1. Introduction

Synoptic and climatological analyses are very complicated in many high mountain regions, such as the Tibetan Plateau, the Rocky Mountains, and western United States (Sangster 1960, 1987; Harrison 1970; Hill 1993). These analyses are also very difficult over Greenland and Antarctica due to the topography of the ice sheets. The difficulty is caused by the fact that no single elevation or pressure surface can be used to track systems across the mountain region. Methods used to reduce station pressure to sea level pressure (SLP) may not be successful at capturing the isobaric pattern that would exist if the earth’s surface of the mountain region were at sea level. Thus, such an SLP pattern is often not representative of the true circulation that exists at ground level across the elevated terrain.

The situation is perhaps worst over the Tibetan Plateau. The topography of east Asia is shown by Fig. 1a and it is derived from the U.S. Navy high-resolution 10’ × 10’ dataset provided by the National Center for Atmospheric Research (NCAR). Figure 1a shows that the average elevation of the Tibetan Plateau is more than 4000 m. Routine synoptic analyses at sea level and at the 850-and 700-hPa levels are all extrapolated from the upper troposphere and cannot track weather systems over or moving across the Tibetan Plateau. However, such systems do exist and they often produce precipitation over Sichuan Basin and other downstream regions after they move away from the plateau.

The vertical coordinate \( \sigma \) (Phillips 1957) defined by \( \sigma = p/p_a \), where \( p_a(x, y, t) \) is the pressure at the earth’s surface, has been widely and successfully used to represent orographic effects in numerical models, but it has
never been used in synoptic analysis. This is because the contours of the geopotential height \( \phi(x, y, \sigma, t) \) in \( \sigma \) coordinates (i.e., on constant \( \sigma \) surfaces) represent primarily topography rather than weather systems. These contours have no explicit relation to the pressure and wind fields. Figure 1b shows the geopotential height for \( \sigma = 0.735 \) at 1200 UTC 16 April 1988 over east Asia, derived from analyses of the European Centre for Medium-Range Weather Forecasts (ECMWF) at 2.5° × 2.5° resolution Tropical Ocean and Global Atmosphere (TOGA) Archive II from NCAR. Comparing Figs. 1a and 1b, it is seen that the pattern of the geopotential height contours at \( \sigma = 0.735 \) in Fig. 1b is very similar to that of the topography shown by Fig. 1a. Weather systems cannot be identified by the contours of the geopotential height on constant \( \sigma \) surfaces, as shown in Fig. 1b.

Harrison (1957, 1970) and Shuman (1957) devised methods for computing functions that could be used to approximate the horizontal pressure gradient force at the earth’s surface instead of at sea level. Sangster (1960, 1987) devised a method for using station pressures and surface virtual temperatures to produce stream and potential functions of the surface geostrophic wind, and the examples showed that much of the distortion of the geostrophic wind that results from reduction of pressures to sea level can be avoided by using the reconstructed geostrophic wind at the earth’s surface.

The streamfunction and velocity potential have been used successfully in diagnosis over the globe (e.g., Krishnamurti 1985; Krishnamurti and Ramanathan 1982; Knutson and Weickmann 1987). These variables can provide valuable insight into the dynamics of the flow. However, the streamfunction and velocity potential in a limited region are rarely used in diagnosis and numerical models. To reconstruct a horizontal vector in a limited region, Sangster (1960) also proposed a boundary value method. This boundary method used by a number of investigators (Hawkins and Rosenthal 1965; Shukla and Saha 1974; Bijlsma et al. 1986) may not obtain good results in limited-area wind partitioning and reconstruction problems. This led Bijlsma et al. (1986) to seek an alternative solution method, in which the two Poisson equations for the streamfunction and velocity potential are solved simultaneously. Lynch (1988) re-examined their (Bijlsma et al. 1986) method and found that the method does not converge in general. This is due to the fact that, if the streamfunction and velocity potential are derived from the horizontal wind in a limited region, they are not unique (Lynch 1989; Chen and Kuo 1992a). To avoid the nonuniqueness, the streamfunction and velocity potential can be separated into their inner and harmonic parts (Chen and Kuo 1992a). The inner parts of the streamfunction and velocity potential in a limited region are uniquely determined by the vorticity and divergence within the region, and they can be used in the same way as the streamfunction and
In this paper, a new variable, \( \phi_e \), is proposed instead of the geopotential height in \( \sigma \) coordinates. The horizontal pressure gradient force in \( \sigma \) coordinates can be separated into its irrotational and rotational parts, where the irrotational part is denoted by \(-\nabla \phi_e\). If the vorticity and divergence equations are used, the irrotational and rotational parts are always present independently in these two equations. In \( p \) coordinates, \(-\nabla \phi(x, y, p, t)\) is also the irrotational part of the horizontal pressure gradient force; the irrotational part \(-\nabla \phi_e\) in \( \sigma \) coordinates can be used in the same manner as \(-\nabla \phi(x, y, p, t)\) is used in \( p \) coordinates. Thus, the variable \( \phi_e \) is referred to as an equivalent isobaric geopotential height in \( \sigma \) coordinates. Its analytic expression is given in section 3.

The geostrophic wind relation holds approximately between the equivalent geopotential \( \phi_e \) and the horizontal wind in \( \sigma \) coordinates for synoptic-scale atmospheric motion in middle and high latitudes. The equivalent geopotential \( \phi_e \) can be used to analyze weather systems on constant \( \sigma \) surfaces. This method is very useful near high mountains. Several synoptic examples of this method over Greenland and the Tibetan Plateau are presented in section 4.

The equivalent geopotential \( \phi_e \) in \( \sigma \) coordinates can be used not only in synoptic analysis but also in dynamic studies. Use of the equivalent geopotential \( \phi_e \) in the vorticity and divergence equations is discussed in section 5. In section 6, the equation for the equivalent geopotential \( \phi_e \) is developed. These equations can be used in numerical models, initialization, and other dynamical problems. As a simple example, it is shown how these equations are used in deriving a generalized \( \omega \) equation in \( \sigma \) coordinates and how precipitation over Greenland is retrieved using this derived equation.

Recently, climate change has been studied by analyzing layers of ice from deep cores drilled from the Greenland Ice Sheet (e.g., Johnsen et al. 1992; Alley et al. 1993; Taylor et al. 1993). To help understand such events, it is necessary to investigate the present precipitation characteristics over Greenland. The observations of Greenland precipitation are limited and generally inaccurate (Bromwich and Robasky 1993). In particular, gauge measurements of the mostly solid precipitation are strongly contaminated by wind effects and are primarily confined to the complex coastal environment. Snow accumulation (the net result of precipitation, sublimation/evaporation, and drifting) determinations from the ice sheet are limited in space and time. However, the analyzed wind, geopotential height, and moisture fields are available for recent years. The precipitation should be retrievable from these fields by a dynamic approach.

A natural method for retrieving precipitation over Greenland from the observed wind, geopotential height, and moisture fields is using a four-dimensional data as-
The National Centers for Environmental Prediction (NCEP) and NCAR (Kalnay et al. 1996) are cooperating in a project (denoted “reanalysis”) to produce a 40-yr (1957–96) record of global analyses of atmospheric fields, which includes the global recovery of precipitation. The model used for data assimilation has a resolution of T62 (triangular truncation at 62 waves), which is equivalent to about 210 km, with 28 nonuniformly spaced vertical levels.

The mean annual precipitation over the Greenland region for 1982–94 from the reanalysis (Kalnay et al. 1996) is shown in Fig. 1c. The observed mean annual precipitation pattern over Greenland given by Ohmura and Reeh (1991) is shown in Fig. 1d, which is based on accumulation measurements of 251 pits and cores on the ice sheet and precipitation measurements at 35 meteorological stations in the coastal region. From Fig. 1c, it can be seen that a wave train pattern of the mean annual precipitation extends toward the southeast from the southeast coast. The topography over Greenland is presented in Fig. 1e, and it shows that the slopes are very steep near the southeast coast of Greenland. This quasi-stationary small-scale feature of the reanalysis precipitation seems to be closely related to these steep slopes. Prevailing wisdom is that this precipitation pattern is the so-called spectral rain and it is a consequence of the model’s inexact horizontal diffusion of humidity in the vicinity of steep mountain slopes (W. M. Ebisuzaki 1996, personal communication).

It is seen that the highest amount of the mean annual precipitation along the southeast coast in Fig. 1c is 388 cm yr$^{-1}$, and the amounts are about twice those observed along the southeast and southwest coasts (Fig. 1d). There is also a high precipitation area with a maximum of 110 cm yr$^{-1}$ in the central region of Greenland, where the observed accumulation is about 20 cm yr$^{-1}$ (Fig. 1d). The NCEP–NCAR Reanalysis precipitation errors over Greenland are much larger than those over the southeastern United States (Higgins et al. 1996). Except for some effects of inaccurate observation over Greenland, these larger errors are probably caused, as mentioned before, by the numerical model’s inability to deal properly with the steep slopes near the coast of Greenland.

It is interesting to know whether the capability of a numerical model in dealing with complicated mountains is improved if the equivalent isobaric geopotential is used. However, FDDA systems for limited regions are still under development, and they are computationally very demanding. Because we only want to determine some basic features of precipitation over Greenland, the relatively simple approach used by Bromwich et al. (1993) is adopted. Bromwich et al. (1993) applied a simplified quasigeostrophic $\omega$ equation in $p$ coordinates with a statistical method to calculate the annual precipitation over the Greenland Ice Sheet. The major shortcoming of their simulated precipitation over Greenland is caused by orographic effects.

Because the quasigeostrophic approximation is not accurate enough, a number of generalized $\omega$ equations with fewer assumptions have been developed (e.g., Krishnamurti 1968a,b; Tarbell et al. 1981; DiMego and Bosart 1982; Iversen and Nordeng 1984; Hirschberg and Fritsch 1991; Pauley and Nieman 1992; Raisanen 1995). However, the quasigeostrophic $\omega$ equation and its generalizations always use $p$ coordinates, and its boundary condition at the earth’s surface is usually given by that at $p = 1000$ hPa, $w = V \cdot \nabla H_s$, where $w$ is the vertical velocity and $H_s$ is the elevation of the earth’s surface. Under this boundary condition, it is necessary to calculate the vorticity and temperature advections at the 850 hPa and other lower levels in deriving the solution of the $\omega$ equation. These advections cannot be computed correctly over Greenland because the 850 hPa and other lower levels are actually below the earth’s surface. Thus, the solution of the quasigeostrophic $\omega$ equation in $p$ coordinates cannot be derived accurately over Greenland.

By means of the equivalent geopotential $\phi_e$, a velocity potential form of the generalized $\omega$ equation without the quasigeostrophic approximation in $\sigma$ coordinates to improve the retrieval of precipitation over Greenland.

2. Inner part analysis

a. Use of the inner parts of the streamfunction and velocity potential in a limited region

In coordinates of a conformal map projection, let $U$, $V$, $\Omega$, $D$ be

$$U = \frac{u}{m}, \quad V = \frac{v}{m}, \quad \Omega = \frac{\zeta}{m^2}, \quad D = \frac{\delta}{m^2}, \quad (2.1)$$

where $u$ and $v$ are the horizontal components of the wind field, $\zeta$ the vertical component of the vorticity, $\delta$ the horizontal divergence, and $m$ the map scale factor. The horizontal wind components are expressed by:

$$U = -\frac{\partial \psi}{\partial y} + \frac{\partial \chi}{\partial x}, \quad V = \frac{\partial \psi}{\partial x} + \frac{\partial \chi}{\partial y}, \quad (2.2)$$

where the streamfunction $\psi$ and velocity potential $\chi$ satisfy the following Poisson equations,

$$\nabla^2 \psi = \Omega, \quad \nabla^2 \chi = D, \quad (2.3)$$

within the limited region, and at the boundary they satisfy the conditions:

$$\mathbf{s} \cdot \mathbf{V} = \frac{\partial \psi}{\partial n} + \frac{\partial \chi}{\partial s} = V_s, \quad n \cdot \mathbf{V} = \frac{\partial \psi}{\partial s} + \frac{\partial \chi}{\partial n} = V_n. \quad (2.4)$$
Here s and n are tangential and normal unit vectors, respectively, and s and n are distances along and normal to the boundary.

The partitioning problem in a limited region is to solve the streamfunction and velocity potential from the two Poisson equations (2.3) with boundary conditions (2.4). The conditions (2.4) are referred to as coupled boundary conditions because the normal and tangential derivatives are coupled together (Chen and Kuo 1992b). They are neither Dirichlet nor Neumann boundary conditions. This partitioning problem is a special problem of mathematical physics.

Lynch (1989) and Chen and Kuo (1992a) found that partitioning the wind in a limited region into nondivergent and irrotational components is not unique. The nonuniqueness is closely related to a portion of the wind being both nondivergent and irrotational in a limited region. A limited region, R, is only a portion of the global area. The region over the globe outside region R is referred to as an external region. From a global perspective, the streamfunction and velocity potential, as well as the horizontal wind, in region R depend upon the vorticity and divergence not only within region R but also in the external region. The partitioning problem in a limited region is to deal with the same as the wind field in a limited region.

To enable the streamfunction and velocity potential in a limited region to be used as conveniently as those used on the globe and to avoid the nonuniqueness, Chen and Kuo (1992a) separated the streamfunction and velocity potential in a limited region into the inner and harmonic parts as

\[ \psi = \psi_i + \psi_h, \quad \chi = \chi_i + \chi_h, \quad (2.5) \]

where the inner parts, \( \psi_i \) and \( \chi_i \), satisfy the Poisson equations

\[ \nabla^2 \psi_i = \Omega, \quad \nabla^2 \chi_i = D \quad (2.6) \]

with zero Dirichlet boundary value. The wind computed from the inner part as

\[ U_i = -\frac{\partial \psi_i}{\partial y} + \frac{\partial \chi_i}{\partial x}, \quad V_i = \frac{\partial \psi_i}{\partial x} + \frac{\partial \chi_i}{\partial y}. \quad (2.7) \]

is called an internal wind. Because the solution of a Poisson equation with a Dirichlet boundary value is unique, these inner parts and the internal wind are uniquely determined by the vorticity and divergence within the limited region R. No matter how the vorticity and divergence vary in the external region (outside the region R), the inner parts of the streamfunction and velocity potential, as well as the internal wind, do not vary.

The difference between the wind and internal wind can be expressed by

\[ U_E = U - U_i, \quad V_E = V - V_i. \quad (2.8) \]

According to (2.2) and (2.7), the relationship between \( V_E \) and the harmonic parts of the streamfunction and velocity potential is expressed by

\[ U_E = -\frac{\partial \psi_h}{\partial y} + \frac{\partial \chi_h}{\partial x}, \quad V_E = \frac{\partial \psi_h}{\partial x} + \frac{\partial \chi_h}{\partial y}. \quad (2.9) \]

From (2.3) and (2.6), the harmonic parts satisfy Laplace equations

\[ \nabla^2 \psi_h = 0, \quad \nabla^2 \chi_h = 0 \quad (2.10) \]

with the coupled boundary conditions

\[ U_E|_{\Sigma} = \left( -\frac{\partial \psi_h}{\partial y} + \frac{\partial \chi_h}{\partial x} \right)|_{\Sigma}, \quad (2.11) \]

The wind \( V_E \) that satisfies (2.9), (2.10), and (2.11) must be both nondivergent and irrotational within the limited region R, thus it must be the external wind. By using the above method, the external wind can be separated from the wind field in a limited region.

The above method is used to separate the horizontal wind into the internal and external wind. The internal wind can be further separated into its internal nondivergent, internal irrotational, and external components based on (2.7). Thus, the horizontal wind in a limited area is expressed by a sum of three parts as

\[ \mathbf{V} = \mathbf{k} \times \nabla \psi_i + \nabla \chi_i + \mathbf{V}_E. \quad (2.12) \]

where \( \mathbf{k} \times \nabla \psi_i \) is the internal nondivergent component and \( \nabla \chi_i \) is the internal irrotational component. Because the inner parts of the streamfunction and velocity potential are uniquely determined in a limited region, they can be used in the same way as the streamfunction and velocity potential are used on the globe. The internal wind in a limited region is dealt with the same as the horizontal wind on the globe. Using the above method in (2.12), any horizontal vector in a limited region can be separated into its internal nondivergent, internal irrotational, and external components.

b. The inner part of the geopotential height

The geopotential height on the constant \( \sigma \) surface in a limited region can also be separated into the harmonic and inner parts,
\[ \phi = \phi_h(x, y, \sigma, t) + \phi_i(x, y, \sigma, t), \]  
where its harmonic part satisfies the Laplace equation  
\[ \nabla^2 \phi_h(x, y, \sigma, t) = 0 \]  
with the Dirichlet boundary value of the geopotential height itself as  
\[ \phi_h|_{\partial x} = \phi_i|_{\partial x}, \]  
and its inner part is derived from the difference between \( \phi \) and its harmonic part,  
\[ \phi_i = \phi - \phi_h. \]  
If the Laplacian of geopotential height, \( \nabla^2 \phi \), is known in a limited region, the inner part can also be derived from the Poisson equation  
\[ \nabla^2 \phi_i = (\nabla^2 \phi)(x, y) \]  
with zero Dirichlet boundary value. Here the right-hand side of (2.17) is a known function in the region.

If an equation \( A = B \) is satisfied in a limited region and both sides of the equation are separated into their inner and harmonic parts, the following equations  
\[ A_h = B_h \quad \text{and} \quad A_i = B_i, \]  
are satisfied in the limited region based on the uniqueness of a harmonic function with a Dirichlet boundary value.

c. Inner part analysis

If a limited-area model is formulated in terms of the vorticity and divergence equations and is nested in a global model by the one-way method, at each time step only the vorticity and divergence within the region can be predicted by the limited-area model. This means that only the internal wind and inner parts can be predicted from the limited-area model, whereas the external wind and harmonic parts must be derived from the prediction of the global model. In this case, the external wind and harmonic parts can be used as the lateral boundary condition for the limited-area model. This method has been successfully used in the harmonic-Fourier spectral limited-area model studied by Chen et al. (1997a) (hereafter referred to as CBB), and very good predicted results have been obtained.

The predicted variables of this limited-area model are the inner parts of the streamfunction, velocity potential, and geopotential height, and they describe the development of motion systems very well in the model. Thus, these inner parts must be able to be used in synoptic analysis of a limited region. A method of using the inner parts of the streamfunction, velocity potential, and other variables to describe motion systems in a limited region is referred to as inner part analysis.

The inner part variables are dependent on the horizontal scale of the limited region chosen for study, thus, the horizontal scale of the studied region must be chosen to be large enough so that the studied motion systems are completely described by the vorticity and divergence within the region. The harmonic parts and the components of the external wind are all harmonic functions in the limited domain. These harmonic functions attain their largest and smallest values at the boundary, and their horizontal scale must be larger than that of the limited region. Thus, the horizontal scale of the harmonic parts and external wind are much larger than those of the motion systems studied in the limited region.

3. An equivalent isobaric geopotential height in \( \sigma \) coordinates

a. Definition of equivalent isobaric geopotential height

The horizontal pressure gradient force in \( p \) coordinates is expressed by \( -\nabla \phi(x, y, p, t) \), where \( \phi(x, y, p, t) \) is the geopotential height in \( p \) coordinates. When it is transformed into \( \sigma \) coordinates, it is expressed by  
\[ G = -\nabla \phi(x, y, \sigma, t) \]  
\[ -RT(x, y, \sigma, t) \nabla \ln p, (x, y, t), \]  
where \( G \) denotes the horizontal pressure gradient force, and \( \phi(x, y, \sigma, t) \) is the geopotential height in \( \sigma \) coordinates. It is seen that the vector \( -\nabla \phi(x, y, p, t) \) is irrotational, but the total vector on the right-hand side of (3.1) is not irrotational because \( T \) is a function of \( x \) and \( y \).

Similar to the pressure gradient force in \( p \) coordinates, we can look for a vector expressed by \( -\nabla \phi_i(x, y, \sigma, t) \) that represents the irrotational part of the horizontal pressure gradient force in \( \sigma \) coordinates. In this case, the irrotational part of \( -\nabla \phi_i(x, y, \sigma, t) \) must be equal to the irrotational part of the total vector on the right-hand side of (3.1), thus we have  
\[ \nabla^2 \phi_i = \nabla^2 \phi + \frac{\partial}{\partial x} \left( RT \frac{\partial \ln p}{\partial x} \right) + \frac{\partial}{\partial y} \left( RT \frac{\partial \ln p}{\partial y} \right) \]  
(3.2)

The total vector on the right-hand side of (3.1) has a rotational part. The irrotational and rotational parts of the horizontal pressure gradient force are independent of each other in the divergence and vorticity equations. The rotational part is very small, and it is proved in the next section that the magnitude of the rotational part is one order smaller than the irrotational part even over high mountain regions. The irrotational part \( -\nabla \phi_i \) can be used in \( \sigma \) coordinates in the same way as \( -\nabla \phi \), but it is used in \( p \) coordinates. Thus, the variable \( \phi_i \) is referred to as an equivalent isobaric geopotential height, or briefly, an equivalent geopotential in \( \sigma \) coordinates.

Over the globe, the Poisson equation (3.2) is easily
solved without lateral boundary conditions. If equation (3.2) needs to be solved in a limited region, similar to (2.13), the equivalent geopotential in $\sigma$ coordinates can be separated into its inner and harmonic parts as

$$\phi_\sigma = \phi_\sigma(x, y, \sigma, t) + \phi_{ei}(x, y, \sigma, t).$$  \hspace{1cm} (3.3)

Based on (3.2), the inner part of the equivalent geopotential, $\phi_{ei}$, in $\sigma$ coordinates can be derived from the solution of the following Poisson equation

$$\nabla^2 \phi_{ei} = \nabla^2 \phi_i + \frac{\partial}{\partial x} \left( RT \frac{\partial \ln p_*}{\partial x} \right) + \frac{\partial}{\partial y} \left( RT \frac{\partial \ln p_*}{\partial y} \right)$$

(3.4)

with zero Dirichlet boundary value. The solution of (3.4) can be expressed by

$$\phi_{ei} = \phi_i + \nabla^2 \left( \frac{\partial}{\partial x} \left( RT \frac{\partial \ln p_*}{\partial x} \right) + \frac{\partial}{\partial y} \left( RT \frac{\partial \ln p_*}{\partial y} \right) \right),$$

(3.5)

where $\nabla^2$ is an integral operator shown in the appendix to denote the solution of the Poisson equation with zero Dirichlet boundary value. From (3.5), the inner part of the equivalent geopotential $\phi_{ei}$ can be derived.

b. Physical interpretation of the equivalent geopotential height

To show the physical basis for $\phi_\sigma$, (3.4) can be rewritten as

$$\nabla^2 (\phi_i - \phi_\sigma) = \nabla^2 \phi_i = -\frac{\partial}{\partial x} \left( RT \frac{\partial \ln p_*}{\partial x} \right) - \frac{\partial}{\partial y} \left( RT \frac{\partial \ln p_*}{\partial y} \right)$$

$$= \nabla \cdot (-RT \nabla \ln p_*),$$

(3.6)

where $\phi_{ei} = \phi_i - \phi_\sigma$ is a potential function of the vector $-RT \nabla \ln p_*$. The horizontal gradient of the surface pressure $p_\sigma(x, y, t)$ caused by topography is much larger than that caused by synoptic disturbances, especially near high mountain regions. Thus, the contours of the height of the earth’s surface, $H_\sigma(x, y)$, are roughly parallel to the contours of $p_\sigma$ or $\ln p_*$. A low value center of $p_\sigma$ or $\ln p_*$ is located in the region where $H_\sigma(x, y)$ is high. The vector $-RT \nabla \ln p_*$ has the direction from high values of $\ln p_*$ to low values. Thus, over a high mountain region, the vector $-RT \nabla \ln p_*$ must converge, that is, $\nabla \cdot (-RT \nabla \ln p_*) < 0$. If this distribution of $\nabla \cdot (-RT \nabla \ln p_*)$ is solved from the Poisson equation (3.6), $\phi_{ei}$ must have a positive center over the high mountain region. (It is easy to estimate the solution of the Poisson equation because the solution is an inverse operator of the Laplacian operator $\nabla^2$.) The potential function $\phi_{ei}$ has the property that $\nabla \phi_{ei} = -RT \nabla \ln p_*$. Thus, $\nabla \phi_{ei} = \nabla \phi_i - \nabla \phi_\sigma$. The potential function $\phi_{ei}$ acts to reduce the value of $\phi(x, y, \sigma, t)$ to $\phi_i(x, y, \sigma, x)$, and to make $\nabla \phi_i(x, y, \sigma, t)$ equal to the irrotational part of the horizontal pressure gradient force.

The physical implication of $\phi_\sigma$ can also be understood from a comparison between $\phi(x, y, \sigma, t)$ and $\phi(x, y, p, t)$. In $\sigma$ coordinates, the earth’s surface is a coordinate surface. The major difference between $\phi(x, y, \sigma, t)$ and $\phi(x, y, p, t)$ is that $\nabla \phi(x, y, \sigma, t)$ is not equal to the horizontal pressure gradient force because $\phi(x, y, \sigma, t)$ has been altered by orography. The value of $\phi(x, y, \sigma, t)$ is computed from the hydrostatic equation in $\sigma$ coordinates. The topography has two ways to affect the computation of $\phi(x, y, \sigma, t)$. One is through the initial value $\phi_\sigma$, where $\phi_\sigma = gH_\sigma$ is used in the integration of the hydrostatic equation. This is the direct effect from the topography itself. The second effect is indirectly through the temperature $T(x, y, \sigma, t)$, where the temperature used in the integration in $\sigma$ coordinates has been altered by orography.

Equation (3.6) can be rewritten as

$$\nabla^2 \phi_{ei} = -RT \nabla^2 \ln p_* - RT \nabla T \cdot \nabla \ln p_*.$$  \hspace{1cm} (3.7)

As mentioned above, a low value center of $p_\sigma$ or $\ln p_*$ is located over a mountain region where the elevation is high. Over the high mountain region, $\nabla^2 \ln p_* > 0$ and $-RT \nabla^2 \ln p_* < 0$ due to the low value center of $\ln p_*$. The temperature $T(x, y, \sigma, t)$ in $\sigma$ coordinates can be separated into two parts as

$$T(x, y, \sigma, t) = T_{wa} + T_{SL},$$  \hspace{1cm} (3.8)

where $T_{SL}(x, y, \sigma, t)$ is the temperature that is not affected by orography. It is the temperature in $\sigma$ coordinates if $p_\sigma = p_\sigma(x, y, t)$, where $p_\sigma(x, y, t)$ is the SLP at $H_\sigma = 0$, and it is equivalent to the temperature in $p$ coordinates. From (3.8), we have $T_{wa} = T(x, y, \sigma, t) - T_{SL}$, where $T_{wa}$ is the temperature caused by orography. If $p_\sigma = p_\sigma(x, y, t)$ at $H_\sigma = 0$, then $T(x, y, \sigma, t) = T_{SL}$, and $T_{wa} = 0$. The temperature $T_{wa}$ is the major difference between $T(x, y, \sigma, t)$ and $T(x, y, p, t)$. The temperature $T_{SL}$ can further be divided into three parts as

$$T_{SL}(x, y, \sigma, t) = T_x(\sigma) + T_y(x, y, \sigma)$$

$$= T_x(\sigma) + T_y(x, y, \sigma),$$  \hspace{1cm} (3.9)

where $T_x(\sigma)$ is the mean value of $T_{SL}$ averaged over the constant $\sigma$ surface, $T_y$ is temperature deviation caused by climate without the effect of orography (equivalent to the climate state in $p$ coordinates), and $T_y$ is caused by the transient synoptic disturbances.

The temperature caused by orography, $T_{wa}$, is produced through temperature decrease with height. The horizontal temperature gradient caused by orography, $\nabla T_{wa}$, is the major part of the temperature gradient near high mountain regions. The isotherms for $T_{wa}$ are almost parallel to the contours of $\ln p_*$ over high mountain regions. The cold center of $T_{wa}$ is located in the region where $\ln p_*$ is low (or the elevation is high). Over a high mountain region, the vectors $\nabla T_{wa}$ and $\nabla \ln p_*$ have the
solute values of both negative value is over steep slopes, where the absolute values of both \( \nabla T_{ai} \) and \( \nabla \ln \rho_a \) are very large.

If the sum of the two terms on the right-hand side of (3.7) is solved from this Poisson equation, \( \phi_{ai} \) has a positive center over the high mountain region. In the region where the effect of topography is small, \( \nabla \rho_a(x, y, t) \) and \( \nabla T_{ai} \) are also very small, and vice versa. Thus, the distribution of \( \phi_a \) produced by the two terms on the right-hand side of (3.7) mainly represents the part of \( \phi(x, y, \sigma, t) \) that is produced by the effects of topography. These effects include those directly from terrain itself, \( \nabla \rho_a \), and indirectly from the temperature field, \( \nabla T_{ai} \).

The equation \( \phi_{ai} = \phi_a - \phi_{ci} \) means that the part of geopotential produced by the effect of topography, \( \phi_{ai} \), is removed from \( \phi(x, y, \sigma, t) \), and the remainder is \( \phi_{ci} \). Thus, the equivalent geopotential \( \phi_{ei} \) is similar to \( \phi(x, y, p, t) \), which is not affected by topography. The equivalent geopotential \( \phi_{ei} \) is a variable in \( \sigma \) coordinates, but it plays the same role as the geopotential height \( \phi(x, y, p, t) \) does in \( p \) coordinates.

c. The geostrophic approximation for the rotational wind in \( \sigma \) coordinates

In \( \sigma \) coordinates, where \( \sigma = \rho/p \), the momentum equation is expressed by

\[
\frac{\partial V}{\partial t} + V \cdot \nabla V + \frac{\partial V}{\partial \sigma} + f/k \times V = \nabla \phi - RT \nabla \ln \rho_a + \mathbf{P},
\]

where \( \mathbf{P} \) is the friction caused by the vertical fluxes of momentum due to convection and boundary layer turbulence. The geostrophic wind approximation in \( \sigma \) coordinates can be denoted by

\[
f/k \times V \equiv -\nabla \phi_{ei} - RT \nabla \ln \rho_a.
\]

The horizontal pressure gradient force in the right-hand side of (3.11) can be partitioned into the irrotational and rotational parts, and (3.11) can be written as

\[
f/k \times V \equiv -\nabla \phi_{ei} - k \times \nabla \eta.
\]

where \( -k \times \nabla \eta \) is the rotational part of the horizontal pressure gradient force, and \( \eta \) is referred to as a geostrophic function. Equation (3.12) is similar to that used by Sangster (1960) to compute the surface geostrophic wind over mountain regions. The geostrophic function satisfies

\[
\nabla^2 \eta = R \left( \frac{\partial T}{\partial x} \frac{\partial \ln \rho_a}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial \ln \rho_a}{\partial x} \right)
\]

without the boundary condition over the globe. If equation (3.13) is solved in a limited region, the geostrophic function can also be separated into its inner and harmonic parts, then the inner part of the geostrophic function, \( \eta_i \), can be derived from the following Poisson equation

\[
\nabla^2 \eta_i = R \left( \frac{\partial T}{\partial x} \frac{\partial \ln \rho_a}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial \ln \rho_a}{\partial x} \right)
\]

with zero Dirichlet boundary value. Equation (3.12) can be rewritten as

\[
U = \frac{u}{m} \equiv -\frac{1}{f_0} \frac{\partial \phi_{ei}}{\partial \sigma} + \frac{1}{f_0} \frac{\partial \eta_i}{\partial x},
\]

\[
V = \frac{v}{m} \equiv -\frac{1}{f_0} \frac{\partial \phi_{ei}}{\partial \sigma} + \frac{1}{f_0} \frac{\partial \eta_i}{\partial y}.
\]

From (3.15), the vorticity is approximately expressed by

\[
\Omega = \nabla^2 \phi_{ei} \equiv \frac{1}{f_0} \nabla^2 \phi_{ei}.
\]

In (3.16), only the irrotational part of the horizontal pressure gradient force is considered, and the rotational part is eliminated automatically. Equation (3.16) can be solved with zero Dirichlet boundary value in a limited region, and its solution is

\[
f_0 \psi_i \equiv \phi_{ai}.
\]

Equation (3.17) is the geostrophic approximation for the rotational wind in \( \sigma \) coordinates.

4. Some examples of synoptic analysis using the equivalent isobaric geopotential in \( \sigma \) coordinates

a. A synoptic example over Greenland analyzed by the inner part of the equivalent isobaric geopotential at \( \sigma = 0.995 \)

The equivalent isobaric geopotential height can be used in synoptic analysis over orographically complicated regions, such as Greenland and the Tibetan Plateau. The topography of the studied region near Greenland is presented in Fig. 1e. Data from ECMWF at 2.5° × 2.5° resolution (TOGA Archive II from NCAR) are interpolated to 16 \( \sigma \) levels in the vertical, at \( \sigma = 0.015, 0.045, 0.075, 0.105, 0.140, 0.200, 0.285, 0.390, 0.510, 0.630, 0.735, 0.825, 0.900, 0.950, 0.980, \) and 0.995. The grid size of the limited region used in analysis is 111 × 81 and the grid spacing is 50 km.

In Greenland, snow and ice surfaces typically have an albedo larger than 0.80, implying that more than three-quarters of the incident shortwave radiation from the sun are reflected. Little of this reflected solar radiation is absorbed by the overlying atmosphere. In addition, the ice surface acts nearly like a blackbody for longwave radiation. Thus, the ice surface efficiently radiates thermal energy to outer space. The cold source at the ice and snow surface will reduce the air temperature in the boundary layer by downward sensible heat...
FIG. 2. (a) The inner part of the equivalent isobaric geopotential height at 0000 UTC 8 January 1992 at $\sigma = 0.995$ (in 10 m$^2$ s$^{-2}$ and contour interval: $40 \times 10$ m$^2$ s$^{-2}$). (b) Same as (a) except for 1200 UTC 8 January 1992. (c) Same as (a) except for 0000 UTC 9 January 1992. (d) Same as (a) except for 1200 UTC 9 January 1992. (e) SLP at 0000 UTC 8 January 1992 (5-hPa isobar spacing). (f) Same as (e) except for 1200 UTC 8 January 1992. (g) Same as (e) except for 0000 UTC 9 January 1992. (h) Same as (e) except for 1200 UTC 9 January 1992. (i) The inner part of the geo-streamfunction at 1200 UTC 8 January 1992 at $\sigma = 0.995$ (in 10 m$^2$ s$^{-2}$ and contour interval: $20 \times 10$ m$^2$ s$^{-2}$). (j) Same as (i) except for 1200 UTC 9 January 1992. (k) Schematic diagram of a mountain with an idealized elliptical shape and a temperature field that decreases with latitude.
transfer. Thus, a cold high pressure system is often located over Greenland if the circulation is not disturbed by synoptic systems.

The inner part of the equivalent isobaric geopotential height computed from (3.5) at $\sigma = 0.995$ for 0000 UTC 8 January 1992 is shown in Fig. 2a. In Fig. 2a, there is an anticyclone over most of Greenland. A cyclone located in the Labrador Sea is moving eastward toward the southwest coast of Greenland, and another cyclonic center is over Baffin Bay.

The inner part of the equivalent isobaric geopotential height at $\sigma = 0.995$ for 1200 UTC 8 and 0000 and 1200 UTC 9 January 1992 are presented in Figs. 2b–d, respectively. The Labrador Sea cyclone reaches the southwest coast of Greenland in the next 12 h as shown in Fig. 2b, and the anticyclone weakens to a high pressure ridge over the east coast. In Figs. 2c and 2d, the center of the equivalent geopotential low has moved to the southeast coast and develops on the lee side of the southern part of Greenland, respectively. At the same time, the cyclone in Baffin Bay moves northward, weakens, and dissipates.

The SLP analyses at 0000 UTC and 1200 UTC 8 January and 0000 and 1200 UTC 9 January 1992 are presented in Figs. 2e–h, respectively. These analyses are directly based on ECMWF analysis and we do not separate the inner and harmonic parts of the SLP. Because the harmonic parts are the harmonic functions and attain their largest and smallest values at the boundary, their absolute values in the inner region are relatively small. Thus, only the inner region of Figs. 2e–h can be used for comparison with Figs. 2a–d, which represent only
We cannot find clear relationship between the geostrophic approximation in \( \sigma \) coordinates and the inner parts of the equivalent geopotential solved from (3.4) with zero Dirichlet boundary value. The inner region of Figs. 2e–h over the oceanic region outside Greenland are very similar to Figs. 2a–d, respectively. The development and displacement of the cyclone initially located over the Labrador Sea are the same as those shown in Figs. 2a–d. However, there are many small but strong high centers (as well as low centers) in these SLP maps over Greenland shown in Figs. 2e–h. These are major differences between the analyses of the equivalent isobaric geopotential and of the SLP over Greenland. These small but strong high pressure systems are caused by reduction of pressure to sea level using cold temperatures at the top of the ice sheet. These systems do not exist in the real atmosphere. If the equivalent isobaric geopotential at \( \sigma = 0.995 \) is used as shown in Figs. 2a–d, these small but strong high pressure systems over Greenland are all removed. The circulation at \( \sigma = 0.995 \) becomes very smooth and it represents the true circulation that exists near the ground of the Greenland Ice Sheet.

Details about how to separate a variable into its inner and harmonic parts and how to derive the solution of the Poisson equation to perform synoptic analyses are the same as those discussed by Chen and Kuo (1992a) and Chen et al. (1997a). The harmonic-sine spectral method is used in all of these computations. These computations can also be implemented by a finite-difference method.

### b. Some characteristics of the geo-streamfunction and the geostrophic approximation in \( \sigma \) coordinates

Figures 2i and 2j are the inner part of the geo-streamfunction at \( \sigma = 0.995 \) level for 1200 UTC 8 and 9 January 1992, respectively. They are derived from (3.14) with zero Dirichlet boundary value. The times of Figs. 2i and 2j are the same as those of Figs. 2b and 2d. From Figs. 2b and 2d, it can be seen that a cyclone develops in Fig. 2d on the east coast of Greenland in place of the high pressure ridge shown in Fig. 2b, whereas Figs. 2i and 2j do not show important changes in these two features. The fields shown in Figs. 2i and 2j look the same; there is a low of the geo-streamfunction on the east coast of Greenland and a high in the west. We cannot find clear relationship between the geo-streamfunction and the current synoptic systems.

Equation (3.14) can be rewritten as

\[
\nabla^2 \eta = k \cdot (R \nabla T \times \nabla \ln p_g), \tag{4.1}
\]

Based on (3.8) and (3.9), the temperature on the constant \( \sigma \) surface can be divided into four parts: \( T_{\sigma\sigma} \), \( T_{\sigma\sigma}(\sigma) \), \( T_{\sigma\sigma} \), and \( T_{\sigma\sigma} \). The temperature gradient caused by topography, \( \nabla T_{\sigma\sigma} \), is a major part of the temperature gradient near high mountain regions. The isotherm of \( T_{\sigma\sigma} \) is almost parallel to the isobaric contours of \( p_g \). The colder area is located in the region where \( p_g \) is lower. Thus, the vectors \( \nabla T_{\sigma\sigma} \) and \( \nabla \ln p_g \) have the same direction, and the quantity \( \nabla T_{\sigma\sigma} \times \nabla \ln p_g \) is nearly zero.

In general, the temperature gradient caused by synoptic disturbances, \( \nabla T_{\sigma\sigma} \), is smaller than that caused by climate, \( \nabla T_{\sigma\sigma} \). The primary part of \( \nabla T_{\sigma\sigma} \) in the high and middle latitudes is that the temperature decreases from south to north. If a mountain has an idealized elliptical shape with its long axis along the north–south direction, its contours of \( p_g \) are shown in Fig. 2k. The temperature decrease from south to north is also shown in Fig. 2k. The quantity \( k \cdot (\nabla T \times \nabla \ln p_g) \) can easily be deduced based on the distribution of \( \ln p_g \) and \( T \) shown in Fig. 2k. In this case, it is positive on the east slope of the mountain and negative on the west slope. If this distribution of \( k \cdot (\nabla T \times \nabla \ln p_g) \) is solved from the Poisson equation (4.1), a negative region of the geo-streamfunction will be located over the east side of the mountain and a positive region over the west side. The distribution of the geo-streamfunction shown in Figs. 2i and 2j is similar to this idealized case. Thus, the distribution of the geo-streamfunction is determined by the shape of the mountain and basic climate temperature distribution.

Noting that the contour interval in Figs. 2b and 2d is double of that in Figs. 2i and 2j, it is seen that the magnitude of \( \nabla \phi_m \) shown in Figs. 2b and 2d is much larger than that of \( \nabla \eta \) in Figs. 2i and 2j. To compare them quantitatively, the mean values of the irrotational and rotational parts of the horizontal pressure gradient force are computed, respectively, by

\[
|\nabla \phi_m| = \frac{1}{MN} \sum_{l=1}^{M} \sum_{j=1}^{N} \left( \frac{\partial \phi_m}{\partial x} \right)_{l,j}^2 + \left( \frac{\partial \phi_m}{\partial y} \right)_{l,j}^2 \tag{4.2}
\]

and

\[
|\nabla \eta| = \frac{1}{MN} \sum_{l=1}^{M} \sum_{j=1}^{N} \left( \frac{\partial \eta}{\partial x} \right)_{l,j}^2 + \left( \frac{\partial \eta}{\partial y} \right)_{l,j}^2 \tag{4.3}
\]

where \( MN \) is the total number of grid points in the computed limited region. The computed mean values of \( |\nabla \phi_m| \) and \( |\nabla \eta| \) over and near the Greenland region for different \( \sigma \) levels at 1200 UTC 8 January 1992 are shown in Table 1. The results of many days have been computed, but they are all the same as those shown in Table 1. The quantity \( (|\nabla \eta|)/(|\nabla \phi_m|) \) denotes the ratio of the rotational part to the irrotational one. From Table 1 it is seen that the ratio \( (|\nabla \eta|)/(|\nabla \phi_m|) \) decreases with height, and its largest value is at the 0.995 \( \sigma \) level. Even at this level, the magnitude of the rotational part is one order smaller than that of the irrotational one. The mean values of the irrotational and rotational parts averaged over all \( \sigma \) levels are 21.616 \( \times 10^{-4} \) and 1.348 \( \times 10^{-4} \) m s\(^{-2} \), respectively. If they are divided by \( f_0 \), using a mean latitude of 70°N, the two mean values become 15.78 and 0.984 m s\(^{-1} \), respectively. These two mean values show that the rotational part of the horizontal pressure gradient force in \( \sigma \) coordinates is more
than one order of magnitude smaller than its irrotational part. Thus, the geostrophic approximation (3.12) can be approximatively rewritten as

$$f \mathbf{k} \times \mathbf{V} \cong -\nabla \phi_e.$$  \hspace{1cm} (4.4)

The irrotational and rotational parts of the horizontal pressure gradient force can be independent of each other. The geostrophic approximation for the irrotational pressure gradient force (or the rotational wind) is expressed by (3.17). Figures 2l and 2m are the inner part of the streamfunction at the $\sigma = 0.995$ level for 1200 UTC 8 and 9 January 1992, respectively. They are derived from (2.6) based on the wind field of the ECMWF analysis. Comparing Figs. 2l and 2m with Figs. 2b and 2d, respectively, it is seen that they are very similar over the region including Greenland. Thus, from qualitative inspection, the geostrophic relation for the rotational wind (3.17) is approximately satisfied at $\sigma = 0.995$ over Greenland.

The geostrophic approximation for the rotational pressure gradient force (or the divergent wind) is expressed by

$$D = \nabla^2 \chi = \frac{1}{f_0} \nabla^2 \eta.$$  \hspace{1cm} (4.5)

Equation (4.5) can be solved with zero Dirichlet boundary value in a limited region; its solution becomes

$$f_0 \eta = \chi.$$  \hspace{1cm} (4.6)

Figures 2n and 2o are the inner part of the velocity potential derived from (2.6) based on the wind field at the $\sigma = 0.995$ level for 1200 UTC 8 and 9 January 1992, respectively. From Fig. 2n, it is seen that the velocity potential is very small over the Greenland region. When the cyclone develops over the east coast of Greenland at 1200 UTC 9 January 1992, an area of high velocity potential also develops over that region at the $\sigma = 0.995$ level as shown in Fig. 2o. A center of high velocity potential corresponds to wind convergence. Comparing Figs. 2n and 2o with Figs. 2l and 2m, respectively, we cannot find any relation satisfying approximation (4.6). In Fig. 2o, there is a region of high velocity potential over the east coast of Greenland. According to (4.6), the geo-streamfunction should also have a high value region over that area. However, the computed geo-streamfunction shown in Fig. 2m is a low center over that region. It is seen that the approximation (4.6) between the inner parts of the geo-streamfunction and divergent wind does not exist in the real atmosphere.

The geostrophic approximation (3.11) is the first-order approximation of (3.10), which means that the magnitudes of the last term on the left-hand side of (3.10) and first two terms on its right-hand side are one order larger than that of the other terms in the equation. If the horizontal pressure gradient force is separated into its irrotational and rotational parts, and the magnitude of the rotational part is one order smaller than that of the irrotational part, the magnitude of the terms in (3.10), which correspond to the two terms in (4.6), is no longer one order larger than that of the other terms in (3.10). Thus, approximation (4.6) does not exist. The geostrophic approximation in $\sigma$ coordinates can be expressed only by (3.17) and (4.4) but not by (4.6).

The above computed example further shows that, because the rotational part of the horizontal pressure gradient force is so small, it need not be considered in the geostrophic approximation and synoptic analysis.

The purpose of this paper is to propose a new variable $\phi_e$ that is used in $\sigma$ coordinates in the same manner as $\phi(x, y, p, t)$ is used in $p$ coordinates, rather than just calculating the geostrophic wind over mountain regions. The equivalent geopotential $\phi_e$ can also be used in tropical and equatorial regions. However, the geostrophic approximation is a useful tool for geopotential analysis on isobaric surfaces. The geostrophic approximation shown by (4.4) and (3.17) for $\phi_e$ is the same as that for $\phi(x, y, p, t)$, thus it must be very useful for equivalent geopotential analysis on constant $\sigma$ surfaces.

c. Systems over the Tibetan Plateau analyzed by the inner part of the equivalent isobaric geopotential at $\sigma = 0.995$

Because weather systems over the Tibetan Plateau are very difficult to identify on lower-tropospheric synoptic analyses (e.g., SLP, 850 hPa, and 700 hPa), the equivalent isobaric geopotential analysis on the constant $\sigma$ surface is especially useful for this plateau region. Figure 1a shows the topography over east Asia, in which the largest mountain is the huge Tibetan Plateau, whose maximum height is greater than 5000 m. The Altai-Sayan Mountains, with a height of about 2200 m, are located to the southwest of Lake Baikal. The height and horizontal scale of the Altai-Sayan Mountains are com-
parable to those of the Alps. In the lee of the Altai-Sayan Mountains, lee cyclogenesis often occurs. In the analysis of the constant $\sigma$ surface, 16 $\sigma$ levels shown above are used. The mesh size of the limited region is $61 \times 51$ and the grid spacing is 170 km.

The inner part of the equivalent isobaric geopotential at $\sigma = 0.995$ for 1200 UTC 14 April 1988 is shown in Fig. 3a. It is seen from Fig. 3a that a mature cyclone moves eastward to the north of Lake Baikal, which may be referred to as the parent cyclone according to Palmén and Newton (1969). A front trailing from the cyclone is oriented from northeast to southwest with a cold high behind. The inner part of the equivalent geopotential at $\sigma = 0.995$ for 0000 and 1200 UTC 15 April 1988 is presented in Figs. 3b–d, respectively. In Fig. 3b, a pressure ridge is formed on the windward side of Altai-Sayan when the front moving eastward is retarded by mountains. At the same time, the equivalent isobaric geopotential on the lee side of the mountains starts to fall and a pressure trough appears over the Mongolian Plateau ahead of the front. Twelve hours later as shown in Fig. 3c, a lee cyclone has formed with the central height of about $-75.9$ m. The value of the inner part of the equivalent isobaric geopotential

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**Fig. 3.** (a) The inner part of the equivalent isobaric geopotential height at 1200 UTC 14 April 1988 at $\sigma = 0.995$ (in 10 m$^2$/s$^2$ and contour interval: 30 $\times$ 10 m$^2$/s$^2$; the heavy dashed line shows the sketch of the Tibetan Plateau and the fronts have also been presented). (b) Same as (a) except for 0000 UTC 15 April 1988. (c) Same as (a) except for 1200 UTC 15 April 1988. (d) Same as (a) except for 0000 UTC 16 April 1988. (e) SLP at 1200 UTC 14 April 1988 (3-hPa isobar spacing). (f) Same as (e) except for 0000 UTC 15 April 1988. (g) Same as (e) except for 1200 UTC 15 April 1988. (h) Same as (e) except for 0000 UTC 16 April 1988. (i) The inner part of the streamfunction at 1200 UTC 14 April 1988 at the level $\sigma = 0.995$ (in 10$^5$ m$^2$/s$^1$ and contour interval: 30 $\times$ 10$^5$ m$^2$/s$^1$). (j) A schematic diagram for the early stage of an SW vortex formed on the eastern flank of the Tibetan Plateau.
Fig. 3 (Continued)
Fig. 4. (a) The geopotential height (solid line at 80 gpm (geopotential meter) spacing), wind (arrows with scale in m s$^{-1}$ at bottom), and temperature (dashed line in 4 K increments) at the 850-hPa level at 0000 UTC 15 April 1988; the heavy dashed line shows the outline of the boundary. At 0000 UTC 16 April as shown in Fig. 3d, this cyclone has grown to its mature state. However, the central value does not decrease, but rather increases and then stays unchanged at $-70$ m. The lee cyclone separates from the parent cyclone as the latter moves away from the limited region.

The SLP analyses at 1200 UTC 14 April, 0000 and 1200 UTC 15 April, and 0000 UTC 16 April 1988 are presented in Figs. 3e–h, respectively. These analyses are directly based on ECMWF analysis and do not separate the inner and harmonic parts. Thus, only the inner region of Figs. 3e–h can be compared with Figs. 3a–d. The heavy dashed line is the outline of the Tibetan Plateau. These analyses are very similar to Figs. 3a–d, respectively, in the region far away from the Tibetan Plateau. They have a similar depiction of the lee cyclone development. However, over the Tibetan Plateau, the analyses in Figs. 3e–h are different from those in Figs. 3a–d, respectively. In Figs. 3e–h over the Tibetan Plateau, there are many anomalous high and low centers in these SLP maps. These high and low systems are caused by reduction of pressure to sea level using the data from upper-atmospheric levels. Some of them look like the topography, and there is no weather significance attached to these anomalous systems. However, if the equivalent isobaric geopotential at $\sigma = 0.995$ is used as shown in Figs. 3a–d, these artificial systems over the Tibetan Plateau are all removed.

From 1200 UTC 14 April (Fig. 3a) to 0000 16 April...
1988 (Fig. 3d), the anticyclone behind the lee cyclone moves toward the southeast, and it causes cold air to affect the eastern part of the Tibetan Plateau. The cold air is separated from the major anticyclone to form a secondary high over the Tibetan Plateau at 0000 UTC 16 April 1988. This secondary high can be seen clearly in Fig. 3d over the plateau, but cannot be identified in Fig. 3h.

Figure 3i is the inner part of the streamfunction derived from the wind field at \( s = 0.995 \) level for 1200 UTC 14 April 1988. Comparing Fig. 3i with Fig. 3a, it is seen that they are similar over the region including the Tibetan Plateau. The geostrophic relation for the rotational wind on the constant \( s \) surface (3.17) is approximately satisfied at the \( s = 0.995 \) level over the Tibetan Plateau, although this approximation is not as good as that in high latitudes, such as the approximation between Figs. 2b and 2l and that between Figs. 2d and 2m.

d. A southwest vortex analyzed by the equivalent isobaric geopotential at \( \sigma = 0.825 \) and 0.735 near the Tibetan Plateau

Over the eastern flank of the Tibetan Plateau, a low pressure system called a southwest (SW) vortex (due to its origin in southwestern China) is frequently observed between the 700- and 850-hPa levels, and a schematic three-dimensional diagram for the early stage of an SW vortex is shown in Fig. 3j. The SW vortices are cyclonic and their horizontal scale is several hundred kilometers. It is not clear what is the major factor causing these low vortices at their early stage. The vortices normally form in regions with horizontally uniform temperature and away from major baroclinic zones. Some remain in this environment for several days and then decay locally. Others are affected by cold fronts arriving from the north along the eastern edge of the Tibetan Plateau. If the vortices are affected by cold fronts, they move away from their place of origin, develop further, and produce precipitation in the downstream region (Kuo et al. 1986; Ma and Bosart 1987; Wang et al. 1993). This is an important type of rain-producing system in China. It is easy to see from Fig. 3j that the early stage of an SW vortex is very difficult to identify on the isobaric analyses at the 850- and 700-hPa levels. Sometimes we can find only half of the wind circulation on these isobaric surfaces. However, the equivalent isobaric geopotential analysis is very useful for identifying the early stage of SW vortices.

We compare the equivalent isobaric geopotential analyzed at the levels \( \sigma = 0.825 \) and 0.735 with the geopotential, temperature, and wind analyses at the 850- and 700-hPa levels. The examples are the same as those studied in section 4c. The geopotential, temperature, and wind fields at the 850- and 700-hPa levels from 0000 UTC 15 April to 0000 UTC 16 April 1988 are shown in Figs. 4a–4f, respectively. These figures are analyzed from the data of ECMWF. Because the 850- and 700-hPa levels over the Tibetan Plateau are below the earth’s surface, the data at these levels are all extrapolated from the upper troposphere.

The inner part of the equivalent geopotential at \( \sigma = 0.825 \) and 0.735 levels for 0000 and 1200 UTC 15 April and 0000 UTC 16 April 1988 is presented in Figs. 5a–
where $a$. Application of the equivalent geopotential to the
near the Tibetan Plateau. It is assumed that
f, respectively. In Figs. 5a and 5b, there is a high geopotential region over the major part of the Tibetan Plateau, but a weak low value region is located over its eastern flank and the Sichuan Basin. In Figs. 5c and 5d, in addition to a low geopotential system developing over the Mongolian Plateau associated with lee cyclogenesis, there is an SW low vortex present over the eastern flank of the Tibetan Plateau and the Sichuan Basin. However, as shown in Figs. 4c and 4d, this SW low vortex is difficult to identify on the corresponding 850- and 700-hPa analyses.

In Figs. 5e and 5f, the troughs associated with the lee cyclone at $\sigma = 0.825$ and 0.735 levels are further developed, and their southern parts affect the SW low. The main parts of the Tibetan Plateau shown in Fig. 3d can also be seen in Figs. 5a and 5b. On the other hand, the secondary high over the Mongolian Plateau associated with lee cyclogenesis, in addition to a low geopotential system developing over the major part of the Tibetan Plateau shown in Fig. 3d can also be seen in Figs. 5a and 5b. In this situation, the shortwave trough often moves north along the eastern edge of the Tibetan Plateau, and it becomes the shortwave trough shown in Figs. 5e and 5f. In this situation, the shortwave trough often moves eastward and develops further. Its further development is away from the Tibetan Plateau, and it will be the same as the analyses in $p$ coordinates and is not discussed here.

The early stages of an SW low can be identified by the equivalent isobaric geopotential analysis at $\sigma = 0.825$ and 0.735 as shown more clearly by Figs. 5c and 5d. On the other hand, the secondary high over the Tibetan Plateau shown in Fig. 3d can also be seen in the analyses for the same time at $\sigma = 0.825$ as shown in Fig. 5e. This secondary high is associated with a relatively cold area in the temperature field over the Tibetan Plateau as revealed by Figs. 4e and 4f.

It is seen from the above that equivalent isobaric geopotential analysis at $\sigma = 0.825$ and 0.735 levels can identify weather systems more clearly than the conventional analyses at the 850- and 700-hPa levels over and near the Tibetan Plateau.

5. Application of the equivalent isobaric geopotential to the inner part equations of streamfunction and velocity potential

a. Application of the equivalent geopotential to the vorticity and divergence equations

It is assumed that

$$f = f_0 + f', \quad m^2 = m_0^2 + (m^2)', \quad (5.1)$$

where $f_0$ and $m_0^2$ are the averaged values of the Coriolis parameter and $m^2$, respectively, over the integration region, and $f'$ and $(m^2)'$ are their deviations, respectively. The temperature is divided into a stationary portion and its deviation, and is expressed by

$$T = T_0(x, y, \sigma) + T'(x, y, \sigma, t) = T_0(x, y, \sigma) + (T(x, y, \sigma, t) - T_0(x, y, \sigma)). \quad (5.2)$$

If the model is used for a diagnostic study or a short time integration, the stationary portion is assumed to be the initial value as

$$T_0(x, y, \sigma) = T(x, y, \sigma, t_0). \quad (5.3)$$

The vorticity and divergence equation are derived from (3.10) and written in the form

$$\frac{d\zeta}{dt} = -f_0\delta + m^2\Omega_{adv}, \quad (5.4)$$

$$\frac{d\delta}{dt} = f_0\zeta - m^2\nabla^2\phi - m^2\nabla \cdot (RT_0(x, y, \sigma)\nabla \ln p_0)$$

$$+ m^2D_{adv} - m^2\nabla^2 E, \quad (5.5)$$

where

$$\zeta = k \cdot \nabla \times \mathbf{V} = m^2\left[\frac{\partial}{\partial x}\left(\frac{v}{m}\right) - \frac{\partial}{\partial y}\left(\frac{u}{m}\right)\right],$$

$$\delta = \nabla \cdot \nabla \times \mathbf{V} = m^2\left[\frac{\partial}{\partial x}\left(\frac{u}{m}\right) + \frac{\partial}{\partial y}\left(\frac{v}{m}\right)\right],$$

$$\Omega_{adv} = -\left(\frac{u}{m} \frac{\partial f^0 + \zeta}{\partial x} + \frac{v}{m} \frac{\partial f^0 + \zeta}{\partial y}\right) - (f^0 + \zeta) \frac{\partial}{m^2},$$

$$+ R\left(\frac{\partial T_0(x, y, \sigma)}{\partial x} \frac{\partial \ln p_0}{\partial x} - \frac{\partial T_0(x, y, \sigma)}{\partial y} \frac{\partial \ln p_0}{\partial y}\right)$$

$$+ \frac{\partial F_1}{\partial x} - \frac{\partial F_1}{\partial y},$$

$$D_{adv} = \frac{\partial}{\partial x}\left(\frac{u}{m} \frac{\partial f^0 + \zeta}{\partial x} - \frac{u}{m} \frac{\partial f^0 + \zeta}{\partial y}\right) + \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}, \quad E = \frac{u^2 + v^2}{2}. \quad (5.6)$$

Here

$$F_u = -\alpha \frac{\partial u}{\partial \sigma} - RT'(x, y, \sigma, t) \frac{\partial \ln p_0}{\partial x} + \frac{p_0}{m},$$

$$F_v = -\alpha \frac{\partial v}{\partial \sigma} - RT'(x, y, \sigma, t) \frac{\partial \ln p_0}{\partial y} + \frac{p_0}{m}. \quad (5.8)$$

The main parts of $\Omega_{adv}$ and $D_{adv}$ on the right-hand side of (5.6) and (5.7) are advection, so that $\Omega_{adv}$ and $D_{adv}$ are briefly termed the generalized vorticity and diver-
gence advection, respectively. Using the definition similar to (3.13), (5.6) can be written as

\[
\Omega_{\text{adv}} = - \left( \frac{u \partial (f^* + \xi)}{m \partial x} + \frac{v \partial (f^* + \xi)}{m \partial y} \right) \\
- \left( f^* + \xi \right) \frac{\delta}{m^2} - \nabla^2 \eta + \frac{\partial E^*}{\partial x} - \frac{\partial E^*}{\partial y} \quad (5.6')
\]

Based on Figs. 2i and 2j, it is seen from (5.6a) that the effect of the rotational part of the horizontal pressure gradient force increases anticyclonic and cyclonic vorticity in the east and west flanks of mountains, respectively, in the middle and high latitudes of the Northern Hemisphere. Using (2.6), (5.4) and (5.5) in a limited region are rewritten as

\[
\frac{\partial \nabla^2 \psi_i}{\partial t} = - f_i \nabla^2 \chi_i + \Omega_{\text{adv}} \quad (5.9)
\]

\[
\frac{\partial \nabla^2 \chi_i}{\partial t} = f_i \nabla^2 \psi_i - \nabla^2 \phi_i - \nabla \cdot (RT_0(x, y, \sigma) \nabla \ln p_*) - \nabla^2 E_i + D_{\text{adv}}. \quad (5.10)
\]

In \( \sigma \) coordinates, the thermodynamic, continuity, and hydrostatic equations are expressed by

\[
\frac{\partial T}{\partial t} = - \nabla \cdot \nabla T - \sigma \frac{\partial T}{\partial \sigma} + \frac{RT}{C_p} \omega + \frac{1}{C_p} \left[ L(C - E) + Q_x \right] \quad (5.11)
\]

\[
\frac{\partial \ln p_*}{\partial t} = \nabla \cdot \nabla \ln p_* - \nabla \cdot \nabla - \frac{\partial \phi}{\partial \sigma} \quad (5.12)
\]

\[
\frac{\partial \phi}{\partial \sigma} = - \frac{RT}{\sigma}, \quad (5.13)
\]

where \( C \) and \( E \) are the rate of condensation and evaporation per unit mass, respectively, \( Q_x \) is the heating rate per unit mass due to the turbulent transfer and radiation.

b. The inner part equations of the streamfunction and velocity potential

Let us introduce two variables, \( \psi_{\text{adv},i} \) and \( \chi_{\text{adv},i} \), which satisfy the following equations:

\[
\nabla^2 \psi_{\text{adv},i} = \Omega_{\text{adv}}, \quad \nabla^2 \chi_{\text{adv},i} = D_{\text{adv}} \quad (5.14)
\]

with the homogeneous Dirichlet boundary values. The variables \( \psi_{\text{adv},i} \) and \( \chi_{\text{adv},i} \) are referred to as the variation rates caused primarily by advection for the inner parts of the streamfunction and velocity potential, respectively.

If the integral operator \( \nabla^{-2} \), shown in the appendix, is used for the solution of the Poisson equation with zero Dirichlet boundary value, then the solutions of (5.14) are expressed by

\[
\psi_{\text{adv},i} = \nabla^{-2} \Omega_{\text{adv}}, \quad \chi_{\text{adv},i} = \nabla^{-2} D_{\text{adv}}. \quad (5.15)
\]

For a synoptic analysis or a diagnostic study, \( T' \) vanishes in (5.2) because only the time at \( t_0 \) is considered. Thus, the inner part of the equivalent isobaric geopotential height (3.5) can be further rewritten in the form

\[
\phi_{\text{adv}} = \phi_i + \nabla^{-2} \left[ \frac{\partial}{\partial x} \left( RT_0(x, y, \sigma) \frac{\partial \ln p_*}{\partial x} \right) \right. \\
\left. + \frac{\partial}{\partial y} \left( RT_0(x, y, \sigma) \frac{\partial \ln p_*}{\partial y} \right) \right]. \quad (5.16)
\]

where \( T_0(x, y, \sigma) \) satisfies (5.3).

In a general dynamical study, such as short-range weather prediction, \( T' \) does not vanish. Equation (5.16) can be considered a new definition for the inner part of the equivalent isobaric geopotential in the general case. Comparing (5.2) with (3.8) and (3.9), \( T'(x, y, \sigma, t) \) corresponds to \( T_0 \), the temperature caused by the transient synoptic disturbances. The stationary portion, \( T_0(x, y, \sigma) \), corresponds to the sum of \( T_0(\sigma) \), \( T_{\text{st}} \), and \( T_1 \). As mentioned in section 3, the temperature caused by topography, \( T_{\text{top}} \), is a major part of \( T(x, y, \sigma, t) \) over high mountain regions, and it makes an important contribution to \( \phi_{\text{adv}} \). Many spectral models and CBB use the generalized geopotential derived based on \( T_0(\sigma) \) only. Over high mountain regions, the equivalent geopotential derived from (5.16) is quite different from the generalized geopotential derived based only on \( T_0(\sigma) \).

Utilizing (5.15) and (5.16), (5.9) and (5.10) can be solved with zero Dirichlet boundary values, and their solutions become

\[
\frac{\partial \psi_i}{\partial t} + f_i \chi_i = \psi_{\text{adv},i} \quad (5.17)
\]

\[
\frac{\partial \chi_i}{\partial t} + \phi_i - f_i \psi_i = \chi_{\text{adv},i} - E_i. \quad (5.18)
\]

Equations (5.17) and (5.18) are the inner part equations of streamfunction and velocity potential, respectively.

During computations with the primitive equations in \( \sigma \) coordinates, the absolute values of the first two terms on the right-hand side of (3.10) are very large near high mountains. The horizontal pressure gradient force is only a small difference between these two large terms, and its computational errors become very large near steep slopes. A number of authors (e.g., Corby et al. 1972; Gary 1973; Sundqvist 1976; Nakamura 1978; Mesinger et al. 1988) examined how these errors can be reduced during computations with primitive equation models. If the horizontal pressure gradient force is separated into its irrotational and rotational parts, the first two terms on the right-hand side of (3.10) are replaced by the two terms on the right-hand side of (3.12). In this case, the computational problem in primitive equation models is automatically eliminated.
6. The inner part equation of the equivalent isobaric geopotential height

a. The vertical difference form of the continuity equation and hydrostatic equation

To develop the equivalent isobaric geopotential equation, the vertical difference forms of the continuity, hydrostatic, and thermodynamic equations need to be used. The vertical distribution of variables is shown in Fig. 6. Based on (2.24) of CBB, the vertical difference form of the continuity equation (5.12) can be rewritten as

$$\frac{\partial \ln p^*}{\partial t} + m^2 \Pi \mathbf{D} \downarrow = P_{\text{adv}},$$

(6.1)

where $P_{\text{adv}}$ is dominated by the surface pressure advection and it is expressed by

$$P_{\text{adv}} = -m^2 \sum_{j=1}^{N} \left( U_j \frac{\partial \ln p^*}{\partial x} + V_j \frac{\partial \ln p^*}{\partial y} \right) \Delta \sigma_j$$

$$- (m^2)’ \Pi \mathbf{D} \downarrow,$$

(6.2)

and $\mathbf{D} \downarrow$ and $\Pi$ indicate the column vector and row vector, respectively, as

$$\mathbf{D} \downarrow = (D_1, \ldots, D_k, \ldots, D_N)^T,$$

$$\Pi = (\Delta \sigma_1, \ldots, \Delta \sigma_k, \ldots, \Delta \sigma_N),$$

(6.3)

where $(\ldots)^T$ is for transpose. As shown in CBB, both the sigma vertical velocity, $\sigma_{\text{adv},k+1/2}$, and local pressure tendency, $(\partial \ln p^*/\partial t)$, can be divided into their divergent part and surface pressure advective part, and the vertical difference forms of these two parts are expressed by

$$\sigma_{\text{adv},k+1/2} = \sigma_{k+1/2} \sum_{j=1}^{N} \delta_j \Delta \sigma_j - \sum_{j=1}^{k} \delta_j \Delta \sigma_j,$$

$$\sigma_{\text{adv},k+1/2} = \sigma_{k+1/2} \sum_{j=1}^{N} V_j \cdot \nabla \ln p^* \Delta \sigma_j$$

$$- \sum_{j=1}^{k} V_j \cdot \nabla \ln p^* \Delta \sigma_j,$$

(6.4)

$$\frac{\partial \ln p^*}{\partial t} = - \sum_{j=1}^{N} \delta_j \Delta \sigma_j,$$

(6.5)

Based on (A.17) of CBB, the pressure vertical velocity is written by

$$\left( \begin{array}{c} \omega \\ \rho \end{array} \right) \downarrow = m^2 (I - \mathbf{C}) \left( U_j \frac{\partial \ln p^*}{\partial x} + V_j \frac{\partial \ln p^*}{\partial y} \right)$$

$$- m^2 \mathbf{C} \mathbf{D} \downarrow,$$

(6.6)

where $I$ is an unit matrix, and $\mathbf{C}$ is a lower-triangular matrix denoted by (A.15) of CBB.

Based on (A.28) of CBB, the finite difference form of the hydrostatic equation is expressed by

$$\phi_{\downarrow} = \phi_{\downarrow} + R B \mathbf{T} \downarrow,$$

(6.7)

where matrix $\mathbf{B}$ is an upper-triangular matrix expressed by (A.29) of CBB, and $\phi_{\downarrow} = \phi_{\downarrow} I$. Here $\phi_{\downarrow} = gH$, and $H$ is the height of the earth’s surface.

b. The vertical difference form of the thermodynamic equation

The thermodynamic equation (5.11) can be rewritten as

$$\frac{\partial T^*}{\partial t} = - m^2 \left[ \frac{u}{m} \frac{\partial T^*}{\partial x} + \frac{v}{m} \frac{\partial T^*}{\partial y} \right] - \sigma^* \frac{\partial T^*}{\partial \sigma}$$

$$+ \kappa T^* \ln p^* + P_T,$$

(6.8)

where $\kappa = R/C_p$ and $P_T = (L(C - E) + Q_T)C_p$. The stationary portion of temperature can further be separated into two parts:

$$T_0(x, y, \sigma) = T_{00}(\sigma) + T_0’(x, y, \sigma)$$

$$+ T_{00}(\sigma) + (T_0(x, y, \sigma) - T_{00}(\sigma)),$$

(6.9)

where $T_{00}(\sigma)$ is the averaged value of $T_0$ over the constant $\sigma$ surface, and $T_0’$ is its deviation.

Based on the separations (6.4) and (6.5), Eq. (6.8) can be rewritten as

$$\frac{\partial T^*}{\partial t} = - \sigma^* \frac{\partial T_{00}}{\partial \sigma} - \phi_{\text{adv}} + \kappa T_{00} \left( \frac{\partial \ln p^*}{\partial t} \right)_{\text{adv}}$$

$$+ T_{\text{adv}} + P_T,$$

(6.10)
and utilizing (6.9), we have
\[
\frac{\partial T'}{\partial t} = T_{adv,k} - m^2 \sum_{l=1}^{N} F_{i,k} D_j + P_{T,i},
\]  
where \(T_{adv,k}\) is the vertical difference form of (6.11), and the elements of matrix \(F\) are defined as
\[
F_{i,j} = \frac{\sigma_{x,i}}{2\Delta \sigma_{x}} \left[ (T_{00,i,k+1} \sigma_{x,i} - T_{00,i,k} \sigma_{x,i}^*) (\sigma_{x,i+1/2} \Delta \sigma_{x} - \epsilon_1) + (T_{00,i,k+1} \sigma_{x,i}^* - T_{00,i,k} \sigma_{x,i}^*) (\sigma_{x,i-1/2} \Delta \sigma_{x} - \epsilon_2) \right] + \kappa T_{00,i,k} \Delta \sigma_{x},
\]  
Here
\[
\epsilon_1 = \begin{cases} 
0, & \text{if } l > k, \\
\Delta \sigma_{x}, & \text{if } l \leq k,
\end{cases}
\]
\[
\epsilon_2 = \begin{cases} 
0, & \text{if } l > k - 1, \\
\Delta \sigma_{x}, & \text{if } l \leq k - 1.
\end{cases}
\]

### c. The equation of the inner part of the equivalent isobaric geopotential height

Substituting the inner part of the hydrostatic equation (6.7) into (5.16), the vertical difference form of (5.16) is expressed by
\[
\phi_{adv,i} = \phi_{adv,i} + R B T_{adv,i} + \nabla^{-1} \left[ \frac{\partial}{\partial x} \left( R T_{adv,i} \frac{\partial \ln p_{x}}{\partial x} \right) + \frac{\partial}{\partial y} \left( R T_{adv,i} \frac{\partial \ln p_{y}}{\partial y} \right) \right].
\]  
Taking the partial derivative of (6.15) with respect to \(t\), and utilizing (6.9), we have
\[
\frac{\partial \phi_{adv,i}}{\partial t} = \frac{\partial \phi_{adv,i} + R B T_{adv,i}}{\partial t} + R T_{adv,i} \frac{\partial \ln p_{x}}{\partial t} + R T_{adv,i} \frac{\partial \ln p_{y}}{\partial t}
\]  
Utilizing (2.18), the inner part equations of (6.1) and (6.12) can be derived. Substituting (6.1) and the derived inner part equations of (6.1) and (6.12) into (6.16), then (6.16) becomes
\[
\frac{\partial \phi_{adv,i}}{\partial t} + m_i \boldsymbol{A} D_j = \Phi_{adv,i} = \Phi_{adv,i} + m_i \boldsymbol{A} D_j.
\]  
where matrix \(\boldsymbol{A}\) is
\[
\boldsymbol{A} = R (B \mathbf{F} + T_{o0}) \Pi
\]
and \(\Phi_{adv,i} = \Phi_{adv,i} + m_i \mathbf{A} D_j\).

Equations (5.17), (5.18), and (6.21) are the basic equations for application of the equivalent isobaric geopotential to numerical prediction, initialization, and other dynamic studies. As a simple example, these equations are used to derive a generalized \(\omega\) equation in \(\sigma\) coordinates presented in the next section.

### 7. A velocity potential form of the generalized \(\omega\) equation in the balanced ageostrophic approximation and its solution

#### a. An ageostrophic geopotential height in \(\sigma\) coordinates

Utilizing the wind partitioning into the streamfunction and velocity potential, the inner part of the equivalent isobaric geopotential can be separated into a geostrophic part and an ageostrophic part as
\[
\phi_{i} = \phi_{i,g} + \phi_{i,a}.
\]  
where the geostrophic part based on (3.17) is defined by
\[
\phi_{i,g} = f_0 \psi_i,
\]  
and the ageostrophic part is defined by the difference between the equivalent geopotential and its geostrophic part, and expressed by
\[ \phi_{\text{geo}} = \phi_u - f_0 \psi, \] (7.3)

In definitions (7.2) and (7.3), \( f_0 \) is not present in the denominator. In the equatorial region where \( f_0 \) approaches zero, the geostrophic part becomes zero, that is, \( \phi_{\text{geo}} = 0 \); and the ageostrophic part (7.3) becomes the equivalent geopotential, that is, \( \phi_{\text{ageo}} = \phi_u \). Thus, definitions (7.2) and (7.3) are different from the traditional definitions of the geostrophic and ageostrophic winds, and they can be used all over the globe including the equatorial region.

The vertical difference form of the inner part equations for streamfunction and velocity potential (5.17) and (5.18) can be rewritten, respectively, as

\[ \frac{\partial \psi}{\partial t} + f_0 \chi = \psi_{\text{adv}}, \] (7.4)

\[ \frac{\partial \chi}{\partial t} + \phi_{\text{ageo}} = \chi_{\text{adv}} - E. \] (7.5)

b. The equation of the inner part of the ageostrophic geopotential height

From (7.4) and (6.21), the equation of the inner part of the ageostrophic geopotential can be derived and it is denoted by

\[ \frac{\partial \psi_{\text{ageo}}}{\partial t} + m^2 A \nabla^2 \chi - f_0 \chi = \Phi_{\text{hadiu}} + m^2 A D_y, \] (7.6)

where

\[ \Phi_{\text{hadiu}} = \Phi_{\text{hadiu}} - f_0 \psi_{\text{adv}} \] (7.7)

is referred to as the variation rate of the ageostrophic geopotential caused by advection and heating.

Equations (7.4), (7.5), and (7.6) are the basic equations in a limited region for three variables: the inner parts of the streamfunction, velocity potential, and ageostrophic geopotential.

c. A velocity potential form of the generalized \( \omega \) equation in \( \sigma \) coordinates with the balanced ageostrophic approximation

Recently, an approximation where the tendencies of the ageostrophic and geopotential and velocity potential in (7.6) and (7.5) vanish has been used in a initialization by Chen et al. (1996) and referred to as the balanced ageostrophic approximation. They proved that the balanced ageostrophic approximation is much more accurate than the quasigeostrophic approximation.

If the tendency of the ageostrophic geopotential is neglected in (7.6), (7.6) becomes

\[ m^2 A \nabla^2 \chi - f_0 \chi = \Phi_{\text{hadiu}} + m^2 A D_y. \] (7.8)

Equation (7.8) is a velocity potential form of the generalized \( \omega \) equation in \( \sigma \) coordinates with the balanced ageostrophic approximation. The advection terms computed by the ageostrophic wind in (7.8) are the same as those in the generalized \( \omega \) equation in \( p \) coordinates (Pauley and Nieman 1992), but the effect of orography on the vertical motion is better described by (7.8).

If the variation rate of the ageostrophic geopotential caused by advection and diabatic heating is computed from the geostrophic wind, and it is expressed by \( \Phi_{\text{hadiu}} \), then (7.8) becomes

\[ m^2 A \nabla^2 \chi - f_0 \chi = \Phi_{\text{hadiu}} + m^2 A D_y. \] (7.9)

Equation (7.9) is a velocity potential form of the quasigeostrophic \( \omega \) equation in \( \sigma \) coordinates.

d. The solution of the velocity potential form of the generalized \( \omega \) equation

Equation (7.8) can easily be solved by transforming it into vertical mode space. For this purpose, we introduce a matrix, \( E \), in order that the following relation is satisfied

\[ E^{-1} A E = G = \text{diag}(gh_1, gh_2, \ldots, gh_N). \] (7.10)

The matrix \( E \) is derived from the eigenvectors of the matrix \( A \), and these eigenvectors are denoted by \( E_j \), \( j = 1, \ldots, N \). The matrix \( E \) is an eigenvector matrix with each column representing an eigenvector \( E_j \). The matrix \( E^{-1} \) is the inverse of the matrix \( E \). The matrix \( G \) is a diagonal matrix with the diagonal elements given by the \( N \) eigenvalues, \( (gh_1, gh_2, \ldots, gh_N) \), of the matrix \( A \).

The transformation of velocity potential between the vertical mode and physical space is expressed by

\[ \chi_{\theta} = E^{-1} \chi \quad \text{and} \quad \chi = E \chi_{\theta}. \] (7.11)

If Eq. (7.8) is multiplied from left by the matrix \( E^{-1} \), then (7.8) becomes

\[ m^2 G \nabla^2 \chi_{\theta} - f_0 \chi_{\theta} = \Phi_{\text{hadiu}} + m^2 G D_y, \] (7.12)

where

\[ \Phi_{\text{hadiu}} = E^{-1} \Phi_{\text{hadiu}}, \quad D_y = E^{-1} D_y. \] (7.13)

Equation (7.12) can be rewritten as

\[ \nabla^2 \chi_{\theta} - \frac{1}{L_{\theta}^2} \chi_{\theta} = \frac{1}{C_i^2} (\Phi_{\text{hadiu}} + C_i^2 D_y), \] (7.14)

where

\[ C_i = m_i \sqrt{gh_i}, \quad \text{and} \quad L_{\theta}^2 = \frac{m_i^2 gh_i}{f_0^2} \left( \frac{C_i}{f_0} \right)^2 \] (7.15)

are the gravity wave phase speed and radius of deformation of the \( k \)th vertical mode, respectively.

Because the lateral boundary value of the inner part of the velocity potential is homogeneous, the solution
of the Helmholtz equation (7.14) can be derived from the double sine series. Based on (A.2) and (A.3) in the appendix, the solution of (7.14) is written in the form

$$X_{v_{i}} = - F \left[ \frac{1}{C_{i}^{2}} \Phi_{v_{i}} + C_{i}^{2} D_{v_{i}} \right] \left( \frac{1}{L_{mn}^{2}} + \frac{1}{L_{i}^{2}} \right)^{-1}.$$  

(7.16)

After the vertical mode of the velocity potential, $X_{v_{i}}$, is derived, we have

$$X_{v_{i}} = F^{-1}[X_{v_{i}0},]_{i}, \quad X_{v_{i}} = E_{i} X_{v_{i}}.$$  

(7.17)

Based on (6.6), the pressure vertical velocity is determined by

$$\left( \frac{\omega}{p} \right)_{v} = m^{2}(1 - C) \left( U_{v} \frac{\partial \ln p_{v}}{\partial x} + V_{v} \frac{\partial \ln p_{v}}{\partial y} \right)$$

$$- m^{2} \nabla^{2} X_{v}.$$  

(7.18)

The horizontal divergence in (7.18) can also be computed directly from the observed wind, and it is referred to as a kinematic method. However, the horizontal divergence derived from the kinematic method is sensitive to small errors in the observed wind. The divergence derived from (7.16) is determined from terms of similar magnitude and is less sensitive to observational errors than the kinematic method; it is referred to as an $\omega$-equation method. Because the $\omega$-equation method involves computations of higher-order derivatives than the kinematic method, this diminishes the advantages if the computation of derivatives is not accurate enough. Thus, in order to reduce computational errors, a harmonic-sine special method (Chen and Kuo 1992a) is used here.

8. Some test results of the annual precipitation over Greenland

The computation procedure for precipitation based on computed vertical motion is given by Chen et al. (1997b). Only large-scale condensation is considered in the precipitation. The computation procedure for the large-scale condensation and precipitation is similar to that discussed by Arakawa and Lamb (1977). In this computation procedure, the temperature variation in a time step is computed from horizontal and vertical advection and adiabatic variation based on the thermodynamic equation (6.11) and (6.12). The specific humidity variation in a time step is deduced from the continuity equation for specific humidity. In these equations, the sigma vertical velocity, $\sigma_{v_{i}1/2}$, is computed from (6.4), in which the horizontal divergence is obtained from solution (7.16). The precipitation rate is only computed for one time step, which is 30 min, and then it is applied to a 12-h period. The precipitation is calculated twice per day based on the analyzed data at 0000 and 1200 UTC from ECMWF. The annual precipitation is derived by summing precipitation day-by-day for one year. The mesh size of the limited region shown in Fig. 1e is 111 x 81 and the grid spacing is 50 km. In the vertical, the 16 sigma levels are the same as used in section 4.

Of course, the above method for retrieving precipitation is not rigorous, but it is very simple in comparison to a limited-area FDDA system. As we only want to determine some basic features of the annual precipitation over Greenland, this simple method is used to see whether or not precipitation is reasonably retrieved if the equivalent isobaric geopotential height is used in the generalized $\omega$ equation.

The retrieved precipitation distributions for 1987 and 1988 are shown in Figs. 7a and 7b, respectively. To determine whether the computed annual precipitation distribution is reasonable, we compare it with the observed mean annual accumulation over Greenland. The large area of very low precipitation (less than 20 cm yr$^{-1}$), which dominates the central interior region of Greenland, is also shown in Figs. 7a and 7b. Thus, the retrieved precipitation captures the main features. A secondary band of relatively high precipitation is present just along the western edge of Greenland. A large area of very low precipitation (less than 20 cm yr$^{-1}$), which dominates the central interior region of Greenland, is also shown in Figs. 7a and 7b. Thus, the retrieved precipitation captures the main features. A secondary band of relatively high precipitation is present just along the western edge of Greenland. Thus, this method also needs to be further enhanced.

The annual precipitation over Greenland simulated by the NCEP–NCAR Reanalysis for 1987 and 1988 is shown in Figs. 7c and 7d, respectively. It can be seen that the annual precipitation pattern over Greenland for an individual year shown in Figs. 7c or 7d is very similar to that of the mean annual precipitation for 1982–94 in Fig. 1c. A wave train pattern extending southeastward, caused by the steep slopes on the southeast coast of Greenland, is also found in the annual precipitation in Figs. 7c and 7d for 1987 and 1988, but does not occur in Figs. 7a and 7b.

Comparing Figs. 7c and 7d with Figs. 1d and 7a and 7b, it can be seen that the precipitation amounts from the reanalysis are about twice those observed in Fig. 1d and those retrieved in Figs. 7a and 7b. There are high precipitation areas in the central region of Greenland with maxima of 111 and 101 cm yr$^{-1}$ in Figs. 7c and 7d, respectively. They are much larger than the observed amount and the retrieved values in Figs. 7a and 7b. The shortcomings in the mean precipitation reanalysis over Greenland shown by Fig. 1c are also present in Figs. 7c and 7d.

It is seen from the above that the precipitation distributions over Greenland retrieved for 1987 and 1988 by the above simple method are superior to those obtained by NCEP–NCAR Reanalysis. This simple method has been used to retrieve precipitation over Green-
land with reasonable accuracy, and the retrieved results have been used to study the basic features of the precipitation over Greenland (Chen et al. 1997b).

9. Conclusions

Based on the studies shown in the above sections, the following conclusions can be reached.

1) A variable $\phi(x, y, \sigma, t)$ whose horizontal gradient $-\nabla \phi$ on the constant $\sigma$ surface is equal to the irrotational part of the horizontal pressure gradient force in $\sigma$ coordinates is referred to as the equivalent isobaric geopotential height. It plays the same role in $\sigma$ coordinates as the geopotential $\phi(x, y, p, t)$ does in $p$ coordinates. The analytic expression for the inner part of the equivalent isobaric geopotential is given by (3.5), and it is also shown by (5.16) if $T_0(x, y, \sigma)$ is equal to the initial value. The horizontal pressure gradient force in $\sigma$ coordinates can be separated into its irrotational and rotational parts, but the magnitude of the rotational part is one order smaller than that of the irrotational part even over high mountain regions.

2) To enable the streamfunction and velocity potential in a limited region to be used in the same way as the streamfunction and velocity potential on the globe and to avoid nonuniqueness, it is necessary to use the inner parts of the streamfunction and velocity potential instead of the streamfunction and velocity potential themselves.
A horizontal vector in a limited region can be separated into its internal nondivergent, internal irrotational, and external components as shown by (2.12). The inner part analysis uses the inner parts of the streamfunction and velocity potential and other inner part variables to describe motion systems in a limited region.

3) It is very difficult to use single elevation or pressure surface to track systems across a mountain region at “ground” level. Equivalent isobaric geopotential analysis on the constant $\sigma$ surface can be used to solve this problem. For example, if the equivalent isobaric geopotential is used at $\sigma = 0.995$ over Greenland as shown in Figs. 2a–d, the small but strong high pressure systems caused by pressure reduction to sea level using lower temperatures at the top of the Greenland Ice Sheet are removed and the circulation becomes very smooth. The geostrophic relation (3.17) between the equivalent geopotential and streamfunction on the constant $\sigma$ surface is approximately satisfied at $\sigma = 0.995$.

4) Because weather systems are very difficult to track on synoptic analyses at sea level and at the 850- and 700-hPa levels over the Tibetan Plateau, equivalent isobaric geopotential analysis in $\sigma$ coordinates is especially useful over this area. An example of equivalent geopotential analysis at $\sigma = 0.995$ (Fig. 3d) shows that cold air separated from a major anticyclone to form a secondary high over the Tibetan Plateau at 0000 UTC 16 April 1988, but this secondary high is hard to see in the SLP analysis (Fig. 3b). It is also found that the early stage of an SW low (or vortex) is more clearly identified by equivalent isobaric geopotential analysis at $\sigma = 0.825$ and 0.735 (Figs. 5c and d) than by conventional analyses at the 850- and 700-hPa levels over and near the Tibetan Plateau.

5) The equivalent isobaric geopotential height in $\sigma$ coordinates can be used not only in synoptic analysis, but also in a dynamic study. Use of the equivalent isobaric geopotential in the vorticity and divergence equations and the corresponding inner part equations of the equivalent isobaric geopotential is analyzed at $\sigma = 0.995$, as shown in Figs. 3a–d.

6) The vertical motion over some high mountain regions, such as Greenland, can be computed by this $\omega$ equation method. Some test results of precipitation over Greenland retrieved by this $\omega$ equation method show that the precipitation distributions for 1987 and 1988 are in good agreement with the observed annual accumulation distribution over the Greenland Ice Sheet, and they are superior to the precipitation distribution derived from the NCEP–NCAR Reanalysis. The computed precipitation also shows that a secondary band of relatively high precipitation is present along the western coast of Greenland. The major shortcoming in the simulation by Bromwich et al. (1993) is corrected. Thus, the $\omega$-equation method in $\sigma$ coordinates is very useful for retrieving the basic features of precipitation over Greenland.

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APPENDIX

An Integral Operator $\nabla^{-2}$ of the Poisson Equation with Zero Dirichlet Boundary Value

The inner part of the streamfunction satisfies the Poisson equation

$$\nabla^2 \psi = \Omega$$  (A.1)

within the rectangular region $[0 \leq x \leq L_x, 0 \leq y \leq L_y]$ with zero Dirichlet boundary value.

Let $F$ denote a finite Fourier sine transform operator in the two-dimensional rectangular region; that is,

$$F[\Omega(x, y)] = \Omega_{mn}$$

$$= \frac{4}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \Omega(x, y) \sin \frac{m \pi x}{L_x} \sin \frac{n \pi y}{L_y} dx dy.$$  (A.2)

where $\Omega(x, y)$ can be an arbitrary function defined in the rectangular region. The double Fourier sine series of $f_i$ are expressed by

$$f_i(x, y) = F^{-1}[F_{i,m}]$$

$$= \sum_{m=1}^{M} \sum_{n=1}^{N} F_{l,m} \sin \frac{m \pi x}{L_x} \sin \frac{n \pi y}{L_y},$$  (A.3)

where $F^{-1}$ is the inverse Fourier sine transform operator from wave to physical space, $f_i(x, y)$ is a function in
the rectangular region with zero boundary value, and \( F_{l,mn} \) is the Fourier transform of \( f \).

The solution of (A.1) in wave space is expressed by

\[
\Psi_{l,mn} = -\mathbf{\Omega}_{mn} \left( \frac{1}{L_{mn}^2} \right)^{-1}, \quad (A.4)
\]

where \( L_{mn} \) is referred to as the horizontal scale of a wavenumber \((m, n)\), and it is expressed by

\[
\frac{1}{L_{mn}^2} = \frac{m^2 \pi^2}{L_x^2} + \frac{n^2 \pi^2}{L_y^2}, \quad (A.5)
\]

and \( \Psi_{l,mn} \) is the Fourier transform of \( \psi \). Then, the solution of (A.1) in physical space is denoted by

\[
\psi_l(x, y) = -F^{-1} \left[ \Omega_{mn} \left( \frac{1}{L_{mn}^2} \right)^{-1} \right]. \quad (A.6)
\]

The expression (A.6) can also be written in another form

\[
\psi_l = \nabla^{-2} \Omega, \quad (A.7)
\]

thus,

\[
\nabla^{-2} \Omega = -F^{-1} \left[ \Omega_{mn} \left( \frac{1}{L_{mn}^2} \right)^{-1} \right], \quad (A.8)
\]

where \( \nabla^{-2} \) is referred to as an integral operator of the Poisson equation with zero Dirichlet boundary value.

REFERENCES


Pauley, P. M., and S. J. Nieman, 1992: A comparison of quasigeo-