambda = 1\) in our new scheme.

For \(k = M\), assuming \(T'_0 = 0\) yields

\[
\nabla^2 T'_M + \lambda b_1 \nabla \cdot (T'_M \nabla \ln p_\ast) \\
= b_2 \left( T_{0M} - T_{0M-1} \right) \nabla^2 \ln p_\ast \\
- \frac{G_M - G_{M-1/2}}{R},
\]

(16)

where

\[
b_1 = -\frac{1}{\ln \sigma_M}, \quad b_2 = -\frac{1}{\ln \sigma_{M-1/2}}.
\]

(17)

It is obvious that both (13) and (16) are elliptical equations so that numerical solutions can be obtained by using a relaxation method. First, we obtain the surface pressure fields \(p_\ast\) by solving a Poisson equation similar to (4) and (7),

\[
\nabla^2 \ln p_\ast = \frac{1}{R T_{0M}} (G_\ast - g \nabla^2 z_\ast),
\]

(18)

then the temperature anomaly \(T'_M\) is obtained. Finally, we obtain the temperature anomalies at all other levels from level \(M - 1\) to level \(1\) in Fig. 1, in succession.

It can be seen from (13) and (16) that the temperature anomaly in the second term of the right-hand side in (1) is an unknown in our proposed scheme, while it is either omitted or considered as a known field in the scheme used by KB or KBR. This improvement is achieved by incorporating the hydrostatic relation into the inverse balance equation in our scheme. It is expected that the proposed scheme would result in more accurate and more consistent solutions for the inverse balance equation in \(\sigma\) coordinate.

4. Some numerical results

To evaluate the accuracy of the new scheme proposed in the previous section, we introduce idealized surface pressure fields as a circular low or equivalently those corresponding to a circular mountain terrain, which takes the form

\[
p_\ast(r) = p_{\ast 0} - \Delta p_\ast \frac{\exp\left[ -(r/R_0)^2 \right]}{[1 + 1.5(r/r_m)^2]^{1/2}},
\]

(19)

where \(\Delta p_\ast\) is the minimum surface pressure at the low center or the surface pressure at the top of the terrain, \(r_m\) is the radius of the maximum tangential wind of the low or the radius at which the sharpest slope occurs on the mountain range, \(R_0\) is the radius of the low or the mountain terrain, and \(r\) is the radius from the low center or from the top of the mountain. The analytical temperature anomaly is given as

\[
T'(r, \sigma) = \Delta T' \sin(\pi \sigma) \exp\left[ -\left( \frac{r}{r_0} \right)^2 \right].
\]

(20)

To evaluate the accuracy of the new scheme, we substitute (20) into the left-hand side of (12), which determines exactly the value of the forcing term on the right-hand side. Then the force term is used to calculate the numerical solution of (12) by (13) and (16). As an example, we take \(p_{\ast 0} = 1008.7\text{ hPa}, R_0 = 600\text{ km}, r_m = 100\text{ km}, r_0 = 200\text{ km},\) and \(\Delta T' = 20^\circ\text{C}.\) The calculation domain is 3000 km \(\times\) 3000 km with grid spacing of 20 km. The model atmosphere is equally divided into 10 layers, that is, \(M = 10.\) Figures 2 and 3 show the difference fields of temperature anomaly between that obtained by numerical methods and that given by (20) in \(\Delta p_\ast = 30\) and 60 hPa in (19), respectively. It is obvious that the numerical solution calculated by the new scheme \((\lambda = 1,\) Figs. 2b and 3b) is much more accurate than that calculated by the scheme used by KB \((\lambda = 0,\) Figs. 2a and 3a). Interestingly, we can see from Figs. 2 and 3 that the error from the scheme used by KB is proportional to \(\Delta p_\ast.\) This demonstrates that in cases of strong weather systems or very high terrain, the scheme used by KB is not satisfactory, while the new scheme can still give accurate solutions to (1).

Considering the new scheme is initially designed for baroclinic vortex initialization, axisymmetric vortices are used in the next calculations. We take the tangential wind fields of a baroclinic vortex as

\[
V(r, \sigma) = V_m r \frac{r}{r_m} \sin\left( \frac{\pi \sigma}{2} \right) \exp\left[ \frac{1}{b} \left( 1 - \left( \frac{r}{r_m} \right)^b \right) \right],
\]

(21)
where $V_m$ is the maximum wind at the surface, $r_m$ is the radius of the maximum wind, $r$ is the radius from the vortex center, and $b$ is a parameter determining the horizontal shape of the wind profile. The vertical profile of the tangential wind is given as a sine function of $\sigma$. Figures 4a and 4b show two examples of the tangential wind profiles for cyclones in the $r-\sigma$ plane described by (21). In all the following calculations, we take the Coriolis parameter $f$ as a constant at 20°N. The undisturbed temperature profile is obtained from the clear-air western North Pacific sounding of Gray (1975).

Next we compare the thermal structure calculated by (13) and (16) for $\lambda = 0$ and $\lambda = 1$. Since the cyclonic circulation decreases with height, it is expected that the vortex described by (21) has a warm core structure. Figure 5 shows the temperature anomaly in the $r-\sigma$ plane for $V_m = 40$ m s$^{-1}$ and $b = 0.7$ vortex. It is observed that for the weak vortex, the difference between $\lambda = 0$ and $\lambda = 1$ solutions is very small. A larger difference occurs in the lower levels for the strong and large vortices (Figs. 6 and 7). The temperature anomaly calculated by the new scheme ($\lambda = 1$) has a stronger warm core at the lower levels than that calculated by KB ($\lambda = 0$). For example, for $V_m = 60$ m s$^{-1}$ and $b = 0.7$ vortex, the largest difference reaches about 4°C at the lowest level (Fig. 6c). At the upper levels, the warm core calculated by the new scheme is somewhat weaker, but the value (1°C at $\sigma = 0.35$) is very small compared to the lower levels. It would be
Fig. 4. Azimuthal wind profiles for cyclones described by Eq. (21). \( V_m = 40 \text{ m s}^{-1} \) and \( r_m = 100 \text{ km} \). (a) for \( b = 0.7 \); (b) for \( b = 1.0 \).

Fig. 5. The temperature anomaly in the \( r-\sigma \) plane for \( b = 0.7 \) and \( V_m = 40 \text{ m s}^{-1} \), calculated by (a) the new proposed scheme and (b) the scheme used by KB, respectively. (c) Difference fields of (b) – (a).
expected that the new scheme can give an accurate balance between the wind fields and the mass and thermal fields. This has been confirmed by some recent applications of the scheme to baroclinic vortex initialization (E. A. Ritchie, personal communication).

5. Summary and concluding remarks

A relatively accurate and consistent numerical scheme has been proposed in the present paper to solve the inverse balance or divergence equation in multi-level primitive equation models using $\sigma$ as the vertical coordinate. The new scheme incorporates the hydrostatic relation with the inverse balance equation so that some approximations made in previous studies are eliminated. Several numerical calculations have shown that the new scheme is more accurate and more consistent than that used by Kurihara and Bender (1980) and Kurihara et al. (1993).

It should be pointed out that the model-consistent initial conditions are usually model-dependent. The numerical results given in the last section are quite simplistic. If the model includes the planetary boundary layer and other physical processes, the initialization of the numerical model will be much more complicated. But the technique described in the present paper can be extended to include some of the physical processes. For example, the irrotational wind component and the surface friction or diffusion of the momentum can be incorporated into the force term $G$ in (1), as described by Kurihara et al. (1993).

Another important aspect is that the proposed scheme is based on a constant surface temperature. This may be not true for real situations. However, if we have the observed surface temperature fields, we can easily obtain $T_s$ and incorporate it into (16) and (18). Therefore, nonuniform conditions, such as a variable sea surface temperature, can be easily treated by the new scheme. In addition, although our numerical calculations given in section 4 only include a circular vortex, the variable large-scale environmental flow can also be obtained in our scheme if reasonable lateral boundary conditions are specified. Further studies of the bogussing of tropical cyclones in numerical models and the
dynamics of baroclinic vortex motion in variable environments by this initialization scheme will be carried out in the near future.

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Fig. 7. As in Fig. 6 but for $b = 1.0$ vortex.

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