Comments on "Contraction Rate and Its Relationship to Frontogenesis, the Lyapunov Exponent, Fluid Trapping, and Airstream Boundaries"

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In Cohen and Schultz (2005, hereafter CS05), a new concept for kinematic diagnosis has been presented. Overall, I find their presentation fascinating and the ideas contained therein are both stimulating and of considerable value for kinematic applications. However, I found their derivation to be somewhat confusing. Some brief review of their derivation is necessary to set the stage for an alternative derivation that confirms their results. They considered the time evolution of the separation vector $\delta\mathbf{r}(t) = \delta x + i\delta y$ [their Eq. (3)] between two fluid parcels, assuming that it could be represented for a small period of time (during which the linear form of the Taylor series expansion of the velocity field was valid) as their Eq. (2):

$$\delta\mathbf{r}(t) = \delta\mathbf{r}(0)\exp[(\sigma + i\omega)t].$$

In the first use of the complex representation, the real and imaginary parts correspond to the components of a vector. In the second use of the complex representation, the real and imaginary parts correspond to the growth rate and the frequency. Equating the real and imaginary parts of these two apparently different applications of complex representation was rather confusing to me. Hence, I chose to follow a different path and see if it would produce a corresponding result.

Consider Fig. 1, which illustrates the geometry of the separation vector, and in which I have chosen to represent the separation vector as $\delta\mathbf{r}(t) = \delta x\mathbf{i} + \delta y\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the x and y directions, respectively. Therefore, the time rate of change of the separation vector is

$$\frac{d(\delta\mathbf{r})}{dt} = \frac{d(\delta x)}{dt}\mathbf{i} + \frac{d(\delta y)}{dt}\mathbf{j}. \quad (1)$$

Observe that this can be written as

$$\frac{d(\delta\mathbf{r})}{dt} = \frac{d}{dt}(\mathbf{r}_2 - \mathbf{r}_1) = \frac{d}{dt}[(x_2\mathbf{i} + y_2\mathbf{j}) - (x_1\mathbf{i} + y_1\mathbf{j})]$$
$$= \left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)\mathbf{i} + \left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right)\mathbf{j}$$
$$= (u_2 - u_1)\mathbf{i} + (v_2 - v_1)\mathbf{j}. \quad (2)$$

Using the same assumption employed by CS05 that the initial distance between the points is within the neighborhood over which the following expansion is valid:

$$u_2 = u_1 + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y,$$
$$v_2 = v_1 + \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y, \quad (3)$$

then substitution into (2) gives

$$\frac{d(\delta\mathbf{r})}{dt} = \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\mathbf{i} + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\mathbf{j}. \quad (4)$$

Now the time rate of change in the vector connecting two points can be broken into two parts: one due to changes in the *length* of that vector, and the second due to changes in the *orientation* of that vector. The orientation of the separation vector at any time is the angle ϕ (see Fig. 1), noting that

$$\delta x = \delta r\cos\phi,$$
$$\delta y = \delta r\sin\phi. \quad (5)$$

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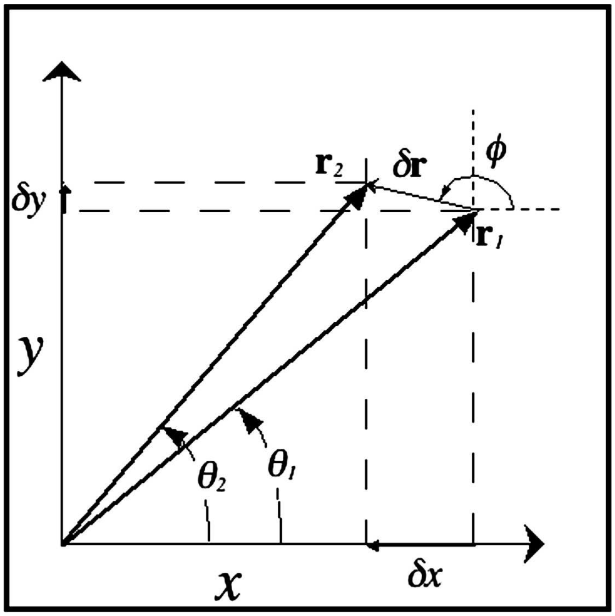

FIG. 1. Schematic illustration showing the components (δx, δy) of the separation vector $\delta\mathbf{r}$ in (x, y) Cartesian space, which has a magnitude δr and an orientation angle ϕ. Position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ for the fluid parcels are also shown.

which also means that

$$\tan\phi = \frac{\delta y}{\delta x} \Rightarrow \phi = \tan^{-1}\left(\frac{\delta y}{\delta x}\right). \tag{6}$$

Next, the length of the separation vector is $\delta r \equiv |\delta\mathbf{r}| = \sqrt{\delta x^2 + \delta y^2}$, so taking the time derivative of δr results in

$$\frac{d(\delta r)}{dt} = \frac{1}{2}\frac{1}{\sqrt{\delta x^2 + \delta y^2}}\left[2\delta x\frac{d(\delta x)}{dt} + 2\delta y\frac{d(\delta y)}{dt}\right]$$

$$= \frac{1}{\delta r}\left[\delta x\frac{d(\delta x)}{dt} + \delta y\frac{d(\delta y)}{dt}\right]. \tag{7}$$

Using (2) and (3) it follows that

$$\frac{d(\delta x)}{dt} = \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right),$$

$$\frac{d(\delta y)}{dt} = \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right). \tag{8}$$

Using (5), this results in the following differential equation for the time rate of change of the *length* of the separation vector (holding the orientation constant):

$$\frac{d(\delta r)}{dt} = \delta r\left[\left(\frac{\partial u}{\partial x}\right)\cos^2\phi + \left(\frac{\partial v}{\partial y}\right)\sin^2\phi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\phi\sin\phi\right]. \tag{9}$$

By inspection, the solution for this simple differential equation for the change in *length* of the separation vector, δr, is indeed given by an exponential growth rate, σ, where

$$\sigma = \left[\left(\frac{\partial u}{\partial x}\right)\cos^2\phi + \left(\frac{\partial v}{\partial y}\right)\sin^2\phi + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\phi\sin\phi\right],$$

which confirms the assumed form of this part of the solution in CS05.

Now consider the change associated with the *orientation* of the separation vector. If the length of the separation vector is held constant, then time differentiation of (6) gives the result that

$$\frac{d\phi}{dt} = \frac{1}{1 + \left(\frac{\delta y}{\delta x}\right)^2}\frac{d}{dt}\left(\frac{\delta y}{\delta x}\right)$$

$$= \frac{1}{\delta x^2 + \delta y^2}\left[\delta x\frac{d(\delta y)}{dt} - \delta y\frac{d(\delta x)}{dt}\right].$$

Again using the foregoing relations, it follows that

$$\frac{d\phi}{dt} \equiv \omega = \frac{\partial v}{\partial x}\cos^2\phi - \frac{\partial u}{\partial y}\sin^2\phi + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\sin\phi\cos\phi,$$

which also confirms the assumed form of the solution for the change in the orientation of the separation vector in CS05

The preceding alternative derivation might be easier for some readers to follow, but it yields precisely the same result presented in CS05. In summary, I have no issue with their results, having satisfied myself that the derivation they followed is valid.

REFERENCE

Cohen, R. A., and D. M. Schultz, 2005: Contraction rate and its relationship to frontogenesis, the Lyapunov exponent, fluid trapping, and airstream boundaries. *Mon. Wea. Rev.,* **133,** 1353–1369.