Conservation of Potential Vorticity on Lorenz Grids

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ABSTRACT

The quasigeostrophic equations formulated using the Charney–Phillips vertical staggering of variables are well known to possess an analog of the form of conservation of potential vorticity. It is shown that a similar analog is enjoyed by the quasigeostrophic equations formulated using the modified Lorenz staggering of variables.

1. Introduction

Arakawa and Moorthi (1988, referred to as AM hereafter) demonstrated that the quasigeostrophic equations formulated using the ''Charney–Phillips'' vertical staggering of variables possess an analog of conservation of potential vorticity. They also derived a potential vorticity equation formulated using the Lorenz staggering of variables but concluded that rather crude approximations would be needed to derive a similar analog of potential vorticity conservation.

Figure 1a illustrates the Charney–Phillips arrangement of model variables (Charney and Phillips 1953) and Fig. 1b the arrangement introduced by Lorenz (1960), appropriate for a perfect gas when the pressure, p, is chosen for the vertical coordinate. The level number, k, is taken to increase with p, and hence decrease with height z, and the total number of full levels within the domain is denoted by K. In both grids the horizontal velocities, u, v, are stored at the ''full'' model levels indexed by integer level numbers and the material time derivative of the pressure, ω, is stored at the intermediate ''half'' levels. Horizontal boundaries, on which $\omega = 0$, are taken to lie at the half levels $k = ½$ and $k = K + ½$. On the Charney–Phillips grid the potential temperature, θ, is stored on the half levels while on the Lorenz grid it is stored at the full levels.

On the original Lorenz grid, the geopotential, $\phi \equiv gz$, and the streamfunction of the geostrophic flow, $\psi = \phi/f_0$, are stored with the density, ρ, on the full levels; f_0 is the value of the Coriolis parameter, f, at a reference latitude. For the hydrostatic relation, $\partial\phi/\partial p = -1/\rho$, it seems more natural to store ϕ (and ψ) on the half levels as in Fig. 1c. This arrangement of variables is usually referred to as the modified Lorenz grid. In this paper a set of equations will be said to use the modified Lorenz grid when the streamfunction at the kth full level, ψ_k, is given by a linear combination of the streamfunction at the adjacent half levels, $\psi_{k+1/2}$ and $\psi_{k-1/2}$:

$$\psi_k \equiv \alpha_k\psi_{k+1/2} + (1 - \alpha_k)\psi_{k-1/2}, \tag{1}$$

with the parameter α_k depending only on the vertical coordinate. [In (1) and all other equations the index k denotes an integer.]

Tokioka (1978) proposed a formulation using (1) and the formulation of Simmons and Burridge (1981) [on which the European Centre for Medium-Range Weather Forecasts (ECMWF) atmospheric forecasting system has been built] uses the modified Lorenz grid to discretize the primitive equations. Arakawa and Suarez (1983) derived a family of schemes for the primitive equations that use the modified Lorenz grid. By contrast, the original formulations of Lorenz (1960) for balanced models and Arakawa and Lamb (1977) for the primitive equations specified the streamfunction at the half levels as a linear combination of those at adjoining full levels.

This paper derives analogs of the conservation of potential vorticity for quasigeostrophic equations discretized using the modified Lorenz grid, similar to those enjoyed by the Charney–Phillips grid.

The quasigeostrophic and the primitive equations for a perfect gas can be written in a number of vertical coordinate systems. Pressure coordinates have been chosen for the main derivation here to aid comparison with AM. An analogous derivation in height coordinates is outlined in section 3. Section 4 provides a summary.

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	(a)	(b)	(c)
k	ϕ, ψ, u, v	ϕ, ψ, ρ, u, v	ρ, u, v
$k + 1/2$	ρ, ω	ω	ϕ, ψ, ω

(p ↓)

FIG. 1. Distribution in the vertical of the basic model variables for a perfect gas using p coordinates: (a) the Charney–Phillips grid, (b) the original Lorenz grid, and (c) the modified Lorenz grid.

2. Derivation of conservation of potential vorticity in pressure coordinates

a. Statement of the quasigeostrophic form of the equations

Following AM, the quasigeostrophic equations that will be discretized are

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \boldsymbol{\nabla}_h\right)(\nabla_h^2 \psi + f) - f_0 \frac{\partial \omega}{\partial p} = 0 \quad \text{and} \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \boldsymbol{\nabla}_h\right)\frac{\partial \psi}{\partial p} + \frac{\omega}{f_0} S = 0. \tag{3}$$

Here $\partial/\partial t$, $\boldsymbol{\nabla}_h$, and ∇_h^2 are, respectively, the time derivative, horizontal gradient, and horizontal Laplacian operators, evaluated at constant pressure; $\mathbf{v}_g$ is the geostrophic velocity; S is a static stability parameter defined by

$$S \equiv -\frac{1}{\overline{\rho}\,\overline{\theta}} \frac{d\overline{\theta}}{dp}; \tag{4}$$

and $\overline{\rho}$ and $\overline{\theta}$ are the density and potential temperatures fields of the basic stratified state, which depends only on p. The rate of change with time of any variable, φ, following the geostrophic flow $\mathbf{v}_g$ at level k, will be variously written as

$$\frac{D_g \varphi}{Dt} \equiv \frac{\partial \varphi}{\partial t} + (\mathbf{v}_g \cdot \boldsymbol{\nabla})\varphi = \frac{\partial \varphi}{\partial t} + J(\psi, \varphi). \tag{5}$$

In the last of these expressions, $J(\psi, \varphi)$ denotes the Jacobian operator, which in Cartesian horizontal coordinates x, y is given by

$$J(\psi, \varphi) = \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \varphi}{\partial x}. \tag{6}$$

The main property of the Jacobian that will be used is that $J(\psi, \varphi) = -J(\varphi, \psi)$, which implies that $J(\varphi, \varphi) = 0$.

Centering (2) and (3) at full levels, their simplest discretizations (in the vertical) are

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{gk} \cdot \boldsymbol{\nabla}_h\right)(\nabla_h^2 \psi_k + f) - f_0\left(\frac{\omega_{k+1/2} - \omega_{k-1/2}}{\Delta p_k}\right) = 0 \quad \text{and} \tag{7}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{gk} \cdot \boldsymbol{\nabla}_h\right)\left(\frac{\psi_{k+1/2} - \psi_{k-1/2}}{\Delta p_k}\right) + \frac{1}{2 f_0}(\omega_{k+1/2} S^b_{k+1/2} + \omega_{k-1/2} S^c_{k-1/2}) = 0. \tag{8}$$

The superscripted variables S^b and S^c have been introduced in (8) to cover two cases. Arakawa and Moorthi (1988) derive (8) from the flux form of the thermodynamic equation, in which case $S^b_{k+1/2}$ is related to $S_{k+1/2}$ and $S^c_{k-1/2}$ to $S_{k-1/2}$. Setting $S^b_{k+1/2} = S^c_{k-1/2} = S_k$ is a simpler alternative.

b. Derivation of the conservation of potential vorticity

The potential vorticity equation is formed from the combination of (7) and (8) for levels k and $k + 1$ which eliminates $\omega_{k-1/2}$, $\omega_{k+1/2}$, and $\omega_{k+3/2}$. This combination gives

$$a_k \frac{D_{gk}}{Dt}(\nabla_h^2 \psi_k + f) + b_k \frac{D_{gk+1}}{Dt}(\nabla_h^2 \psi_{k+1} + f) + c_k \frac{D_{gk}}{Dt}\left(\frac{\psi_{k+1/2} - \psi_{k-1/2}}{\Delta p_k}\right) + d_k \frac{D_{gk+1}}{Dt}\left(\frac{\psi_{k+3/2} - \psi_{k-1/2}}{\Delta p_{k+1}}\right) = 0, \tag{9}$$

where

$$a_k = \frac{S^c_{k-1/2}}{S^b_{k+1/2} + S^c_{k-1/2}}; \qquad b_k = \frac{\Delta p_{k+1}}{\Delta p_k}\frac{S^b_{k+3/2}}{S^b_{k+3/2} + S^c_{k+1/2}};$$

$$c_k = \frac{-2 f_0^2}{S^b_{k+1/2} + S^c_{k-1/2}}\frac{1}{\Delta p_k};$$

$$d_k = \frac{2 f_0^2}{S^b_{k+3/2} + S^c_{k+1/2}}\frac{1}{\Delta p_k} \quad \text{for } k = 2, \ldots, K-2. \tag{10}$$

Note that when $S^b_{k+1/2} = S^c_{k-1/2} = S_k$, $a_k = ½$ and $b_k = \Delta p_{k+1}/(2\Delta p_k)$. Defining

$$R_{k+1/2} \equiv c_k\left(\frac{\psi_{k+1/2} - \psi_{k-1/2}}{\Delta p_k}\right) + d_k\left(\frac{\psi_{k+3/2} - \psi_{k+1/2}}{\Delta p_{k+1}}\right) \tag{11}$$

by simple substitution of (10) in (11) one obtains

$$R_{k+1/2} \equiv \frac{2 f_0^2}{\Delta p_k}\left\{\left[\frac{\psi_{k+3/2} - \psi_{k+1/2}}{(S^b_{k+3/2} + S^c_{k+1/2})\Delta p_{k+1}}\right] - \left[\frac{\psi_{k+1/2} - \psi_{k-1/2}}{(S^b_{k+1/2} + S^c_{k-1/2})\Delta p_k}\right]\right\} \quad \text{for } k = 2, \ldots, K-2. \tag{12}$$

This is an accurate representation of the partial derivative form of the stretching component of the potential vorticity, $f_0^2\partial/\partial p[(1/S)(\partial\psi/\partial p)]$, evaluated at level $k + ½$.

The main point of this note is that the advection of the vortex stretching terms in the potential vorticity equation can be reexpressed as the advection of $R_{k+1/2}$ by the geostrophic velocity at level $k + ½$. Using (1) and the properties of the Jacobian

$$\begin{aligned}J(\psi_k, \psi_{k+1/2} - \psi_{k-1/2}) &= J[\alpha_k\psi_{k+1/2} + (1 - \alpha_k)\psi_{k-1/2}, \psi_{k+1/2} - \psi_{k-1/2}] \\ &= \alpha_k J(\psi_{k+1/2}, -\psi_{k-1/2}) + (1 - \alpha_k)J(\psi_{k-1/2}, \psi_{k+1/2}) \\ &= -J(\psi_{k+1/2}, \psi_{k-1/2}). \end{aligned} \tag{13}$$

Similarly,

$$J(\psi_{k+1}, \psi_{k+3/2} - \psi_{k+1/2}) = J(\psi_{k+1/2}, \psi_{k+3/2}). \tag{14}$$

Using (13) and (14) in (9) and then the definition (11), one can show that

$$a_k\frac{D_{gk}}{Dt}(\nabla_h^2\psi_k + f) + b_k\frac{D_{gk+1}}{Dt}(\nabla_h^2\psi_{k+1} + f) + \frac{D_{gk+1/2}}{Dt}R_{k+1/2} = 0. \tag{15}$$

This relationship, describing the conservation of potential vorticity, is the main one derived in this paper. At this point it has been derived only for levels $k = 2, \ldots K - 2$. Note that no approximations have been made in deriving (15) from (7) and (8) and that (15) applies whatever choices are made for α_k in (1) and for variables S^b and S^c in (8). The relationship is not quite as desirable as the corresponding one obtained by AM for the Charney–Phillips grid because the vorticities are advected at their own levels (k and $k + 1$) rather than by a common velocity at level $k + ½$. As AM remark, the potential vorticity on the Lorenz grid is defined at the ''half'' levels.

c. Derivation of equations near the boundaries

At the upper boundary $\omega_{1/2} = 0$, so from (7) and (8)

$$\frac{D_{g1}}{Dt}(\nabla_h^2\psi_1 + f) = \frac{f_0\omega_{3/2}}{\Delta p_1} \quad \text{and} \tag{16}$$

$$\frac{D_{g1}}{Dt}\left(\frac{\psi_{3/2} - \psi_{1/2}}{\Delta p_1}\right) = -\frac{\omega_{3/2}S_{3/2}^b}{2f_0}. \tag{17}$$

Eliminating $\omega_{3/2}$,

$$\frac{D_1}{Dt}\left[\nabla_h^2\psi_1 + f + \frac{2f_0^2}{S_{3/2}^b}\left(\frac{\psi_{3/2} - \psi_{1/2}}{\Delta p_1^2}\right)\right] = 0, \tag{18}$$

which can be interpreted as expressing the conservation of the generalized form of potential vorticity at level 1 (AM, section 4).

Conservation relations centered at level 3/2 may be obtained by eliminating $\omega_{5/2}$ from (7) and (8) evaluated with $k = 2$,

$$-S_{5/2}^b\Delta p_2\frac{D_{g2}}{Dt}(\nabla_h^2\psi_2 + f) = (S_{3/2}^a + S_{5/2}^b)f_0\omega_{3/2} + 2f_0^2\frac{D_{g2}}{Dt}\left(\frac{\psi_{5/2} - \psi_{3/2}}{\Delta p_2}\right), \tag{19}$$

and using (16) and/or (17) in (19) to substitute for $\omega_{3/2}$. Arakawa and Moorthi (1988) used (17). An alternative is to use equal contributions from (16) and (17), which gives (9) for $k = 1$ with

$$a_1 = \frac{1}{2}, \qquad b_1 = \frac{\Delta p_2}{\Delta p_1}\frac{S_{5/2}^b}{S_{5/2}^b + S_{3/2}^c},$$

$$c_1 = \frac{-f_0^2}{S_{3/2}^b}\frac{1}{\Delta p_1}, \qquad d_1 = \frac{2f_0^2}{S_{5/2}^b + S_{3/2}^c}\frac{1}{\Delta p_1}. \tag{20}$$

For the choice $S_{k+1/2}^b = S_{k-1/2}^c = S_k$, the coefficients agree with (10) evaluated with $k = 1$ and the expression for $R_{k+1/2}$, (12), also holds for $k = 1$. The derivation of (15) for $k = 1$ can be repeated regardless of the choices of $S_{k+1/2}^b$ and $S_{k-1/2}^c$.

Expressions corresponding to (18) and (20) may be derived for $k = K - 1$.

3. Derivation in height coordinates

White (2002, section 10.2) provides a set of quasi-geostrophic equations in z coordinates for a perfect, hydrostatic gas with which the above derivations can be repeated. His set reduces to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_h\right)(\nabla_h^2\psi + f) - \frac{f_0}{\overline{\rho}}\frac{\partial}{\partial z}(\overline{\rho}\hat{w}) = 0 \quad \text{and} \tag{21}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_h\right)\frac{\partial\psi}{\partial z} + \frac{N^2}{f_0}\hat{w} = 0. \tag{22}$$

Here $\hat{w}$ is an extended vertical velocity related to the standard vertical velocity, w, by

$$\hat{w} = w - \frac{f_0}{g}\frac{\partial\psi}{\partial t}; \tag{23}$$

$N^2 \equiv (g/\overline{\theta})(d\overline{\theta}/dz)$ is the Brunt–Väisälä frequency; g is the local acceleration by gravity and all horizontal derivatives are evaluated at constant height. Choosing the most straightforward discretization with level number increasing with height, one obtains

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_{gk}\cdot\nabla_h\right)(\nabla_h^2\psi_k + f) - \frac{f_0}{\overline{\rho}_k}\left(\frac{\overline{\rho}_{k+1/2}\hat{w}_{k+1/2} - \overline{\rho}_{k-1/2}\hat{w}_{k-1/2}}{\Delta z_k}\right) = 0 \quad \text{and} \tag{24}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_h\right)\left(\frac{\psi_{k+1/2} - \psi_{k-1/2}}{\Delta z_k}\right) + \frac{N_k^2}{2f_0}(\hat{w}_{k+1/2} + \hat{w}_{k-1/2}) = 0, \tag{25}$$

which are similar but not quite identical in form to (7) and (8). Repeating the derivations (without making approximations) one reobtains (15) for $k = 2, \ldots, K - 2$ with

$$a_k = \frac{\overline{\rho}_k}{\overline{\rho}_{k+1/2} + \overline{\rho}_{k-1/2}}; \quad b_k = \frac{\Delta z_{k+1}}{\Delta z_k}\frac{\overline{\rho}_{k+1}}{\overline{\rho}_{k+3/2} + \overline{\rho}_{k+1/2}};$$

$$\text{for } k = 2, \ldots, K - 2 \quad \text{and} \tag{26}$$

$$R_{k+1/2} = \frac{f_0^2}{\overline{\rho}_{k+1/2}\Delta z_k}\left[\left(\frac{2\overline{\rho}_{k+3/2}\overline{\rho}_{k+1/2}}{\overline{\rho}_{k+3/2} + \overline{\rho}_{k+1/2}}\right)\left(\frac{\psi_{k+3/2} - \psi_{k+1/2}}{N_{k+1}^2\Delta z_{k+1}}\right) - \left(\frac{2\overline{\rho}_{k+1/2}\overline{\rho}_{k-1/2}}{\overline{\rho}_{k+1/2} + \overline{\rho}_{k-1/2}}\right)\left(\frac{\psi_{k+1/2} - \psi_{k-1/2}}{N_k^2\Delta z_k}\right)\right]$$

$$\text{for } k = 2, \ldots, K - 2 \tag{27}$$

in place of (10)–(12). Note that $R_{k+1/2}$ is a good analog for the continuous form of potential vorticity, which is $(f_0^2/\overline{\rho})(\partial/\partial z)[(\overline{\rho}/N^2)(\partial\psi/\partial z)]$. Taking $w_{1/2} = 0$ as the lower boundary coundition, in place of (18) one obtains

$$\frac{D_{g1}}{Dt}\left[\nabla_h^2\psi_1 + f + \frac{2f_0^2\overline{\rho}_{3/2}}{\overline{\rho}_1 N_1^2}\frac{(\psi_{3/2} - \psi_{1/2})}{\Delta z_1^2}\right] = 0. \tag{28}$$

One can also re-obtain (15) for $k = 1$ with

$$a_1 = \frac{1}{2}\frac{\overline{\rho}_1}{\overline{\rho}_{3/2}}; \quad b_1 = \frac{\Delta z_2}{\Delta z_1}\frac{\overline{\rho}_2}{\overline{\rho}_{5/2} + \overline{\rho}_{3/2}}; \tag{29}$$

$$R_{3/2} = \frac{f_0^2}{\overline{\rho}_{3/2}\Delta z_1}\left[\left(\frac{2\overline{\rho}_{5/2}\overline{\rho}_{3/2}}{\overline{\rho}_{5/2} + \overline{\rho}_{3/2}}\right)\left(\frac{\psi_{5/2} - \psi_{3/2}}{N_2^2\Delta z_{k+1}}\right) - \overline{\rho}_{3/2}\left(\frac{\psi_{3/2} - \psi_{1/2}}{N_1^2\Delta z_1}\right)\right]. \tag{30}$$

If instead the natural lower boundary condition, $w_{1/2} = 0$, is used, combination of (24) and (25) yields:

$$\frac{D_{g1}}{Dt}\left[\nabla_h^2\psi_1 + f + \frac{2f_0^2\overline{\rho}_{3/2}}{\overline{\rho}_1 N_1^2}\frac{(\psi_{3/2} - \psi_{1/2})}{\Delta z_1^2}\right] - \frac{f_0^2(\overline{\rho}_{3/2} + \overline{\rho}_{1/2})}{g\overline{\rho}_1\Delta z_1}\frac{\partial\psi_{1/2}}{\partial t} = 0. \tag{31}$$

Using the facts that $\partial\psi_{1/2}/\partial t = D_{g1/2}\psi_{1/2}/Dt$ and $J(\psi_1, \psi_{3/2} - \psi_{1/2}) = J(\psi_{1/2}, \psi_{3/2} - \psi_{1/2})$, (31) can be expressed as a conservation relation:

$$\frac{D_{g1}}{Dt}(\nabla_h^2\psi_1 + f) + \frac{D_{g1/2}}{Dt}\left[\frac{2f_0^2\overline{\rho}_{3/2}}{\overline{\rho}_1 N_1^2}\frac{(\psi_{3/2} - \psi_{1/2})}{\Delta z_1^2} - \frac{f_0^2(\overline{\rho}_{3/2} + \overline{\rho}_{1/2})\psi_{1/2}}{g\overline{\rho}_1\Delta z_1}\right] = 0. \tag{32}$$

For this case $\overline{\rho}_{1/2}$ and N_1^2 must be specified to determine (24) and (25) for $k = 1$, so the derivation of (15) leading to (26) and (27) applies also for $k = 1$ and $k = K - 1$.

4. Summary

Analogs of the conservation of potential vorticity conservation have been derived (without making any approximations) for two sets of discretized quasigeostrophic equations, one formulated in pressure coordinates and the other in z coordinates. Both sets use the modified Lorenz grid in which the streamfunction at full levels is given in terms of those at adjacent half levels by (1). In pressure coordinates the governing equations are (7) and (8) and the analog of potential vorticity conservation is (15), which applies for $k = 1, \ldots, K - 1$. The quantities in (15) are given by (10) and (11) for $k = 2, \ldots, K - 2$ and by (11) and (20) for the upper boundary ($k = 1$). The analog of the conservation of generalized potential vorticity at the upper boundary is given by (18). In height coordinates the governing equations are (24) and (25). The analog of potential vorticity conservation is again (15), which applies for $k = 1, \ldots, K - 1$ with its quantities being given by (26) and (27). The analog of the conservation of generalized potential vorticity at the lower boundary is given by (28) when the boundary condition is taken to be $w_{1/2} = 0$, and by (32) when the natural boundary condition $w_{1/2} = 0$, is used.

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